Monte Carlo studies of the spontaneous rotational symmetry breaking in a matrix model with the complex action

Takehiro Azuma [Visiting Fellowship at Tata Institute of Fundamental Research (TIFR)]

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Collaboration with K.N. Anagnostopoulos and J. Nishimura

Contents

1 Introduction

Matrix models as a constructive definition of superstring theory

IKKT model (IIB matrix model) ⇒ Promising candidate for the constructive definition of superstring theory.

N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, hep-th/9612115.

$$S = \frac{1}{g^2}\left(-\frac{1}{4}\mathrm{tr}\,[A_\mu, A_\nu]^2 + \frac{1}{2}\mathrm{tr}\,\bar{\psi}\Gamma^\mu[A_\mu, \psi]\right).$$

- **Dimensional reduction of $\mathcal{N} = 1$ 10d Super-Yang-Mills (SYM) theory to 0d.**
  A_μ (10d vector) and ψ (10d Majorana-Weyl spinor) are $N \times N$ matrices.
  Eigenvalues of A_μ ⇒ spacetime coordinate.
- **Matrix regularization of Green-Schwarz action of type IIB superstring theory.**
- **$\mathcal{N} = 2$ supersymmetry in 10 dimensions.**
- **Matrices describe the many-body system.**

- **No free parameters:** $A_\mu \to g^{\frac{1}{2}} A_\mu$, $\psi \to g^{\frac{3}{4}} \psi$.

- **Evidences for spontaneous breakdown of SO(10) symmetry to SO(4).**
  J. Nishimura and F. Sugino, hep-th/0111102, H. Kawai, et. al. hep-th/0204240,0211272,0602044,0603146.

- **Complex action (after integrating out fermions) :**

  * **Crucial for spontaneous breakdown of rotational symmetry:**
    J. Nishimura and G. Vernizzi, hep-th/0003223.

  * **Difficulty of Monte Carlo simulation**

2 Simplified IKKT model

Simplified model with spontaneous rotational symmetry breakdown,

J. Nishimura, hep-th/0108070.

$$S = \underbrace{\frac{N}{2}\mathrm{tr}\, A_\mu^2}_{=S_b} \underbrace{- \bar{\psi}_\alpha^f (\Gamma_\mu)_{\alpha\beta} A_\mu \psi_\beta^f}_{=S_f}$$

- A_μ: $N \times N$ hermitian matrices ($\mu = 1, \cdots, 4$)
  $\bar{\psi}_\alpha^f, \psi_\alpha^f$: N-dim vector ($\alpha = 1, 2, f = 1, \cdots, N_f$), N_f = (number of flavors).

$$\Gamma_1 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \ \Gamma_2 = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \Gamma_3 = i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ \Gamma_4 = \sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- SU(N) symmetry and SO(4) rotational symmetry.

- Partition function:

$$Z = \int dA e^{-S_B} (\det \mathcal{D})^{N_f} = \int dA e^{-S_0} e^{i\Gamma}, \text{ where}$$
$$\mathcal{D} = \Gamma_\mu A_\mu = (2N \times 2N \text{ matrices}), \quad e^{-S_0} = e^{-S_B} |\det \mathcal{D}|^{N_f}.$$

Analytical studies of the model

Solvable at $N \to \infty$ using random matrix theory (RMT) technique.

$$\langle \frac{1}{N} \mathrm{tr}\, A_\mu^2 \rangle = \begin{cases} 1 + r, & (\mu = 1,2,3) \\ 1 - r, & (\mu = 4), \end{cases}$$

for small $r = N_f/N$.

Spontaneous breakdown of SO(4) symmetry to SO(3).

For the phase-quenched partition function $Z_0 = \int dA e^{-S_0}$,

$\langle \frac{1}{N} \mathrm{tr}\, A_\mu^2 \rangle = 1 + r/2$ for $\mu = 1,2,3,4$.

The phase plays a crucial role in the spontaneous rotational symmetry breakdown.

Gaussian expansion analysis up to 9th order:

T. Okubo, J. Nishimura and F. Sugino, hep-th/0412194.

Spontaneous breakdown of SO(4) to SO(2) at finite r.

3 Monte Carlo studies of the model

Brief History of the Monte Carlo simulation of large-N reduced models

Bosonic models

- **Simulation of bosonic Yang-Mills model** T. Hotta, J. Nishimura and A. Tsuchiya, hep-th/9811220
- **Simulation of bosonic Yang-Mills-Chern-Simons models**

  ⇒ **Properties of fuzzy manifolds (fuzzy S^2, S^4, CP^2, $S^2 \times S^2$).**

  T. Azuma, S. Bal, K. Nagao and J. Nishimura hep-th/0401038,0405096,0405277,0506205
- **Simulation of finite-temperature BFSS-type (0+1)d models.**

  N. Kawahara, J. Nishimura and S. Takeuchi, in preparation.

Supersymmetric models

Simulation of IIB matrix model is difficult due to sign problem.

