OPE between the energy-momentum tensor and the Wilson loop in $\mathcal{N} = 4$ SYM theory

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Takehiro Azuma
Department of Physics, Kyoto University

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collaborated with Hikaru Kawai

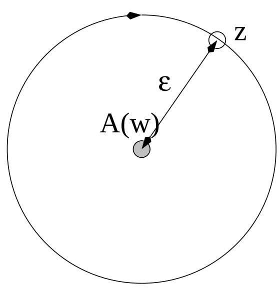
Contents

1 Introduction

Operator product expansion(OPE) is an important notion in conformal field theory.
⇒ OPE extracts the conformal weight of the operator.

The OPE in the 2-dimensional CFT

$$T(z)A(w) = \left(\begin{array}{l}\text{lower-dimensional}\\ \text{operators}\end{array}\right) + \frac{hA(w)}{(z-w)^2} + \frac{\partial A(w)}{(z-w)} + \cdots.$$



- An example of the lower-dimensional operator:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \cdots.$$

- An important special case is a primary field, on which the OPE reduces to

$$T(z)A(w) = \frac{hA(w)}{(z-w)^2} + \frac{\partial A(w)}{(z-w)} + \cdots.$$

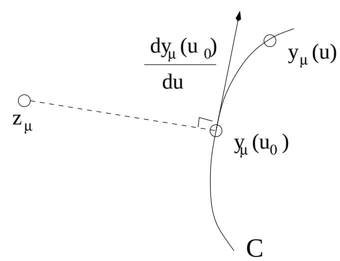
2 The OPE $T_{\mu\nu}(z)W[C]$ in 4-dimensional Euclidean space

We consider the OPE $T_{\mu\nu}(z)W[C]$ in $\mathcal{N}=4$ SYM theory in the Euclidean space.

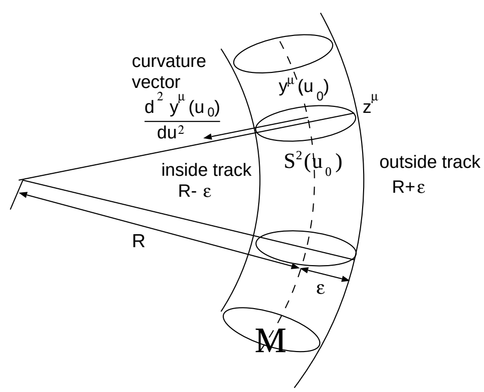
$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2G^2}\left[\frac{1}{2}(F_{\mu\nu})^2 + (D_\mu\phi_i)^2 + ([\phi_i,\phi_j])^2 + \xi(\partial^\mu A_\mu)^2\right], \\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu], \\
D_\mu\phi_i &= \partial_\mu\phi_i - i[A_\mu,\phi_i], \\
W[C] &= \frac{1}{N}\mathrm{Tr}P\exp\left[\oint_C du\left\{iA_\mu(y(u))\frac{dy^\mu(u)}{du} + \phi_i(y(u))\theta^i(u)\right\}\right].
\end{aligned}
$$

- $y_\mu(u)$: coordinate of the Wilson loop
- u : an arc length parameter which satisfies $|\frac{dy_\mu(u)}{du}| = 1$.

$y_\mu(u_0)$: the nearest point on the loop to the point z_μ.



We consider the conformal Ward identity by wrapping the Wilson loop by the enveloping surface of the spheres $S^2(u_0)$.



$$\int_{\mathcal{M}} d^4z \partial^\mu [T_{\mu\nu}(z) W[C] v^\nu(z)]$$
$$= \int du_0 \int_{S^2(u_0)} d\Omega [1 - (z_\alpha - y_\alpha(u_0)\frac{d^2 y^\alpha(u_0)}{du^2})] n^\mu T_{\mu\nu}(z) W[C] v^\nu(z).$$

- $S^2(u_0)$: a S^2 sphere of a fixed radius ϵ which is perpendicular to the loop and has its center at $y_\mu(u_0)$.
- $\mathcal{M}$: the region inside the enveloping surface.
- $v^\mu(z)$: the conformal Killing vector.

