Matrix model with manifest general coordinate invariance

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Contents

1 Introduction

$\boxed{\text{Constructive definition of superstring theory}}$

A large N reduced model has been proposed as a nonperturbative formulation of superstring theory.

$\boxed{\text{IIB matrix model}}$

N.Ishibashi, H.Kawai, Y.Kitazawa and A.Tsuchiya, hep-th/9612115.

For a review, hep-th/9908038

$$S = -\frac{1}{g^2} Tr_{N\times N}\left(\frac{1}{4}\sum_{a,b=0}^{9}[A_a, A_b]^2 - \frac{1}{2}\bar{\psi}\sum_{a=0}^{9}\Gamma^a[A_a,\psi]\right).$$

- A_a and ψ are $N \times N$ Hermitian matrices.
  - * A_a: 10-dimensional vectors
  - * ψ: 10-dimensional Majorana-Weyl (i.e. 16-component) spinors
- This model possesses $SU(N)$ gauge symmetry and $SO(9,1)$ Lorentz symmetry.
- Dimensional reduction of $\mathcal{N} = 1$ 10-dimensional SYM to 0 dimension.
- Matrix regularization of the Green-Schwarz action of type IIB superstring theory.

- $\mathcal{N}=2$ SUSY: This theory must contain spin-2 gravitons if it admits massless particles.
  - * homogeneous : $\delta^{(1)}_{\epsilon} A_a = i\bar{\epsilon}\Gamma_a\psi, \quad \delta^{(1)}_{\epsilon}\psi = \frac{i}{2}\Gamma^{ab}[A_a, A_b]\epsilon.$
  - * inhomogeneous : $\delta^{(2)}_{\xi} A_a = 0, \quad \delta^{(2)}_{\xi}\psi = \xi.$
  - * We obtain the following commutation relations:

$$
\begin{aligned}
&(1)\ \ [\delta^{(1)}_{\epsilon_1}, \delta^{(1)}_{\epsilon_2}]A_a = 0,\ [\delta^{(1)}_{\epsilon_1}, \delta^{(1)}_{\epsilon_2}]\psi = 0,\\
&(2)\ \ [\delta^{(2)}_{\xi_1}, \delta^{(2)}_{\xi_2}]A_a = 0,\ [\delta^{(2)}_{\xi_1}, \delta^{(2)}_{\xi_2}]\psi = 0,\\
&(3)\ \ [\delta^{(1)}_{\epsilon}, \delta^{(2)}_{\xi}]A_a = -i\bar{\epsilon}\Gamma_a\xi,\ [\delta^{(1)}_{\epsilon}, \delta^{(2)}_{\xi}]\psi = 0.
\end{aligned}
$$

We take the following linear combination

$$\tilde{\delta}^{(1)} = \delta^{(1)} + \delta^{(2)},\ \tilde{\delta}^{(2)} = i(\delta^{(1)} - \delta^{(2)}).$$

This gives a shift of the bosonic variables: $(\alpha, \beta = 1, 2)$

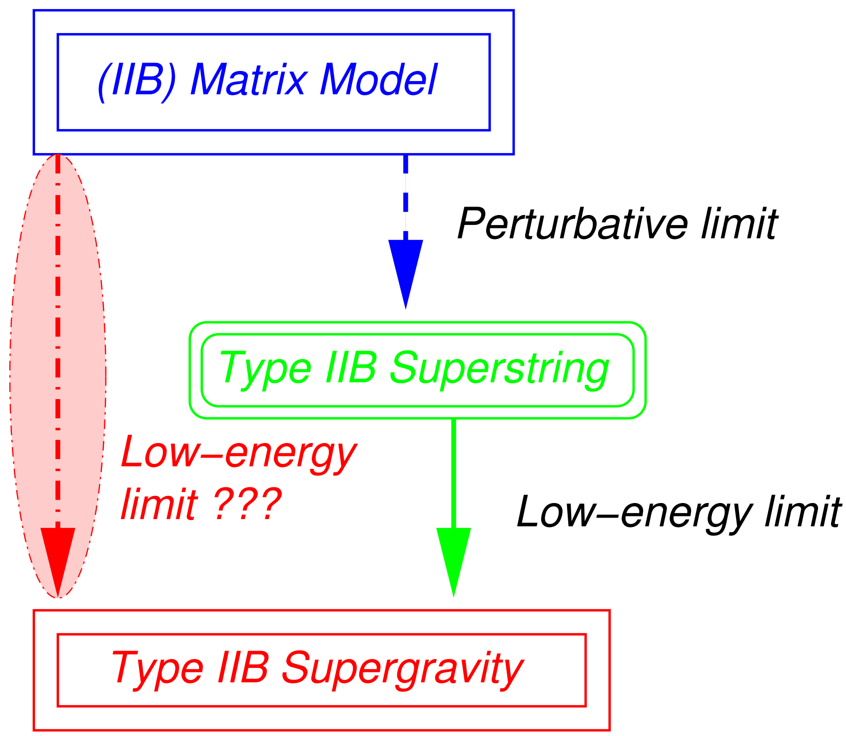
$$
\boxed{
\begin{aligned}
&[\tilde{\delta}^{(\alpha)}_{\epsilon}, \tilde{\delta}^{(\beta)}_{\xi}]\psi = 0,\\
&[\tilde{\delta}^{(\alpha)}_{\epsilon}, \tilde{\delta}^{(\beta)}_{\xi}]A_a = -2i\delta^{\alpha\beta}\bar{\epsilon}\Gamma_a\xi.
\end{aligned}
}
$$

$\Rightarrow$ Therefore, the large N matrices A_a, per se, represent the spacetime coordinate.

Is it possible to formulate a matrix model which describes the gravitational interaction more manifestly?

Can a matrix model describe the physics in the curved space?

- How is the local Lorentz invariance realized in the matrix model?
- Does a matrix model reduce to the (type IIB) supergravity in the low-energy limit?



2 Matrix as differential operator

We identify infinitely large N matrices with differential operator.

The information of spacetime can be embedded to matrices in various ways.

- Twisted Eguchi-Kawai(TEK) model:

  A. Gonzalez-Arroyo and M. Okawa, Phys. Rev. D 27, 2397 (1983).

  A. Gonzalez-Arroyo and C. P. Korthals Altes, Phys. Lett. B 131, 396 (1983).

$$A_a \sim \partial_a + a_a.$$

  The matrices A_a represent the covariant derivative on the spacetime.

- IIB matrix model:

$$A_a \sim X_a.$$

  A_a itself represent the space-time coordinate.

IIB matrix model with noncommutative background

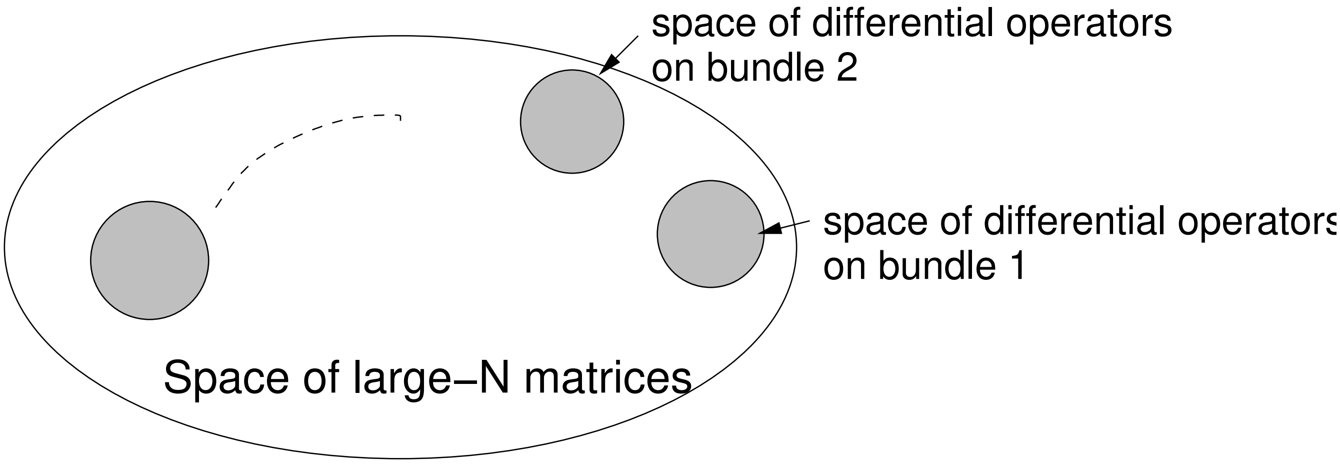
$$[\hat{p}_a, \hat{p}_b] = iB_{ab}, (B_{ab} = \text{real c-numbers})$$

interpolates these two pictures.