- **hybrid R (or hybrid Monte Carlo) simulation of the 4d supersymmetric model (fermion determinant is real positive, O($N^{5,(6)}$) CPU times).**

  J. Ambjorn, K. N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, hep-th/0003208,
  K. N. Anagnostopoulos, T. Azuma, K. Nagao and J. Nishimura, hep-th/0506062.

- **hybrid Monte Carlo simulation of the one-loop effective action of the quenched 10d IIB matrix model, (O(N^3) CPU time).**

  J. Ambjorn, K. N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, hep-th/0005147.

Complex action plays a key role in spontaneous breakdown of Lorentz symmetry:

J. Nishimura and G. Vernizzi, hep-th/0003223.

- **Factorization method to simulate a complex action system.**

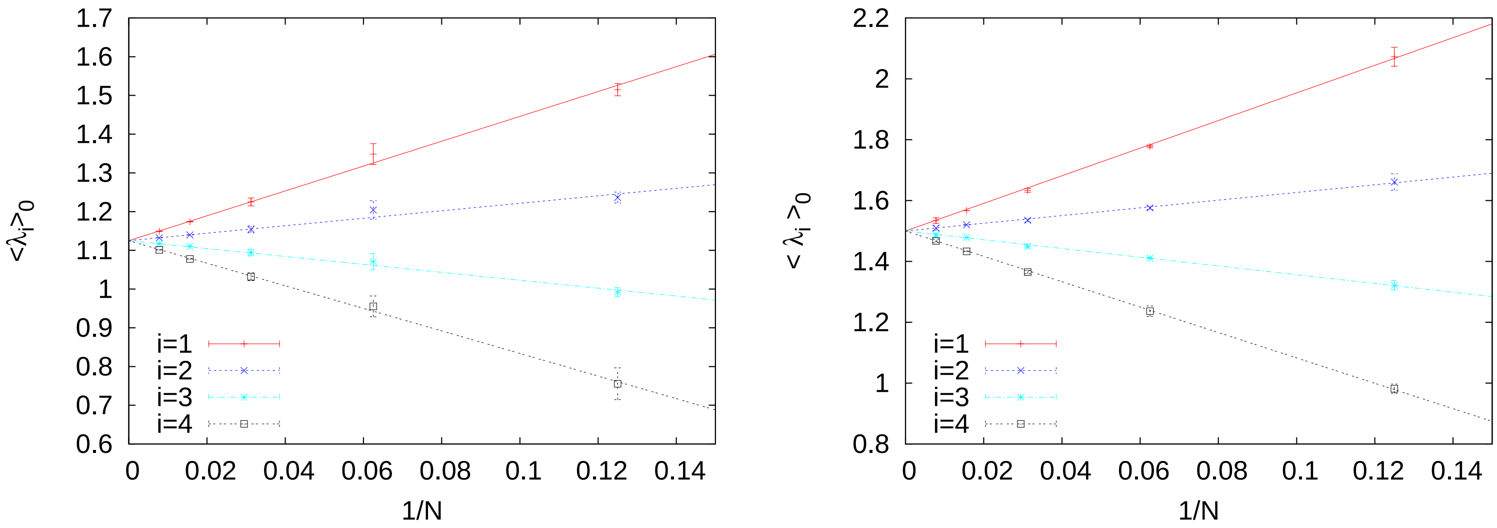
  K. N. Anagnostopoulos and J. Nishimura, hep-th/0108041,
  J. Ambjorn, K. N. Anagnostopoulos, J. Nishimura and J. J. M. Verbaarschot, hep-lat/0208025 .

Hybrid Monte Carlo (HMC) simulation of the phase-quenched model

HMC simulation of the partition function Z_0 with the phase omitted.

Observable for probing dimensionality : $T_{\mu\nu} = \frac{1}{N}\mathrm{tr}\,(A_\mu A_\nu)$.

λ_i $(i = 1, 2, 3, 4)$: eigenvalues of $T_{\mu\nu}$ $(\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4)$



Results for $r = \frac{1}{4}$ (left) and $r = 1$ (right), for $N = 8, 16, 32, 64, 128$.

$$\lambda_1 = \cdots = \lambda_4 \to 1 + \frac{r}{2} \text{ (as } N \to \infty).$$

Factorization method

An approach to the complex action problem in Monte Carlo simulation.

K. N. Anagnostopoulos and J. Nishimura, hep-th/0108041,
J. Ambjorn, K. N. Anagnostopoulos, J. Nishimura and J. J. M. Verbaarschot, hep-lat/0208025.

Overlap problem: Discrepancy of a distribution function between the phase-quenched model Z_0 and the full model Z.

Force the simulation to sample the important region for the full model.

Standard reweighting method:

$$\langle \lambda_i \rangle = \frac{\langle \lambda_i \cos\Gamma \rangle_0}{\langle \cos\Gamma \rangle_0}, \text{ where } \langle * \rangle_0 = (\text{ V.E.V. for the phase-quenched model } Z_0).$$

(Number of configurations required) $\simeq e^{O(N^2)}$. $\Rightarrow$ complex-action problem.