$$\begin{aligned} \text{Translation:} \quad & v^\nu(z) = \xi^\nu, \\ \text{Dilatation:} \quad & v^\nu(z) = \lambda z^\nu, \\ \text{SCT:} \quad & v^\nu(z) = 2z^\nu(b_\alpha z^\alpha) - b^\nu z^2. \end{aligned}$$

- $T_{\mu\nu}(z)$: energy-momentum tensor $T_{\mu\nu}(z) = \frac{2}{\sqrt{g}}\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}}$.
- The weakest singularity in the OPE than contribute to the conformal Ward identity is $\mathcal{O}(z - y(u_0))^{-2}$, since the surface area of $S^2(u_0)$ is $4\pi\epsilon^2$.

We expand the OPE $T_{\mu\nu}(z)W[C]$ as a power series expansion in $z_\mu - y_\mu(u_0)$.

$$T_{\mu\nu}(z)W[C] = (T_{\mu\nu}(z)W[C])_c + (T_{\mu\nu}(z)W[C])_{vec} + (T_{\mu\nu}(z)W[C])_{sca}.$$

- $(T_{\mu\nu}(z)W[C])_c$: the terms containing $W[C]$ itself without any insertion of the fields to it.
- $(T_{\mu\nu}(z)W[C])_{vec,sca}$: the insertion of the fields $A_\mu(y(u_0))$ and $\phi_i(y(u_0))$ to $W[C]$, respectively.

$T(z)A(w)$	lower$\cdots$	$\frac{hA(w)}{(z-w)^2}$	$\frac{\partial A(w)}{z-w}$
$T_{\mu\nu}(z)W[C]$	$-$	$(T_{\mu\nu}(z)W[C])_c$	$(T_{\mu\nu}(z)W[C])_{vec}$ $(T_{\mu\nu}(z)W[C])_{sca}$

The general form is obtained up to the free parameters $q, q', \alpha, \beta, \gamma$ by the following reasonings:

- Dimensional analysis
- Properties of $T_{\mu\nu}(z)$ in conformal field theory

  * $T_\mu{}^\mu(z) = 0$: conformal invariance of the action.

  * $\partial^\mu T_{\mu\nu}(z) = 0$: the conservation law of the energy and the momentum.

- The Wilson loop possesses no conformal weight and only the deformation occurs with respect to translation:

$$\int_{\mathcal{M}} d^4z \partial^\mu [T_{\mu\nu}(z) W[C] \xi^\nu] = -\int ds (\frac{\delta W[C]}{\delta y^\nu(s)}) \xi^\nu.$$

  * s: general parameterization running over $0 \le s \le 2\pi$. The relationship to the arc length parameter is $\frac{du_0}{ds} = |\frac{dy_\mu(s)}{ds}|$.

  * $(\frac{\delta W[C]}{\delta y^\nu(s)})$: deformation of the Wilson loop.

$$\begin{aligned}
(\frac{\delta W[C]}{\delta y^\nu(s)}) &= \frac{1}{N} TrP\hat{w}_{2\pi,s}[iF_{\nu\alpha}(y(s))\frac{dy^\alpha(s)}{ds} \\
&\quad +|\frac{dy_\mu(s)}{ds}|\theta_i(s) D_\nu \phi^i(y(s))]\hat{w}_{s,0} \\
&- \frac{d}{ds}[\frac{1}{N} TrP\hat{w}_{2\pi,s}\{\theta_i(s)\phi^i(y(s))\frac{\frac{dy_\nu(s)}{ds}}{|\frac{dy_\mu(s)}{ds}|}\}\hat{w}_{s,0}], \\
\hat{w}_{b,a} &= \exp[\int_a^b ds\{iA_\mu(y(s))\frac{dy^\mu(s)}{ds} + |\frac{dy_\mu(s)}{ds}|\phi_i(y(s))\theta^i(s)\}].
\end{aligned}$$