H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, hep-th/9908141

$Tr_{N\times N}\bar{\psi}\Gamma^a[A_a, \psi]$ reduces to the fermionic action $\int d^dx\bar{\psi}(x)i\Gamma^a(\partial_i\psi(x) + [a_i(x), \psi(x)])$ in the flat space in the low-energy limit.

- **A differential operator acts on a field in the curved space naturally.**
- **The space of the large N matrices includes the differential operators on an arbitrary spin bundle over an arbitrary manifold simultaneously.**



Attempts for a matrix model with local Lorentz invariance

The fermionic action in the curved space:

$$S_F = \int d^d x e(x) \bar{\psi}(x) i\Gamma^a e_a{}^i(x) \left(\partial_i \psi(x) + [A_i(x), \psi(x)] + \frac{1}{4}\Gamma^{bc}\omega_{ibc}(x)\psi(x)\right).$$

- $a, b, c, \cdots$: indices of the 10-dimensional Minkowskian spacetime.
- $i, j, k, \cdots$: indices of the 10-dimensional curved spacetime.

The correspondence between the matrix model and the continuum limit:

$$\begin{aligned}
Tr_{N\times N} &\rightarrow \int d^d x, \\
\psi &\rightarrow \underbrace{\Psi(x) = e^{\frac{1}{2}}(x)\psi(x)}_{\text{spinor root density}}, \\
\lceil A_a, \quad \rceil &\rightarrow i e^{\frac{1}{2}}(x) e_a{}^i(x)(\partial_i + \lceil A_i(x), \quad \rceil) e^{-\frac{1}{2}}(x), \\
\{A_{a_1 a_2 a_3}, \psi\} &\rightarrow \underbrace{e_{[a_1}{}^i(x)\omega_{i a_2 a_3]}(x)\psi(x)}_{\text{anti-commutator} \Leftrightarrow \text{product}} .
\end{aligned}$$

The rank-3 matrices correspond to the spin connection!

Commutation relations of (anti)-hermitian operators:

$$(1)[h_1, h_2] \in \mathrm{A}, \quad (2)[h, a] \in \mathrm{H}, \quad (3)[a_1, a_2] \in \mathrm{A},$$
$$(4)\{h_1, h_2\} \in \mathrm{H}, \quad (5)\{h, a\} \in \mathrm{A}, \quad (6)\{a_1, a_2\} \in \mathrm{H}.$$

- Hermitian matrices:
  $\mathrm{H} = \{M \in M_{N\times N}(\mathrm{C}) | M^\dagger = M\}$. $h, h_1 h_2 \in \mathrm{H}$.
- Anti-hermitian matrices :
  $\mathrm{A} = \{M \in M_{N\times N}(\mathrm{C}) | M^\dagger = -M\}$. $a, a_1, a_2 \in \mathrm{A}$.

[Proof of (4)] $\{h_1, h_2\}^\dagger = (h_1 h_2 + h_2 h_1)^\dagger = h_2^\dagger h_1^\dagger + h_1^\dagger h_2^\dagger = \{h_1, h_2\}$.

Notation of the gamma matrices:

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}, \text{ where } \eta^{ab} = \mathrm{diag}(-1, +1, \cdots, +1),$$

We take the gamma matrices to be real:

$$(\Gamma^a)^\dagger = ({}^T\Gamma^a) = \begin{cases} -\Gamma^a & (a = 0) \\ +\Gamma^a & (a = 1, 2, \cdots, 9) \end{cases}.$$

$$C = (\text{charge conjugation}) = \Gamma^0, \quad \Gamma^0(\Gamma^a)^\dagger\Gamma^0 = \Gamma^a.$$

$$
\begin{aligned}
S_F &= \int d^dx \bar{\Psi}(x) e^{\frac{1}{2}}(x) i\Gamma^a {e_a}^i(x) \left\{ \partial_i (e^{-\frac{1}{2}}(x)\Psi(x)) \right. \\
&\qquad \left. + [A_i(x), e^{-\frac{1}{2}}(x)\Psi(x)] + \frac{1}{4}\Gamma^{bc}\omega_{ibc}(x) e^{-\frac{1}{2}}(x)\Psi(x) \right\} \\
&= \int d^dx \left\{ \bar{\Psi}(x) i\Gamma^a \left[{e_a}^i(x)\partial_i + \frac{1}{2}{e_c}^i(x)\omega_{ica}(x) \right. \right. \\
&\qquad \left. + {e_a}^i(x) e^{\frac{1}{2}}(x)(\partial_i e^{-\frac{1}{2}}(x)) \right] \Psi(x) \\
&\qquad + i\bar{\Psi}(x)\Gamma^a {e_a}^i(x)[A_i(x), \Psi(x)] \\
&\qquad \left. + \frac{i}{4}\bar{\Psi}(x)\Gamma^{a_1 a_2 a_3} {e_{[a_1}}^i(x)\omega_{i a_2 a_3]}(x)\Psi(x) \right\} \\
&\stackrel{\star}{=} \int d^dx \left\{ \bar{\Psi}(x) i\Gamma^a {e_a}^i(x)(\partial_i \Psi(x) + [A_i(x), \Psi(x)]) \right. \\
&\qquad \left. + \frac{i}{4}\bar{\Psi}(x)\Gamma^{a_1 a_2 a_3} {e_{[a_1}}^i(x)\omega_{i a_2 a_3]}(x)\Psi(x) \right\}.
\end{aligned}
$$

In $\stackrel{\star}{=}$, we have utilized the following relationship (when $\Psi(x)$ is Majorana):

$$
\begin{aligned}
\bar{\Psi}(x)\Gamma^a\Psi(x) &= (\bar{\Psi}(x)\Gamma^a\Psi(x))^\dagger = -\Psi^\dagger(x)(\Gamma^a)^\dagger(\Gamma^0)^\dagger\Psi(x) \\
&= -\Psi^\dagger(x)\Gamma^0(\Gamma^0(\Gamma^a)^\dagger\Gamma^0)\Psi(x) = -\bar{\Psi}(x)\Gamma^a\Psi(x) = 0.
\end{aligned}
$$

The corresponding matrix model is

$$
\begin{aligned}
S_F &\Leftrightarrow \frac{1}{2} Tr \bar{\psi}\Gamma^a[A_a, \psi] + \frac{i}{2}\bar{\psi}\Gamma^{abc}\{A_{abc}, \psi\} \\
&= Tr(\bar{\psi}\Gamma^a A_a \psi + i\bar{\psi}\Gamma^{a_1 a_2 a_3} A_{a_1 a_2 a_3}\psi).
\end{aligned}
$$