$\tilde{\lambda}_i \stackrel{\text{def}}{=} \lambda_i / \langle \lambda_i \rangle_0$: deviation from 1 $\Rightarrow$ effect of the phase.

Distribution function

$$\rho_i(x) \stackrel{\text{def}}{=} \langle \delta(x - \tilde{\lambda}_i) \rangle = \frac{1}{C} \rho_i^{(0)}(x) w_i(x),$$

where

$$C = \langle \cos \Gamma \rangle_0, \quad \rho_i^{(0)}(x) = \langle \delta(x - \tilde{\lambda}_i) \rangle_0, \quad w_i(x) = \langle \cos \Gamma \rangle_{i,x},$$

$$\langle * \rangle_{i,x} = [\text{V.E.V. for the partition function } Z_{i,x} = \int dA e^{-S_0} \delta(x - \tilde{\lambda}_i)].$$

Resolution of the overlap problem: The system is forced to visit the configurations where $\rho_i(x)$ is important.

In practice, we approximate the partition function $Z_{i,x}$ by

$$Z_{i,V} = \int dA e^{-S_0} e^{-V(\lambda_i)}, \text{ where } V(x) = \frac{\gamma}{2}(x-\xi)^2, \quad \gamma, \xi = (\text{parameters}).$$

Monte Carlo evaluation of $\rho_i^{(0)}(x)$ and $w_i(x)$:

$$\rho_{i,V}(x) \stackrel{\text{def}}{=} \langle \delta(x - \tilde{\lambda}_i) \rangle_{i,V} \propto \rho_i^{(0)}(x) \exp(-V(\langle \lambda_i \rangle_0 x)).$$

The position of the peak x_p for the distribution function $\rho_{i,V}(x)$:

$$0 = \frac{\partial}{\partial x} \log \rho_{i,V}(x) = f_i^{(0)}(x) - \langle \lambda_i \rangle_0 V'(\langle \lambda_i \rangle_0 x), \text{ where } f_i^{(0)}(x) \stackrel{\text{def}}{=} \frac{\partial}{\partial x} \log \rho_i^{(0)}(x).$$

- **Determination of x_p: $\rho_{i,V}(x)$ has a sharp peak for large γ**
  $\Rightarrow$ x_p is approximated as $x_p \simeq \langle \tilde{\lambda}_i \rangle_{i,V}$.
- **Determination of $\rho_i^{(0)}(x)$: Vary ξ, and calculate $f_i^{(0)}(x_p)$ for different x_p.**
  Then, evaluate $\rho_i^{(0)}(x) = \exp[\int_0^x dz f_i^{(0)}(z) + \text{const.}]$.

Why such a roundabout way? $\Rightarrow$ to capture the skirt of $\rho_i^{(0)}(x)$.

Monte Carlo evaluation of $\langle \tilde{\lambda}_i \rangle$

$\tilde{\lambda}_i = \lambda_i / \langle \lambda_i \rangle_0$: **deviation from phase-quenched model.**

Direct evaluation:

$$\langle \tilde{\lambda}_i \rangle = \int_0^\infty dx \, x \rho_i(x) = \frac{\int_0^\infty dx \, x \rho_i^{(0)}(x) w_i(x)}{\int_0^\infty dx \, \rho_i^{(0)}(x) w_i(x)}.$$

Difficult because $w_i(x) \simeq 0$ at large N.

The errorbar must be very small ($w_i(x) = 0.04 \pm 0.05$ no longer makes sense).

$w_i(x) > 0 \Rightarrow \langle \tilde{\lambda}_i \rangle$ **is the minimum of $\mathcal{F}_i(x)$:**

$$\mathcal{F}_i(x) = (\text{free energy density}) = -\frac{1}{N^2} \log \rho_i(x).$$

We solve $\mathcal{F}'_i(x) = 0$, namely

$$\frac{1}{N^2} f_i^{(0)}(x) = -\frac{d}{dx}\left(\frac{1}{N^2} \log w_i(x)\right).$$

Result for $r = N_f/N = 1$

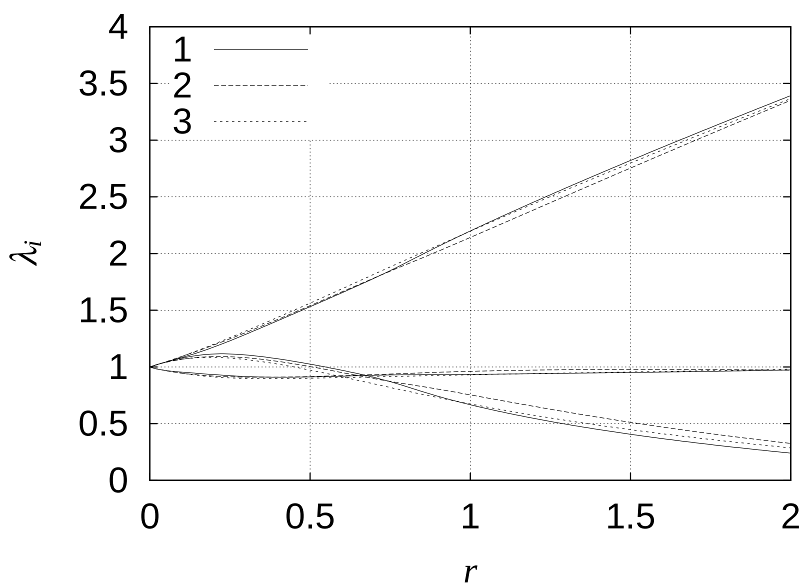
Result for 9th-order Gaussian expansion:

T. Okubo, J. Nishimura and F. Sugino, hep-th/0412194.

$\tilde{\lambda}_{i=1} \simeq 1.4,\ \tilde{\lambda}_{i=2} \simeq 1.4,\ \tilde{\lambda}_{i=3} \simeq 0.7,\ \tilde{\lambda}_{i=4} \simeq 0.5.$

Spontaneous breakdown of the rotational symmetry SO(4) → SO(2).

Quoted from Figure 4 (right) of hep-th/0412194.



Both $\frac{1}{N^2}\log w_i(x)$ and $\frac{1}{N^2} f_i^{(0)}(x)$ scale at large N as

$$\frac{1}{N^2}\log w_i(x) \to \Phi_i(x), \quad \frac{1}{N^2} f_i^{(0)}(x) \to F_i(x).$$

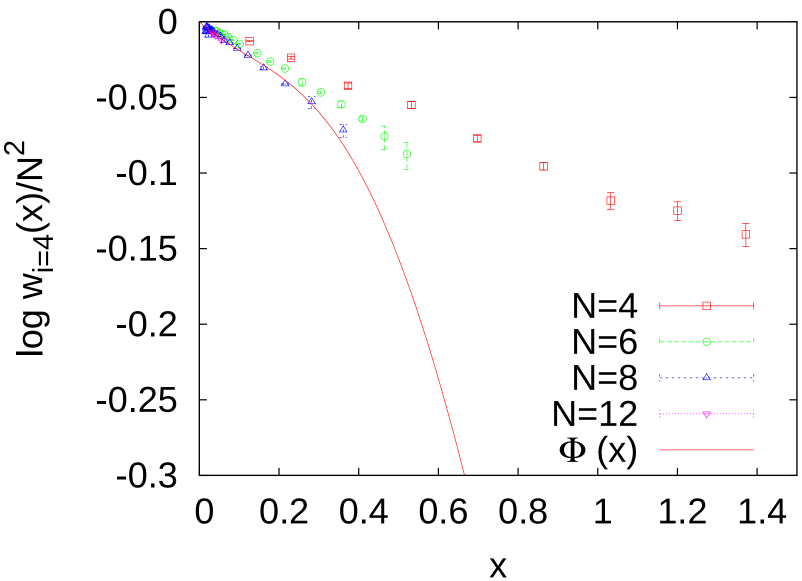
The minimum of "free energy density" is obtained by

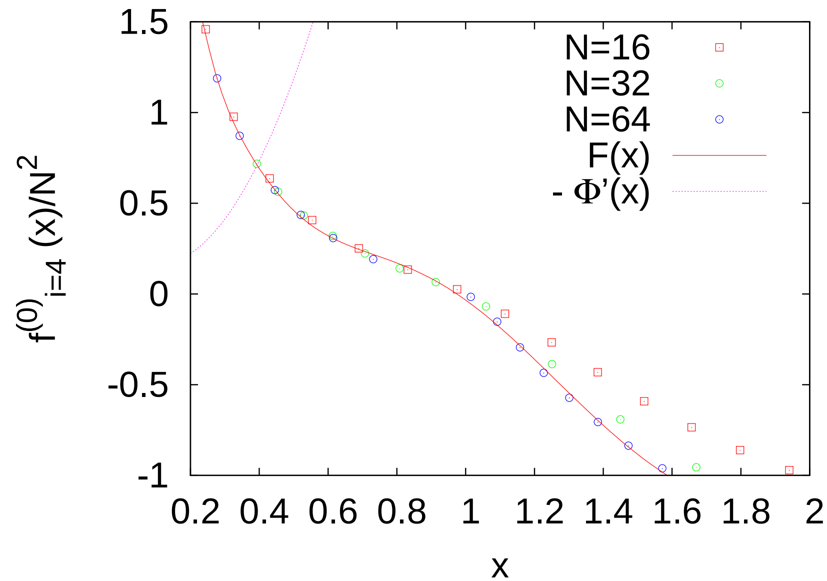
$$F_i(x) + \Phi'(x) = 0.$$

Fitting of $F_i(x)$:

$$F_i(x) \simeq a_{i,0} + (a_{i,1}x + \frac{b_{i,1}}{x}) + \cdots + (a_{i,4}x^4 + \frac{b_{i,4}}{x^4}).$$