- We further assume that the same holds true of dilatation:

$$\int_{\mathcal{M}} d^4z \partial^\mu [T_{\mu\nu}(z) W[C] \lambda z^\nu] = -\int ds (\frac{\delta W[C]}{\delta y^\nu(s)}) \lambda y^\nu(s).$$

$$(T_{\mu\nu}(z)W[C])_c = \frac{q}{24\pi^2}[\frac{1}{|z-y(u_0)|^4}(g_{\mu\nu} - 2\frac{dy_\mu(u_0)}{du}\frac{dy_\nu(u_0)}{du}$$
$$-2\frac{(z_\mu - y_\mu(u_0))(z_\nu - y_\nu(u_0))}{|z-y(u_0)|^2})]W[C]$$
$$+ \quad \frac{1}{24\pi^2}[-2(q+q')\frac{(z_\mu - y_\mu(u_0))(z_\nu - y_\nu(u_0))(z_\alpha - y_\alpha(u_0))\frac{d^2y^\alpha(u_0)}{du^2}}{|z-y(u_0)|^6}$$
$$+(-q+\frac{q'}{2})\frac{(z_\mu - y_\mu(u_0))\frac{d^2y_\nu(u_0)}{du^2} + (z_\nu - y_\nu(u_0))\frac{d^2y_\mu(u_0)}{du^2}}{|z-y(u_0)|^4}$$
$$+(-4q+q')\frac{\frac{y_\mu(u_0)}{du}\frac{y_\nu(u_0)}{du}(z_\alpha - y_\alpha(u_0))\frac{d^2y^\alpha(u_0)}{du^2}}{|z-y(u_0)|^4}$$
$$+2qg_{\mu\nu}\frac{(z_\alpha - y_\alpha(u_0))\frac{d^2y^\alpha(u_0)}{du^2}}{|z-y(u_0)|^4}]W[C] + \mathcal{O}(z-y(u_0))^{-2},$$

$$(T_{\mu\nu}(z)W[C])_{vec} = \frac{1}{N}TrPw_{L,u_0}\frac{i}{4\pi|z-y(u_0)|^3}$$
$$\times[-(z_\mu - y_\mu(u_0))F_{\nu\alpha}(y(u_0))\frac{dy^\alpha(u_0)}{du} - (z_\nu - y_\nu(u_0))F_{\mu\alpha}(y(u_0))\frac{dy^\alpha(u_0)}{du}$$
$$+g_{\mu\nu}(z^\alpha - y^\alpha(u_0))F_{\alpha\beta}(y(u_0))\frac{dy^\beta(u_0)}{du}$$
$$+(z^\alpha - y^\alpha(u_0))[F_{\mu\alpha}(y(u_0))\frac{dy_\nu(u_0)}{du} + F_{\nu\alpha}(y(u_0))\frac{dy_\mu(u_0)}{du}]]w_{u_0,0}$$
$$+\mathcal{O}(z-y(u_0))^{-1},$$