Proof of the equality (only for the boson, when ψ is Majorana):

$$
\begin{aligned}
& \frac{1}{2} Tr(\bar{\psi}\Gamma^a[A_a, \psi]) = \frac{1}{2}\bar{\psi}^A\Gamma^a A_a^B \psi^C Tr(t^A[t^B, t^C]) \\
&= \frac{1}{2}\bar{\psi}^A\Gamma^a A_a^B \psi^C Tr(t^A t^B t^C - t^C t^B t^A) \\
&= \frac{1}{2}(\bar{\psi}^A\Gamma^a A_a^B \psi^C - \bar{\psi}^C\Gamma^a A_a^B \psi^A) Tr(t^A t^B t^C) = Tr(\bar{\psi}\Gamma^a A_a \psi).
\end{aligned}
$$

Local Lorentz transformation and the "gauged" model

The symmetry of IIB matrix model:
$SO(9,1)$ and $U(N)$ symmetry is decoupled.
The $SO(9,1) \times U(N)$ symmetry is a tensor product of the group. For $\zeta \in so(9,1)$ and $u \in u(N)$,

$$\exp(\zeta \otimes 1 + 1 \otimes u) = e^{\zeta} \otimes e^{u}.$$

The spacetime coordinate is embedded in the eigenvalues of the large N matrices.

$\Rightarrow$ If we are to formulate a matrix model with local Lorentz invariance, the $so(9,1)$ Lorentz symmetry and the $u(N)$ gauge symmetry must be unified.

(*) $\mathcal{A}, \mathcal{B}$ = [The Lie algebras whose bases are $\{a_i\}$ and $\{b_j\}$, respectively.]

- $\mathcal{A} \otimes \mathcal{B}$: The space spanned by the basis $a_i \otimes b_j$. This is not necessarily a closed Lie algebra.
- $\mathcal{A} \check{\otimes} \mathcal{B}$: The smallest Lie algebra that includes $\mathcal{A} \otimes \mathcal{B}$ as a subset.

The gauge group must close with respect to the commutator

$$[a \otimes A, b \otimes B] = \frac{1}{2}\left([a,b] \otimes \{A,B\} + \{a,b\} \otimes [A,B]\right).$$

(*) In order to grasp the intuitive image of the unified tensor product, we consider the following simple example.

$$su(6) = su(3) \check{\otimes} su(2).$$

λ^a: basis of $su(3)$ $(a = 1, 2, \cdots 8)$.
σ^i: basis of $su(2)$ $(i = 1, 2, 3)$.

- $\lambda^a \otimes \sigma^i$ (24 dimensions): The basis of $su(3) \otimes su(2)$, which does not constitute a closed Lie algebra.
- $\lambda^a \otimes 1 + 1 \otimes \sigma^i$ (11 dimensions): The generators of the Lie group $SU(3) \times SU(2)$.
- $su(3) \check{\otimes} su(2) = (su(3) \otimes su(2)) \oplus (SU(3) \times SU(2))_{algebra}$ This is a closed 35-dimensional Lie algebra.

$SU(3) \times SU(2)$ is a 11-dimensional Lie group, while $su(3) \check{\otimes} su(2)$ is a 35-dimensional Lie algebra.

$\boxed{\text{Local Lorentz transformation of the matrix model}}$

$$\delta\psi = \frac{1}{4}\Gamma^{a_1a_2}\varepsilon_{a_1a_2}\psi,$$

instead of $\delta\psi = \frac{1}{4}\Gamma^{a_1a_2}\{\varepsilon_{a_1a_2}, \psi\}$ at the cost of the hermiticity of ψ.

At this time, the product $A_a\psi$ does not directly correspond to the covariant derivative $(\partial_a\psi(x) + [A_a(x), \psi(x)])$.

The local Lorentz transformation of the action:

$$\delta S'_F = \frac{1}{4}Tr\bar{\psi}[\Gamma^a A_a + i\Gamma^{a_1a_2a_3}A_{a_1a_2a_3}, \Gamma^{b_1b_2}\varepsilon_{b_1b_2}]\psi.$$

However, this action does not close with respect to the local Lorentz transformation:

$$\begin{aligned}
&[i\Gamma^{a_1a_2a_3}A_{a_1a_2a_3}, \Gamma^{b_1b_2}\varepsilon_{b_1b_2}] \\
= \; &\frac{i}{2}\underbrace{[\Gamma^{a_1a_2a_3}, \Gamma^{b_1b_2}]}_{\text{rank 3}}\{A_{a_1a_2a_3}, \varepsilon_{b_1b_2}\} + \frac{i}{2}\underbrace{\{\Gamma^{a_1a_2a_3}, \Gamma^{b_1b_2}\}}_{\text{rank 1, 5}}[A_{a_1a_2a_3}, \varepsilon_{b_1b_2}].
\end{aligned}$$

We need the terms of all odd ranks in order to formulate a local Lorentz invariant matrix model.

The algebra of the local Lorentz transformation must include all the even-rank gamma matrices:

$$\begin{aligned}
&[\Gamma^{a_1a_2}\varepsilon_{a_1a_2}, \Gamma^{b_1b_2}\varepsilon'_{b_1b_2}] \\
= \; &\frac{1}{2}\underbrace{[\Gamma^{a_1a_2}, \Gamma^{b_1b_2}]}_{\text{rank-2}}\{\varepsilon_{a_1a_2}, \varepsilon'_{b_1b_2}\} + \frac{1}{2}\underbrace{\{\Gamma^{a_1a_2}, \Gamma^{b_1b_2}\}}_{\text{rank-0, 4}}[\varepsilon_{a_1a_2}, \varepsilon'_{b_1b_2}].
\end{aligned}$$

3 Attempts for a matrix model related to the type IIB supergravity

$$S = Tr_{N\times N}[tr_{32\times 32}V(m^2) + \bar{\psi}m\psi]$$

- $Tr(tr)$: the trace for the $N \times N(32 \times 32)$ matrices.
- m includes all odd-rank gamma matrices in 10 dimensions:

$$\begin{aligned} m &= m_a\Gamma^a + \frac{i}{3!}m_{a_1a_2a_3}\Gamma^{a_1a_2a_3} - \frac{1}{5!}m_{a_1\cdots a_5}\Gamma^{a_1\cdots a_5} \\ &\quad - \frac{i}{7!}m_{a_1\cdots a_7}\Gamma^{a_1\cdots a_7} + \frac{1}{9!}m_{a_1\cdots a_9}\Gamma^{a_1\cdots a_9}, \end{aligned}$$

where $m_{a_1\cdots a_{2n-1}}$ are hermitian matrices:

$$m_{a_1\cdots a_{2n-1}} = \frac{i^{n-1}}{32\times(2n-1)!}tr(m\Gamma_{a_1\cdots a_{2n-1}}).$$

m satisfies $\Gamma^0 m^\dagger \Gamma^0 = m$, and the action is hermitian.

We want to identify m with the Dirac operator.

$\Rightarrow$ We introduce $D = [(\text{length})^{-1}]$ as an extension of the Dirac operator.

$$\begin{aligned} m &= \tau^{\frac{1}{2}}D, \text{ where } \tau = [(\text{length})]^2, \\ D &= A_a\Gamma^a + \frac{i}{3!}A_{a_1a_2a_3}\Gamma^{a_1a_2a_3} - \frac{1}{5!}A_{a_1\cdots a_5}\Gamma^{a_1\cdots a_5} \\ &\quad - \frac{i}{7!}A_{a_1\cdots a_7}\Gamma^{a_1\cdots a_7} + \frac{1}{9!}A_{a_1\cdots a_9}\Gamma^{a_1\cdots a_9}. \end{aligned}$$

τ is not an N-dependent cut-off parameter, but a reference scale ($\sim l_s^2$).