$i = 4$ case





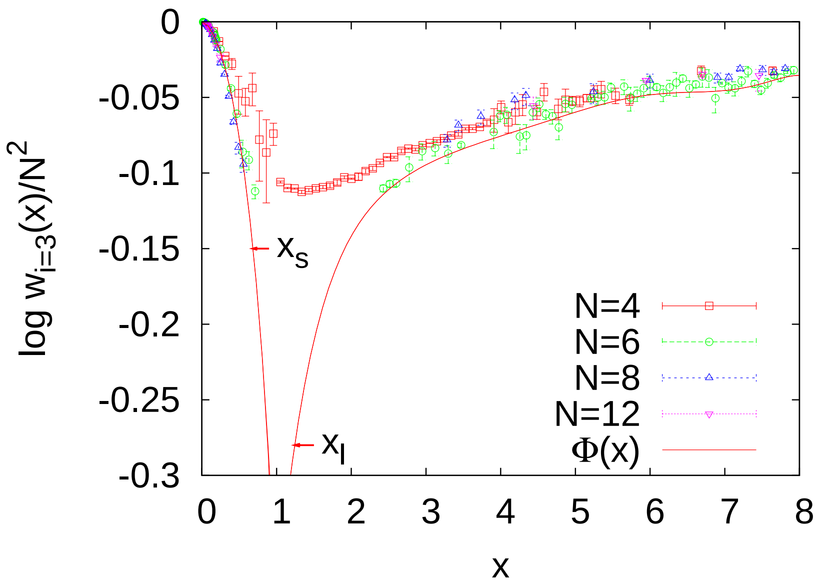
$\Phi_{i=4}(x)$ decreases monotonously $\Rightarrow$ One extremum of "free energy density"

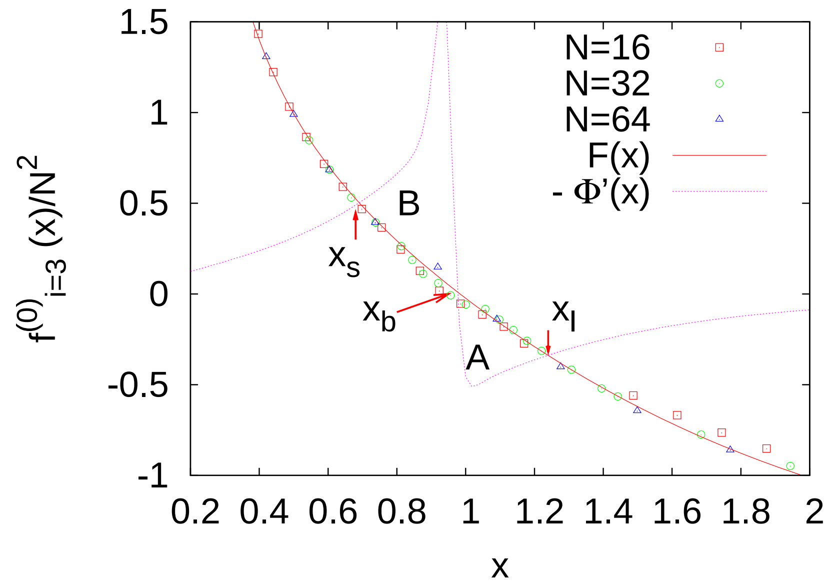
$\Rightarrow$ single-peak structure of $\rho_{i=4}(x)$.

$\Phi_i(x)$: fitted by 4-th order polynomial.

$$\langle \tilde{\lambda}_{i=4} \rangle \simeq 0.4.$$

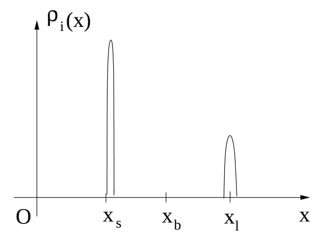
$i = 3$ case





Three extrema of "free energy density" ⇒ double-peak structure of $\rho_{i=3}(x)$.

$x_s \simeq 0.7$, $x_l \simeq 1.2$ $(x_s < x_b < x_l)$.



Which peak is the higher, x_s or x_l?

Extrapolation of $\Phi_i(x)$:

$$\Phi_i(x) \simeq \begin{cases} \phi_{i,s}(x) = c_{i,0} + c_{i,1}x + \cdots + c_{i,4}x^4, \quad (x < x_s), \\ \phi_{i,l}(x) = d_{i,0} + d_{i,1}x + \cdots + d_{i,8}x^8, \quad (x > x_l), \\ \frac{\phi_{i,s}(x)e^{-\mathcal{C}(x-\alpha)} + \phi_{i,l}(x)e^{\mathcal{C}(x-\alpha)}}{e^{-\mathcal{C}(x-\alpha)} + e^{\mathcal{C}(x-\alpha)}}, \\ (x_s < x < x_l). \end{cases}$$

At $x = \alpha$, $\phi_{i,s}(x) = \phi_{i,l}(x)$.