$$
\begin{aligned}
(T_{\mu\nu}(z)W[C])_{sca} &= \frac{1}{N}TrPw_{L,u_0}[\frac{\theta_i(u_0)\phi^i(y(u_0))}{24\pi|z-y(u_0)|^3}(2-\frac{\alpha}{10}+\frac{\beta}{2}) \\
&\times[g_{\mu\nu}-3\frac{(z_\mu-y_\mu(u_0))(z_\nu-y_\nu(u_0))}{|z-y(u_0)|^2}-\frac{dy_\mu(u_0)}{du}\frac{dy_\nu(u_0)}{du}] \\
&+\frac{\theta_i(u_0)\phi^i(y(u_0))}{24\pi|z-y(u_0)|^3}[(1+\frac{\alpha}{10}-\frac{\beta}{2}+\frac{\gamma}{5})((z_\mu-y_\mu(u_0))\frac{d^2y_\nu(u_0)}{du^2}+(z_\nu-y_\nu(u_0))\frac{d^2y_\mu(u_0)}{du^2}) \\
&+(1-\frac{\alpha}{5}+\beta-\frac{\gamma}{5})g_{\mu\nu}(z_\alpha-y_\alpha(u_0))\frac{dy^\alpha(u_0)}{du^2} \\
&+(-3+\frac{3\alpha}{5}-3\beta-\frac{3\gamma}{5})\frac{(z_\mu-y_\mu(u_0))(z_\nu-y_\nu(u_0))(z_\alpha-y_\alpha(u_0))\frac{dy^\alpha(u_0)}{du^2}}{|z-y(u_0)|^3} \\
&+(-3+\gamma)\frac{dy_\mu(u_0)}{du}\frac{dy_\nu(u_0)}{du}(z_\alpha-y_\alpha(u_0))\frac{d^2y^\alpha(u_0)}{du^2}] \\
&+\frac{\frac{d\theta_i(u_0)}{du}}{24|z-y(u_0)|^3}\phi^i(y(u_0))(2+\frac{\alpha}{5}-\beta)[(z_\mu-y_\mu(u_0))\frac{dy_\nu(u_0)}{du}+(z_\nu-y_\nu(u_0))\frac{dy_\mu(u_0)}{du}] \\
&+\frac{\theta_i(u_0)}{24\pi|z-y(u_0)|^3}[(-4+\frac{\alpha}{5})((z_\mu-y_\mu(u_0))D_\nu\phi^i(y(u_0))+(z_\nu-y_\nu(u_0))D_\mu\phi^i(y(u_0))) \\
&+(4-\frac{\alpha}{5})g_{\mu\nu}(z_\alpha-y_\alpha(u_0))D^\alpha\phi^i(y(u_0)) \\
&+(-6-\frac{3\alpha}{5})\frac{(z_\mu-y_\mu(u_0))(z_\nu-y_\nu(u_0))(z_\alpha-y_\alpha(u_0))D^\alpha\phi^i(y(u_0))}{|z-y(u_0)|^2} \\
&+(-2+\alpha)\frac{dy_\mu(u_0)}{du}\frac{dy_\nu(u_0)}{du}(z_\alpha-y_\alpha(u_0))D^\alpha\phi^i(y(u_0)) \\
&+\beta\frac{dy_\alpha(u_0)}{du}(D^\alpha\phi^i(y(u_0)))[(z_\mu-y_\mu(u_0))\frac{dy_\nu(u_0)}{du} \\
&+(z_\nu-y_\nu(u_0))\frac{dy_\mu(u_0)}{du}]]]w_{u_0,0}+\mathcal{O}(z-y(u_0))^{-1}.
\end{aligned}
$$

The Wilson loop possesses no conformal weight also with respect to the special conformal transformation:

$$
\begin{aligned}
&\int_{\mathcal{M}} d^4z\partial^\mu[T_{\mu\nu}(z)W[C]v^\nu(z)] \\
&= -\int ds(\frac{\delta W[C]}{\delta y^\nu(s)})[2y^\nu(s)(b_\alpha y^\alpha(s))-b^\nu(y(s))^2].
\end{aligned}
$$

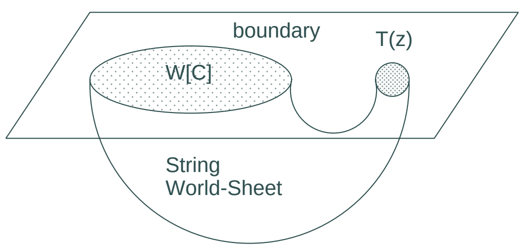
3 Conclusion

Summary

- The Wilson loop possesses no conformal weight with respect to the translation, dilatation and special conformal transformation.
- The Wilson loop undergoes only a deformation with respect to these conformal transformation.

Future Problem

- The computation of $T_{\mu\nu}W[C]$ in terms of supergravity in AdS space



- The computation of the circular $\langle W[C]\rangle$.
  N. Drukker, D. J. Gross and H. Ooguri, hep-th/9904191
  They argued that the expectation value of the circular Wilson loop in $\mathcal{N} = 4$ SYM is

$$\langle W[C_{circ}]\rangle = \frac{1}{N} L^1_{N-1}(-\frac{\lambda}{4N}) \exp(\frac{\lambda}{8N})$$
$$= \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) + \frac{\lambda}{48N^2} I_2(\sqrt{\lambda}) + \frac{\lambda^2}{1280N^4} I_4(\sqrt{\lambda}) + \cdots .$$

Our result may serve to discuss the gauge invariance of this quantity.