$A_{a_1\cdots a_{2n-1}} = \frac{i^{2n-1}}{32\times(2n-1)!} tr(D\Gamma_{a_1\cdots a_{2n-1}})$ are hermitian differential operators.

$\Rightarrow$ They are expanded by the number of the derivatives:

$$A_{a_1\cdots a_{2n-1}} = a_{a_1\cdots a_{2n-1}}(x) + \sum_{k=1}^{\infty} \frac{i^k}{2}\{\partial_{i_1}\cdots\partial_{i_k}, \underbrace{a^{(i_1\dots i_k)}{}_{a_1\cdots a_{2n-1}}(x)}_{[(\text{length})^{-1+k}]}\}.$$

$a_a{}^{(i)}(x)$ is identified with the vielbein $e_a{}^i(x)$ in the background metric.

$$\begin{aligned} D &= e^{\frac{1}{2}}(x)\left[ie_a{}^i(x)\Gamma^a\left(\partial_i + \frac{1}{4}\Gamma^{bc}\omega_{ibc}(x)\right)\right]e^{-\frac{1}{2}}(x) \\ &+ (\text{higher-rank terms}) + (\text{higher-derivative terms}). \end{aligned}$$

The potential $V(m^2)$ is generically $V(m^2) \sim \exp(-(m^2)^\alpha)$.
$\Rightarrow$ The damping factor is naturally included in the bosonic term.
$\Rightarrow$ The trace for the infinitely large N matrices is finite.

ψ is a Weyl fermion, but not Majorana.
We need to introduce a damping factor so that the trace should be finite.

$$\psi = (\chi(x) + \sum_{l=1}^{\infty} \underbrace{\chi^{(i_1\cdots i_l)}(x)}_{[(\text{length})]^l}\ \partial_{i_1}\cdots\partial_{i_l})e^{-(\tau D^2)^\alpha}.$$

Local Lorentz invariance

The action is invariant under the local Lorentz transformation:

$$\begin{aligned}\delta m &= [m,\varepsilon], \quad \delta\psi = \varepsilon\psi, \quad \delta\bar{\psi} = -\bar{\psi}\varepsilon, \text{ where}\\ \varepsilon &= -i\varepsilon_{\emptyset} + \frac{1}{2!}\Gamma^{a_1a_2}\varepsilon_{a_1a_2} + \frac{i}{4!}\Gamma^{a_1\cdots a_4}\varepsilon_{a_1\cdots a_4} - \frac{1}{6!}\Gamma^{a_1\cdots a_6}\varepsilon_{a_1\cdots a_6}\\ &\quad -\frac{i}{8!}\Gamma^{a_1\cdots a_8}\varepsilon_{a_1\cdots a_8} + \frac{1}{10!}\Gamma^{a_1\cdots a_{10}}\varepsilon_{a_1\cdots a_{10}}.\end{aligned}$$

- All even-rank gamma matrices are necessary for the local Lorentz transformation algebra to close.
- ε satisfies $\Gamma^0\varepsilon^\dagger\Gamma^0 = \varepsilon$, and thus the commutator $\delta m = [m,\varepsilon]$ actually satisfies $\Gamma^0(\delta m)^\dagger\Gamma^0 = \delta m$.

The invariance under the local Lorentz transformation:

$$\delta S = 2Tr[tr(V'_S(m^2)m[m,\varepsilon])] + Tr[tr(\bar{\psi}[m,\varepsilon]\psi)] = 0.$$

The cyclic property still holds true of the trace for the large N matrices, if we assume that the coefficients damp rapidly at infinity:

$$\lim_{|x|\to\infty} a^{(i_1\cdots i_k)}{}_{a_1\cdots a_{2n-1}}(x) = \lim_{|x|\to\infty}\chi^{(i_1\cdots i_k)}(x) = 0.$$

[Proof] After integrating in the action, the following commutator vanishes:

$$\begin{aligned}&Tr([\partial_j, a^{(i_1\cdots i_k)}{}_{a_1\cdots a_{2n-1}}(x)])\\ &= \int d^dx\langle x|(\partial_j a^{(i_1\cdots i_k)}{}_{a_1\cdots a_{2n-1}}(x))|x\rangle\\ &= \int d^dx(\partial_j a^{(i_1\cdots i_k)}{}_{a_1\cdots a_{2n-1}}(x))\langle x|x\rangle = 0.\end{aligned}$$

$\boxed{\text{Heat kernel expansion}}$

The trace of the large N matrices is analyzed through **the heat kernel (Seeley de Witt) expansion**, which is the expansion around $e^{-\tau \partial_a \partial^a} = e^{-\tau m_0^2}$.

We seek the answers of the following questions:

- Is $m_0 = i\Gamma^a \partial_a$ (the Dirac operator in the flat space) a classical solution?
  (If so, this model cancels the cosmological constant.)
- Which fields are massive and decoupled in the classical low-energy limit?
  If this model is to reduce to the type IIB supergravity, only the following fields must remain massless:
  * even-rank antisymmetric tensor $a^{(i)}{}_{ia_1 \cdots a_{2n}}(x)$
  * dilatino $\chi(x)$, and gravitino $\chi^{(i)}(x)$

The computation is performed through the Campbell-Baker-Hausdorff (CBH) formula:

$$
\begin{aligned}
Tr(e^{-\tau D^2}) &= \int d^d x \langle x | e^{-\tau D^2} | x \rangle \\
&= Tr\left[\underbrace{\exp\left(\overbrace{(-\tau \partial_a \partial^a)}^{=X} + \overbrace{(-\tau(D^2 - \partial_a \partial^a))}^{=Y} \right) \exp\left(\overbrace{\tau \partial_a \partial^a}^{=-X} \right)}_{CBH} e^{-\tau \partial_a \partial^a} \right] \\
&= Tr\left[\exp\left(Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X+Y, [X+Y, -X]] \right.\right. \\
&\qquad \left.\left. + \frac{1}{12}[-X, [-X, X+Y]] + \cdots \right) e^{-X} \right] \\
&= Tr\left[\left(1 + Y + \frac{1}{2}[X, Y] + \frac{1}{6}[X, [X, Y]] + \frac{1}{2}Y^2 + \frac{1}{8}[X, Y]^2 \right.\right. \\
&\qquad \left.\left. + \frac{1}{3}Y[X, Y] + \frac{1}{6}[X, Y]Y + \cdots \right) e^{-X} \right], \\
\langle x | e^{-X} | y \rangle &= \frac{1}{(2\pi\tau)^{\frac{d}{2}}} \exp\left(-\frac{1}{4\tau}(x^a - y^a)(x^b - y^b)\eta_{ab} \right).
\end{aligned}
$$

The Laplace transformation of $V(u)$:

$$V(u) = \int_0^\infty dsg(s)e^{-su}.$$

Then, the bosonic part is expanded as

$$\begin{aligned} Tr[trV(m^2)] &= \int_0^\infty dsg(s)Tr[tre^{-s\tau D^2}] \\ &= \int \frac{d^dx}{(2\pi\tau)^{\frac{d}{2}}} \left(\sum_{k=-\infty}^{\infty} \left(\int_0^\infty dsg(s)s^{-\frac{d}{2}+k} \right) \tau^k \underbrace{\mathcal{A}_k(x)}_{[(\text{length})]^{-2k}} \right). \end{aligned}$$

If m_0 is to be a classical solution,
$\Rightarrow$ The linear terms of the fluctuation around m_0 should vanish.