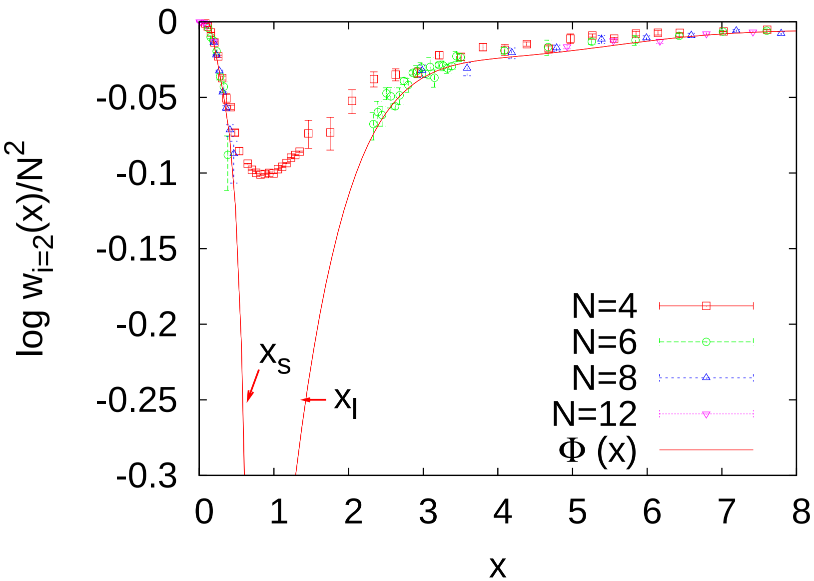
- $\frac{1}{N^2}(\log \rho_i(x_l) - \log \rho_i(x_b)) = \int_{x_b}^{x_l} dx (F_i(x) + \Phi_i'(x)) =$ (A's area).
- $\frac{1}{N^2}(\log \rho_i(x_s) - \log \rho_i(x_b)) = -\int_{x_s}^{x_b} dx (F_i(x) + \Phi_i'(x)) =$ (B's area).

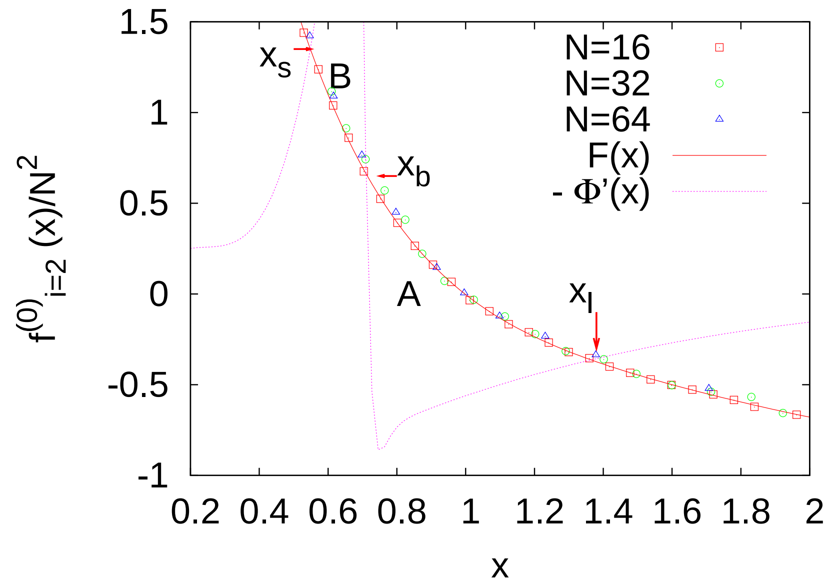
Difference of the height:

$$\Delta_i = \frac{1}{N^2}(\log \rho_i(x_l) - \log \rho_i(x_s)) = (\Phi_i(x_l) - \Phi_i(x_s)) + \int_{x_s}^{x_l} dx F_i(x)$$
$$= \text{(A's area)-(B's area)} \simeq -0.10.$$

The higher peak lies at $x_s \Rightarrow \langle \tilde{\lambda}_{i=3} \rangle \simeq 0.7$.