- The linear terms of the derivatives vanish after integrating in the action:

$$\int d^dx(\partial_{j_1} \cdots \partial_{j_m} a_a{}^{(ai_1i_1\cdots i_li_l)}(x)) = 0.$$

- Only a scalar can constitute a Lorentz invariant linear term.
$\Rightarrow$ We focus on the following terms:

$$\underbrace{a_a{}^{(ai_1i_1\cdots i_li_l)}(x)}_{[(\text{length})]^{2l}} \in \mathcal{A}_{-l}(x).$$

The coefficients $\mathcal{A}_0(x)$, $\mathcal{A}_{-1}(x)$, $\mathcal{A}_{-2}(x)\cdots$ must vanish.
Then, the cosmological constant $\int d^dx\frac{1}{(2\pi\tau)^{\frac{d}{2}}}e(x) \in \mathcal{A}_0(x)$ also vanishes.

Then, the following condition must be satisfied:

$$\boxed{\begin{aligned} &\int_0^\infty dsg(s)s^{-\frac{d}{2}-n} = 0, \quad (n = 0, -1, -2, \cdots) \\ \Leftrightarrow\ &\int_0^\infty duV(u)u^{\frac{d}{2}+n} = 0, \quad (n = -1, 0, 1, 2, \cdots). \end{aligned}}$$

$(\int_0^\infty duV(u)u^{\alpha-1} = \int_0^\infty dudsg(s)e^{-su}u^{\alpha-1} = \Gamma(\alpha)\int_0^\infty dsg(s)s^{-\alpha})$.

$V(u)$ is chosen as, for example,

$$V_0(u) = \frac{\partial^{\frac{d}{2}-1}(e^{-u^{\frac{1}{4}}}\sin u^{\frac{1}{4}})}{\partial u^{\frac{d}{2}-1}}.$$

The model reduces to the Einstein gravity in the classical low-energy limit.

- The linear term of the vielbein ${a_a}^{(a)}(x)$ vanishes.
- The cross terms ${a_a}^{(a)}(x){a_b}^{(bi_1\cdots i_k)}(x)$ also vanish, due to the general coordinate invariance.

$$Tr[tre^{-\tau D^2}] = \int d^dx \frac{32}{(2\pi\tau)^{\frac{d}{2}}}\tau e(x) \underbrace{\frac{R(x)}{6}}_{[(\text{length})]^{-2} \in \mathcal{A}_1(x)} + \cdots.$$

$V(u)$ must be chosen so that $\mathcal{A}_1(x)$ survives in the action.

Which fields are massive or massless?

mass terms: $\underbrace{a^{(i_1\cdots i_k)}{}_{a_1\cdots a_{2n-1}}(x)a^{(j_1\cdots j_l)}{}_{a_1\cdots a_{2n-1}}(x)}_{[(\text{length})]^{-2+k+l},\ \mathcal{A}_{1-\frac{k+l}{2}}(x)}$,

kinetic terms: $\underbrace{\partial_{k_1}a^{(i_1\cdots i_k)}{}_{a_1\cdots a_{2n-1}}(x)\partial_{k_2}a^{(j_1\cdots j_l)}{}_{a_1\cdots a_{2n-1}}(x)}_{[(\text{length})]^{-4+k+l}\in\mathcal{A}_{2-\frac{k+l}{2}}(x)}$,

- odd-rank antisymmetric tensor $a_{a_1\cdots a_{2n-1}}(x)$:
  Mass terms $\in \mathcal{A}_1(x)$, Kinetic terms $\in \mathcal{A}_2(x)$.
  These fields are generically massive.
- even-rank anti-symmetric tensor $a^{(i)}{}_{ia_1\cdots a_{2n}}(x)$:
  Mass terms $\in \mathcal{A}_0(x)$, Kinetic terms $\in \mathcal{A}_1(x)$.
  They may be massless ??
- Higher-spin fields: $a^{(i_1\cdots i_k)}{}_{a_1\cdots a_{2n-1}}(x)$ $(k = 2, 3, \cdots)$:
  The mass terms and the kinetic terms are absent.
  No clue of whether they are massive.

$\boxed{\mathcal{N} = 2 \text{ SUSY}}$

The SUSY transformation of the model:

$$\begin{aligned} \delta\psi &= 2V'(m^2)\epsilon, \quad \delta\bar{\psi} = 2\bar{\epsilon}V'(m^2), \\ \delta m &= \epsilon\bar{\psi} + \psi\bar{\epsilon}. \end{aligned}$$

SUSY invariance of the action

$$\begin{aligned} \delta_\epsilon S &= Tr\left[tr\left(\left(2V'(m^2)m(\epsilon\bar{\psi} + \psi\bar{\epsilon})\right) + \bar{\psi}(\epsilon\bar{\psi} + \psi\bar{\epsilon})\psi \right.\right. \\ &\qquad \left.\left. + 2\bar{\psi}mV'(m^2)\epsilon + 2\bar{\epsilon}mV'(m^2)\psi\right)\right] = 0. \end{aligned}$$

Commutator of the SUSY transformation on shell:
In the following, we assume that the Taylor expansion of $V(u)$ around $u = 0$ is possible.

$$\begin{aligned} &[\delta_\epsilon, \delta_\xi]m = 2[\xi\bar{\epsilon} - \epsilon\bar{\xi}, V'(m^2)], \\ &[\delta_\epsilon, \delta_\xi]\psi = 2\psi\left(\bar{\epsilon}m\frac{V'(m^2) - V'(0)}{m^2}\xi - \bar{\xi}m\frac{V'(m^2) - V'(0)}{m^2}\epsilon\right). \end{aligned}$$

where we have utilized the equation of motion:

$$\frac{\partial S}{\partial\bar{\psi}} = 2m\psi = 0, \quad \frac{\partial S}{\partial\psi} = 2\bar{\psi}m = 0.$$

In order to see the structure of the $\mathcal{N}=2$ SUSY, we separate the SUSY parameters into the hermitian and the antihermitian parts as

$$\epsilon = \epsilon_1 + i\epsilon_2, \;\; \xi = \xi_1 + i\xi_2,$$

($\xi_1, \xi_2, \epsilon_1, \epsilon_2$ are Majorana-Weyl fermions.)

The translation of the bosons is attributed to the quartic term in the Taylor expansion of $V(m) = \sum_{k=1}^{\infty} \frac{a_{2k}}{2k} m^{2k}$.

We assume that the SUSY parameters $\epsilon_{1,2}, \xi_{1,2}$ are c-numbers (proportional to the unit matrix $1_{\mathrm{N}\times\mathrm{N}}$).

$$\begin{aligned}
[\delta_\epsilon, \delta_\xi] A_a &= \frac{1}{16} tr([\delta_\epsilon, \delta_\xi] m \Gamma_a) \\
&= \frac{1}{16} \sum_{k=2}^{\infty} a_{2k} tr(\xi\bar{\epsilon} m^{2k-2}\Gamma_a - \epsilon\bar{\xi} m^{2k-2}\Gamma_a \\
&\qquad -m^{2k-2}\xi\bar{\epsilon}\Gamma_a + m^{2k-2}\epsilon\bar{\xi}\Gamma_a) \\
&= \frac{1}{16} \sum_{k=2}^{\infty} a_{2k} (\bar{\xi}[m^{2k-2}, \Gamma_a]\epsilon - \bar{\epsilon}[m^{2k-2}, \Gamma_a]\xi) \\
&= \frac{a_4}{16} (\bar{\xi}[\Gamma^{b_1}\Gamma^{b_2}, \Gamma_a]\epsilon - \bar{\epsilon}[\Gamma^{b_1}\Gamma^{b_2}, \Gamma_a]\xi) A_{b_1} A_{b_2} + \cdots \\
&= \frac{a_4}{16} (\bar{\xi}\Gamma^i\epsilon - \bar{\epsilon}\Gamma^i\xi)[A_i, A_a] + \cdots \\
&= \frac{a_4}{8} (\bar{\xi}_1\Gamma^i\epsilon_1 + \bar{\xi}_2\Gamma^i\epsilon_2)[A_i, A_a] + \cdots,
\end{aligned}$$