$i = 2$ case





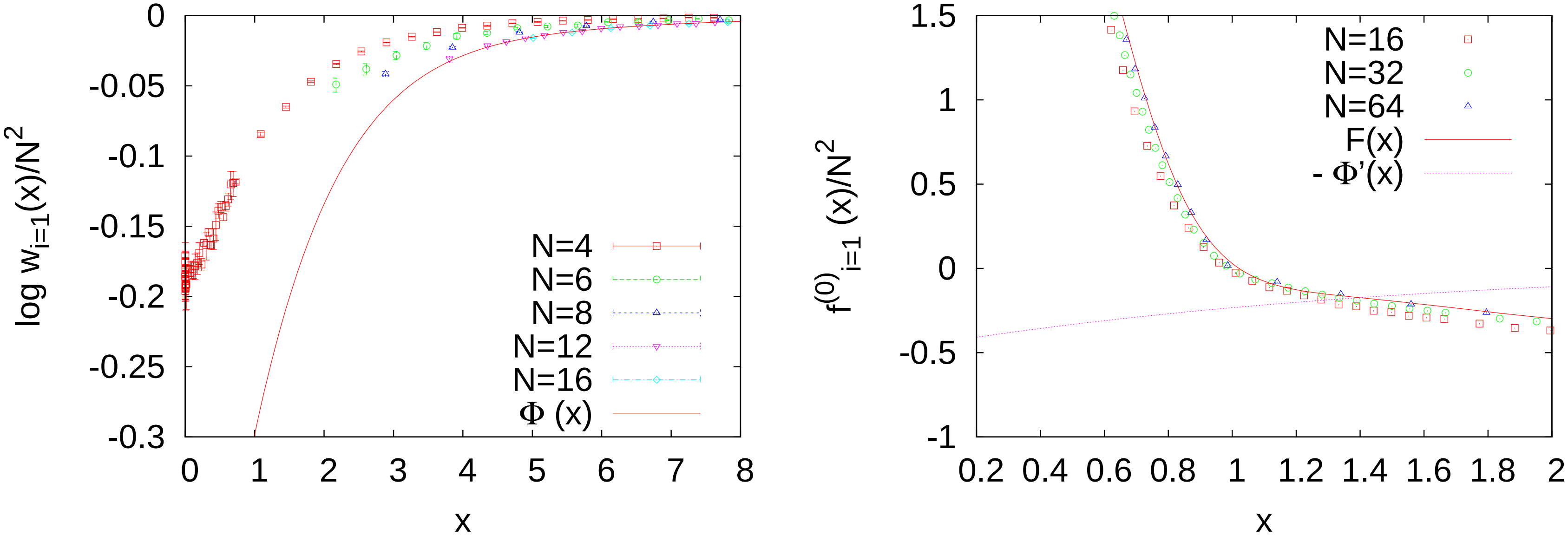
Three extrema of "free energy density" $\Rightarrow$ double-peak structure of $\rho_{i=2}(x)$.

$x_s \simeq 0.6$, $x_l \simeq 1.4$ $(x_s < x_b < x_l)$.

$\Phi_{i=2}(x)$ is fitted similarly to $\Phi_{i=3}(x)$.

$\Delta_{i=2} \simeq 0.12 \Rightarrow$ The higher peak lies at $x_l \Rightarrow \langle \tilde{\lambda}_{i=2} \rangle \simeq 1.4$.

$i = 1$ case



$\Phi_{i=1}(x)$ increases monotonously $\Rightarrow$ One extremum of "free energy density"

$\Rightarrow$ single-peak structure of $\rho_{i=1}(x)$.

$\Phi_i(x)$: fitted by 4-th order polynomial.

$$\langle \tilde{\lambda}_{i=1} \rangle \simeq 1.4.$$

VEV's $\langle \tilde{\lambda}_{i=1,2,3,4} \rangle$ are consistent with 9th order Gaussian expansion method.

Spontaneous breakdown of the rotational symmetry $SO(4) \to SO(2)$.

4 Conclusion

Monte Carlo simulation of the simplified IKKT model via factorization method.

Simulation of the $r = 1$ case $\rightarrow$ symmetry breakdown of SO(4) to SO(2).

Future problems

- **Application of the multi-canonical method to matrix models.**

  B. A. Berg and T. Neuhaus, hep-lat/9202004 .

  Problem of factorization method: Many simulations for different ξ.

  Multicanonical simulation $\Rightarrow$ We can exhaust various ξ with one simulation.

- **Simulation of the 6,10-dimensional IKKT model**

  It costs $O(N^6)$ CPU time.

  However, the effect of the phase may be milder than this simplified model.

Algorithm of Hybrid Monte Carlo (HMC) simulation

Hybrid Monte Carlo simulation ⇒ standard technique to incorporate fermions.

CPU cost: IKKT model (fermion is adjoint rep.) $O(N^6)$,

our simplified model (fermion is vector rep.) $O(N^3)$.

P_μ : (auxiliary bosonic hermitian matrix → conjugate momentum of A_μ)

$$S_{\rm HMC}[P,A] = \frac{1}{2}{\rm tr}\,(P_\mu^2) + S_0[A] + S_{{\rm pot},I}[A], \text{ where}$$

$$S_0[A] = \frac{N}{2}{\rm tr}\,A_\mu^2 - N_f \log|\det \mathcal{D}|, \quad \mathcal{D} = \Gamma_\mu A_\mu, \quad S_{{\rm pot},I} = \frac{\gamma}{2}(\lambda_I - \xi)^2.$$

1. **Update $P_\mu(\tau = 0)$ with a Gaussian random number.**

   Inherit $A_\mu(\tau = 0)$ from the previous sweep.

   τ: fictitious time of the classical system $(0 \le \tau \le T)$.