The field $a_a(x)$ receives the translation and the gauge transformation:

$$\begin{aligned}
[A_i, A_a] &= [i\partial_i + a_i(x), i\partial_a + a_a(x)] + \cdots \\
&= \underbrace{i(\partial_i a_a(x))}_{\text{translation}} \;\; \underbrace{-i(\partial_a a_i(x)) + [a_i(x), a_a(x)]}_{\text{gauge transformation}} + \cdots .
\end{aligned}$$

However, the fermions do not receive the translation.

$$
\begin{aligned}
[\delta_\epsilon, \delta_\xi]\psi &= -\sum_{k=2}^{n} a_{2k}\psi(\bar{\xi}m^{2k-3}\epsilon - \bar{\epsilon}m^{2k-3}\xi) + \cdots \\
&= -a_4(\bar{\xi}\Gamma^j\epsilon - \bar{\epsilon}\Gamma^j\xi)\psi A_j + \cdots \\
&= -2a_4(\bar{\xi}_1\Gamma^j\epsilon_1 + \bar{\xi}_2\Gamma^j\epsilon_2)\psi A_j + \cdots .
\end{aligned}
$$

We explore the term ψA_i more carefully:

$$
\begin{aligned}
\psi A_j &= i\psi\partial_j + \cdots \\
&= \left(\chi(x)\partial_j + \sum_{l=1}^{\infty}\chi^{(i_1\cdots i_l)}(x)\partial_{i_1}\cdots\partial_{i_l}\partial_j\right) e^{-(\tau D^2)^\alpha} + \cdots .
\end{aligned}
$$

Therefore, each fermionic field is transformed as

$$
\begin{aligned}
&[\delta_\epsilon, \delta_\xi]\chi(x) = 0 + \cdots , \\
&[\delta_\epsilon, \delta_\xi]\chi^{(i_1\cdots i_{l+1})}(x) = -2a_4(\bar{\xi}_1\Gamma^j\epsilon_1 + \bar{\xi}_2\Gamma^j\epsilon_2)\chi^{(\{i_1\cdots i_l)}(x)\delta^{i_{l+1}\}j} + \cdots .
\end{aligned}
$$

(*) $\cdots$ denotes the omission of the non-linear terms of the fields.

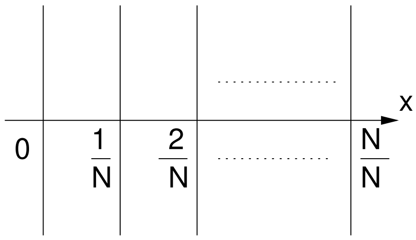
It is a future problem to surmount this difficulty.

4 Conclusion

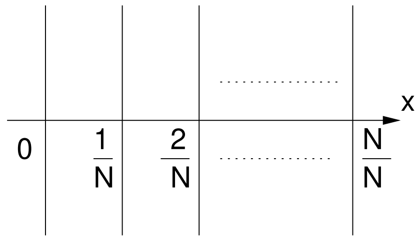
- We have pursued the possibility for a matrix model to describe the gravitational interaction in the curved spacetime.
- We have identified the large N matrices with the differential operators.
- In order to describe the local Lorentz invariance in a matrix model, the following two ideas are essential:
  * We have identified the higher-rank tensor fields with the spin connection.
  * $so(9,1)$ Lorentz symmetry and the $u(N)$ gauge symmetry must be coupled.
- We have attempted to build a model which reduces to the type IIB supergravity in the low-energy limit:
  * We have elucidated that the bosonic part reduce to the Einstein gravity.
  * There are many problems for the supersymmetric model:
    $\mathcal{N}=2$ SUSY, the mass of the fields $\cdots$.

Differential operators in the space of large N matrices

Scalars on S_1



(1) trivial bundle with the periodic condition f(1)=f(0).



(2) Z2–twisted bundle with the antiperiodic condition f(1)=–f(0).

(1) Trivial bundle:

We first consider the trivial bundle with the periodic condition $f(1) = f(0)$. We discritize the region $0 \leq x \leq 1$ into small slices of spacing $\epsilon = \frac{1}{N}$.

$$\begin{aligned} \partial_x f\left(\frac{k}{N}\right) &\rightarrow \frac{1}{2}\left(\frac{f(\frac{k+1}{N}) - f(\frac{k}{N})}{\epsilon} + \frac{f(\frac{k}{N}) - f(\frac{k-1}{N})}{\epsilon}\right) \\ &= \frac{N}{2}\left(f\left(\frac{k+1}{N}\right) - f\left(\frac{k-1}{N}\right)\right). \end{aligned}$$

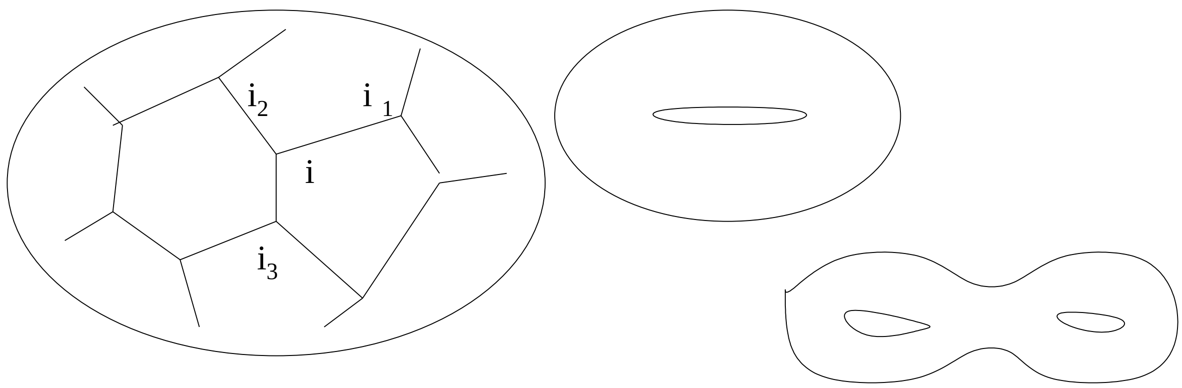
$$\partial_x \rightarrow A = \frac{N}{2}\begin{pmatrix} 0 & 1 & & & & -1 \\ -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & & \ddots & & \\ 1 & & & & -1 & 0 \end{pmatrix}.$$

(2) Z_2-twisted bundle

Now, the periodic condition $f(1) = -f(0)$ is imposed:

$$\partial_x \rightarrow A = \frac{N}{2}\begin{pmatrix} 0 & 1 & & & & 1 \\ -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & & \ddots & & \\ -1 & & & & -1 & 0 \end{pmatrix}.$$

Laplacian on various manifolds



$i_{1,}$ $i_{2,}$ i_3 are the neighbours of i.

$$\triangle \to \boldsymbol{K} = \begin{array}{cc} & \begin{array}{ccccc} & i_1 & i & i_2 & i_3 \end{array} \\ i & \begin{pmatrix} & \vdots & \vdots & \vdots & \vdots \\ \cdots & 1 & -3 & 1 & 1 \\ & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \end{array}.$$

In the space of a large N matrix, the differential operators over various manifolds are embedded.

Hausdorff's moment problem

[Theorem] (Hausdorff) Let $f(x)$ be a continuous function. If

$$\int_0^1 dx f(x) x^n = 0,$$

for $n = 0, 1, 2, \cdots$, then $f(x) = 0$ for all $x \in [0, 1]$.

However, this statement does not hold true if we replace $[0, 1]$ with $[0, \infty]$:

[Example] The continuous function

$$h(x) = \exp(-x^{\frac{1}{4}}) \sin(x^{\frac{1}{4}})$$

satisfy $\int_0^\infty dx h(x) x^n = 0$ for all $n = 0, 1, 2, \cdots$.

[Proof] We note that

$$\int_0^\infty dy y^m e^{-ay} = m! a^{-m-1}$$

for $a = \exp(\frac{i\pi}{4}) = \frac{1+i}{\sqrt{2}}$ and $m = 0, 1, 2, \cdots$. This is a real number when $m - 3$ is a multiple of 4.

Taking the imaginary part of the both hand sides, we obtain

$$\int_0^\infty dy y^{4n+3} \sin(\frac{y}{\sqrt{2}}) \exp(-\frac{y}{\sqrt{2}}) = 0,$$

for $n = 0, 1, 2, \cdots$. We make a substitution $x = \frac{y^4}{4}$ to obtain $\int_0^\infty dx h(x) x^n = 0$. (Q.E.D.)

Proof of the SUSY transformation of IIB matrix model

1. $[\delta^{(1)}_{\epsilon_1}, \delta^{(1)}_{\epsilon_2}]A_a = 0$, $[\delta^{(1)}_{\epsilon_1}, \delta^{(1)}_{\epsilon_2}]\psi = 0$.

The commutation relation for the bosons is obtained by comparing the following two paths:

$$A_a \overset{\delta^{(1)}_{\epsilon_2}}{\to} A_a + i\epsilon_2\Gamma_a\psi \overset{\delta^{(1)}_{\epsilon_1}}{\to} A_a + i(\bar{\epsilon}_1 + \bar{\epsilon}_2)\Gamma_a\psi - \frac{1}{2}\bar{\epsilon}_2\Gamma_a[A_b, A_c]\Gamma^{bc}\epsilon_1,$$
$$A_a \overset{\delta^{(1)}_{\epsilon_1}}{\to} A_a + i\epsilon_1\Gamma_a\psi \overset{\delta^{(1)}_{\epsilon_2}}{\to} A_a + i(\bar{\epsilon}_1 + \bar{\epsilon}_2)\Gamma_a\psi - \frac{1}{2}\bar{\epsilon}_1\Gamma_a[A_b, A_c]\Gamma^{bc}\epsilon_2.$$

Then, the commutator is

$$\begin{aligned}[\delta^{(1)}_{\epsilon_1}, \delta^{(1)}_{\epsilon_2}]A_a &= -\frac{1}{2}\bar{\epsilon}_2\Gamma_a[A_b, A_c]\Gamma^{bc}\epsilon_1 + \frac{1}{2}\bar{\epsilon}_1\Gamma_a[A_b, A_c]\Gamma^{bc}\epsilon_2 \\ &= [A_a, 2\bar{\epsilon}_1\Gamma^c\epsilon_2A_c].\end{aligned}$$

On the other hand, the commutation relation for the fermions is obtained by

$$\psi \overset{\delta^{(1)}_{\epsilon_2}}{\to} \psi + \frac{a}{2}[A_a, A_b]\Gamma^{ab}\epsilon_2 \overset{\delta^{(1)}_{\epsilon_1}}{\to} \psi + \frac{a}{2}[A_a, A_b]\Gamma^{ab}(\epsilon_1 + \epsilon_2) - [A_a, \bar{\epsilon}_1\Gamma_b\psi]\Gamma^{ab}\epsilon_2,$$
$$\psi \overset{\delta^{(1)}_{\epsilon_1}}{\to} \psi + \frac{a}{2}[A_a, A_b]\Gamma^{ab}\epsilon_1 \overset{\delta^{(1)}_{\epsilon_2}}{\to} \psi + \frac{a}{2}[A_a, A_b]\Gamma^{ab}(\epsilon_1 + \epsilon_2) - [A_a, \bar{\epsilon}_2\Gamma_b\psi]\Gamma^{ab}\epsilon_1.$$

By using the formula of Fierz transformation

$$\begin{aligned}\bar{\epsilon}_1\Gamma_b\psi\Gamma^{ab}\epsilon_2 &= (\bar{\epsilon}_1\Gamma^a\epsilon_2)\psi - \frac{7}{16}(\bar{\epsilon}_1\Gamma^c\epsilon_2)\Gamma_c\Gamma^a\psi \\ &- \frac{1}{16\times 5!}(\bar{\epsilon}_1\Gamma^{c_1\cdots c_5}\epsilon_2)\Gamma_{c_1\cdots c_5}\Gamma^a\psi,\end{aligned}$$

and the equation of motion

$$\frac{dS}{d\bar{\psi}} = -\frac{1}{g^2}\Gamma^a[A_a, \psi] = 0,$$

the commutator is computed on shell to be

$$[\delta^{(1)}_{\epsilon_1}, \delta^{(1)}_{\epsilon_2}]\psi = [\psi, 2\bar{\epsilon}_1\Gamma^c\epsilon_2A_c].$$

These commutators are set to be zero by the gauge transformation.

2. $[\delta^{(2)}_{\xi_1}, \delta^{(2)}_{\xi_2}]A_a = 0$, $[\delta^{(2)}_{\xi_1}, \delta^{(2)}_{\xi_2}]\psi = 0$.

This is trivial because the inhomogeneous SUSY transformation is merely a translation of the fermions.

3. $[\delta^{(1)}_{\epsilon}, \delta^{(2)}_{\xi}]A_a = -i\bar{\epsilon}\Gamma_a\xi$, $[\delta^{(1)}_{\epsilon}, \delta^{(2)}_{\xi}]\psi = 0$.

This can be proven by taking the difference of these two transformations:

$$
\begin{aligned}
&A_a \xrightarrow{\delta^{(2)}_{\xi}} A_a \xrightarrow{\delta^{(1)}_{\epsilon}} A_a + i\bar{\epsilon}\Gamma_a\psi \\
&A_a \xrightarrow{\delta^{(1)}_{\epsilon}} A_a + i\bar{\epsilon}\Gamma_a\psi \xrightarrow{\delta^{(2)}_{\xi}} A_a + i\bar{\epsilon}\Gamma_a(\psi + \xi), \\
&\psi \xrightarrow{\delta^{(2)}_{\xi}} \psi + \xi \xrightarrow{\delta^{(1)}_{\epsilon}} \psi + \xi + \frac{a}{2}\Gamma^{ij}[A_a, A_b]\epsilon \\
&\psi \xrightarrow{\delta^{(1)}_{\epsilon}} \psi + \frac{a}{2}\Gamma^{ij}[A_a, A_b]\epsilon \xrightarrow{\delta^{(2)}_{\xi}} \psi + \xi + \frac{a}{2}\Gamma^{ij}[A_a, A_b]\epsilon.
\end{aligned}
$$

Explicit computation of the Seeley de Witt coefficients

We consider the trace of the large N matrices in terms of the heat kernel: The trace of the operators are expressed using the complete system as

$$Trm = \int d^D x \langle x|m|x\rangle, \tag{1}$$

where the bracket $|x\rangle$ and $\langle x|$ satisfies $\sum_x |x\rangle\langle x| = 1$. However, it is difficult to consider the trace of a general operator, and we regard the operator as the sum of the Laplacian and the perturbation around it. This is a famous procedure, and the perturbation is expressed in terms of *Seeley de Witt coefficient*.