2. We solve the Hamiltonian equation of motion.

$$\frac{d(A_\mu)_{ij}}{d\tau} = \frac{\partial S_{\rm HMC}}{\partial (P_\mu)_{ij}} = (P_\mu)_{ji},$$

$$\frac{d(P_\mu)_{ij}}{d\tau} = -\frac{\partial S_{\rm HMC}}{\partial (A_\mu)_{ij}} = -N(A_\mu)_{ji} + \frac{N_f}{2}\left\{ {\rm Tr}(\mathcal{D}^{-1}\frac{\partial \mathcal{D}}{\partial (A_\mu)_{ij}}) + {\rm Tr}(\mathcal{D}^{-1}\frac{\partial \mathcal{D}}{\partial (A_\mu)_{ji}})^* \right\}$$

$$-\frac{2\gamma}{N}(\lambda_I - \xi)\left(\sum_{\nu=1}^{4} v_\mu^{(I)} v_\nu^{(I)} (A_\nu)_{ji}\right).$$

$v_\mu^{(I)}$: eigenvector of $T_{\mu\nu} = \frac{1}{N}{\rm tr}\,(A_\mu A_\nu)$.

$\sum_{\rho=1}^{4} T_{\nu\rho} v_\rho^{(I)} = \lambda_I v_\nu^{(I)}$, normalized as $\sum_{\nu=1}^{4} v_\nu^{(I)} v_\nu^{(I)} = 1$.

3. Old configuration: $[P_\mu^{(\rm old)}, A_\mu^{(\rm old)}] = [P_\mu(\tau=0), A_\mu(\tau=0)]$,

New configuration: $[P_\mu^{(\rm new)}, A_\mu^{(\rm new)}] = [P_\mu(\tau=T), A_\mu(\tau=T)]$.

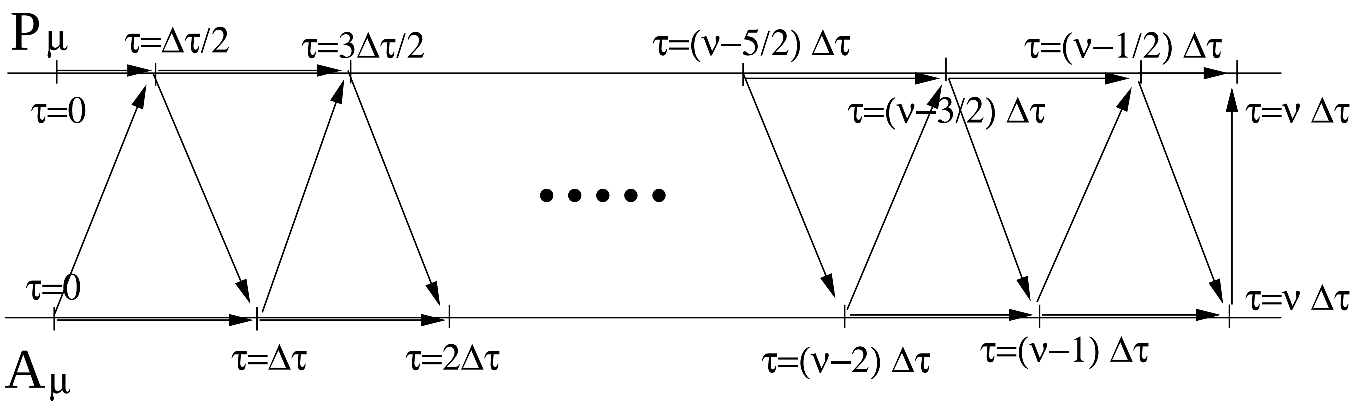
Metropolis accept/reject procedure:

Accept the new configuration with the probability max(1,$e^{-\Delta S_{\rm HMC}}$),

$$\Delta S_{\rm HMC} = S_{\rm HMC}[P_\mu^{(\rm new)}, A_\mu^{(\rm new)}] - S_{\rm HMC}[P_\mu^{(\rm old)}, A_\mu^{(\rm old)}].$$

Leap frog discretization: We solve the discretized Hamiltonian equation of motion.

$\Delta\tau$: step size, $T = \nu\Delta\tau$.



$$
\begin{aligned}
(P_\mu^{(1/2)})_{ij} &= (P_\mu^{(0)})_{ij} - \frac{\Delta\tau}{2}\frac{dS_{\rm HMC}}{d(A_\mu)_{ij}}(A_\mu^{(0)}) \ , \\
(A_\mu^{(1)})_{ij} &= (A_\mu^{(0)})_{ij} + \Delta\tau\,(P_\mu^{(1/2)})_{ji} \ , \\
(P_\mu^{(n+1/2)})_{ij} &= (P_\mu^{(n-1/2)})_{ij} - \Delta\tau\frac{dS_{\rm HMC}}{d(A_\mu)_{ij}}(A_\mu^{(n)}) \ , \\
(A_\mu^{(n+1)})_{ij} &= (A_\mu^{(n)})_{ij} + \Delta\tau\,(P_\mu^{(n+1/2)})_{ji} \ , \\
(P_\mu^{(\nu)})_{ij} &= (P_\mu^{(\nu-1/2)})_{ij} - \frac{\Delta\tau}{2}\frac{dS_{\rm HMC}}{d(A_\mu)_{ij}}(A_\mu^{(\nu)}) \ ,
\end{aligned}
$$