It is well known that the Green function is computed to be

$$\langle x| \exp\left(\tau g^{ij}(y)\frac{d}{dx^i}\frac{d}{dx^j}\right)|y\rangle = \frac{e(y)}{(2\pi\tau)^{\frac{d}{2}}}\exp\left(-\frac{(x-y)^i(x-y)^j g_{ij}(y)}{4\tau}\right). \tag{2}$$

We consider the general elliptic differential operator

$$D^2 = -\left(g_{ij}(x)\frac{d}{dx^i}\frac{d}{dx^j} + A^i(x)\frac{d}{dx^i} + B(x)\right). \tag{3}$$

And we are now interested in the trace

$$Tr\exp(-\tau D^2) = \int d^d x \langle x|\exp(-\tau D^2)|x\rangle. \tag{4}$$

To this end, we compute the following quantity utilizing the Campbell-Hausdorff formula:

$$\langle x|\exp(-\tau D^2)|y\rangle = \langle x|\exp(X+Y)|y\rangle, \text{ where} \tag{5}$$

$$X = \tau\left(g^{ij}(y)\frac{d}{dx^i}\frac{d}{dx^j}\right), \tag{6}$$

$$Y = \tau\left((g^{ij}(x) - g^{ij}(y))\frac{d}{dx^i}\frac{d}{dx^j} + A^i(x)\frac{d}{dx^i} + B(x)\right). \tag{7}$$

The Campbell-Hausdorff formula is

$$e^A e^B = \exp\left(A + B + \frac{1}{2}[A,B] + \frac{1}{12}([A,[A,B]] + [B,[B,A]]) + \cdots\right). \tag{8}$$

Since we know that $\langle x|e^X|y\rangle = \frac{e(y)}{(2\pi\tau)^{\frac{d}{2}}}\exp\left(-\frac{1}{4\tau}(x-y)^i(x-y)^j g_{ij}(y)\right)$, the quantity in question is computed as

$$\begin{aligned} e^{X+Y}e^{-X} &= \exp\left(Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X+Y,[X+Y,-X]] + [-X,[-X,X+Y]]) + \cdots\right) \\ &= \exp\left(Y + \frac{1}{2}[X,Y] + \frac{1}{12}(2[X,[X,Y]] - [Y,[Y,X]]) + \cdots\right) \\ &= 1 + Y + \frac{1}{2}[X,Y] + \frac{1}{6}[X,[X,Y]] + \frac{1}{12}[Y,[X,Y]] + \cdots \\ &\quad + \frac{1}{2}(Y + \frac{1}{2}[X,Y] + \frac{1}{6}[X,[X,Y]] + \frac{1}{12}[Y,[X,Y]] + \cdots)^2 + \cdots \\ &= 1 + Y + \frac{1}{2}[X,Y] + \frac{1}{6}[X,[X,Y]] + \frac{1}{2}Y^2 + \frac{1}{8}[X,Y]^2 + \frac{1}{3}Y[X,Y] + \frac{1}{6}[X,Y]Y + \cdot \end{aligned} \tag{9}$$

Before we enter the computation of the quantity $\langle x|e^{X+Y}|y\rangle$, we summarize the formula of the differentiation of e^X:

$$\begin{aligned}
\frac{de^X}{dx^i} &= -\frac{1}{2\tau}(x-y)^j g_{ij}(y)e^X,\\
\frac{d^2e^X}{dx^{i_1}dx^{i_2}} &= \left(-\frac{1}{2\tau}g_{i_1i_2}(y)+\frac{1}{4\tau^2}(x-y)^{l_1}(x-y)^{l_2}g_{i_1l_1}(y)g_{i_2l_2}(y)\right)e^X,\\
\frac{d^3e^X}{dx^{i_1}dx^{i_2}dx^{i_3}} &= \left(\frac{1}{4\tau^2}(x-y)^l(g_{i_1i_2}(y)g_{i_3l}(y)+g_{i_2i_3}(y)g_{i_1l}(y)+g_{i_3i_1}(y)g_{i_2l}(y))\right.\\
&\quad\left.-\frac{1}{8\tau^3}(x-y)^{l_1}(x-y)^{l_2}(x-y)^{l_3}g_{i_1l_1}(y)g_{i_2l_2}(y)g_{i_3l_3}(y)\right)e^X,\\
\frac{d^4e^X}{dx^{i_1}dx^{i_2}dx^{i_3}dx^{i_4}} &= \left(\frac{1}{4\tau^2}(g_{i_1i_2}(y)g_{i_3i_4}(y)+g_{i_2i_3}(y)g_{i_4i_1}(y)+g_{i_1i_3}(y)g_{i_2i_4}(y))\right.\\
&\quad-\frac{1}{8\tau^3}(x-y)^{l_1}(x-y)^{l_2}(g_{i_1i_2}(y)g_{i_3l_1}(y)g_{i_4l_2}(y)+g_{i_2i_3}(y)g_{i_1l_1}(y)g_{i_4l_2}(y)+g_{i_1i_3}(y)g_{i_2l_1}(y)g_{i_4l_2}(y)\\
&\qquad+g_{i_1i_4}(y)g_{i_2l_1}(y)g_{i_3l_2}(y)+g_{i_2i_4}(y)g_{i_1l_1}(y)g_{i_3l_2}(y)+g_{i_3i_4}(y)g_{i_1l_1}(y)g_{i_2l_2}(y))\\
&\quad\left.+\frac{1}{16\tau^4}(x-y)^{l_1}(x-y)^{l_2}(x-y)^{l_3}(x-y)^{l_4}g_{i_1l_1}(y)g_{i_2l_2}(y)g_{i_3l_3}(y)g_{i_4l_4}(y)\right)e^X.
\end{aligned} \tag{10}$$

Computation of Ye^X

We start with the computation of the easiest case:

$$\begin{aligned}
Ye^X &= \tau\left((g^{ij}(x)-g^{ij}(y))\frac{d}{dx^i}\frac{d}{dx^j}+A^i(x)\frac{d}{dx^i}+B(x)\right)e^X\\
&= \left(\tau B(x)-\frac{1}{2}A^i(x-y)^j g_{ij}(y)+(g^{ij}(x)-g^{ij}(y))(-\frac{1}{2}g_{ij}(y)+\frac{1}{4\tau}(x-y)^{l_1}(x-y)^{l_2}g_{il_1}(y)g_{jl_2}(y))\right)e^X.
\end{aligned} \tag{11}$$

Therefore, the trace is obtained by

$$Tr(Ye^X)=\int d^dx\langle x|Ye^X|x\rangle=\int d^dx\frac{\tau e(x)}{(2\pi\tau)^{\frac{d}{2}}}B(x). \tag{12}$$

Computation of $\frac{1}{2}[X,Y]e^X$

We next go on to a bit more complicated case, and we compute the operator $[X,Y]$ itself:

$$\begin{aligned}
[X,Y] &= \tau^2\left(g^{i_1i_2}(y)\frac{d}{dx^{i_1}}\frac{d}{dx^{i_2}}\right)\times\left((g^{j_1j_2}(x)-g^{j_1j_2}(y))\frac{d}{dx^{j_1}}\frac{d}{dx^{j_2}}+A^j(x)\frac{d}{dx^j}+B(x)\right)\\
&- \tau^2\left((g^{j_1j_2}(x)-g^{j_1j_2}(y))\frac{d}{dx^{j_1}}\frac{d}{dx^{j_2}}+A^j(x)\frac{d}{dx^j}+B(x)\right)\times\left(g^{i_1i_2}(y)\frac{d}{dx^{i_1}}\frac{d}{dx^{i_2}}\right)\\
&= \tau^2\left(2g^{i_1i_2}(y)(\frac{dg^{j_1j_2}(x)}{dx^{i_1}})\frac{d^3}{dx^{i_2}dx^{j_1}dx^{j_2}}+g^{i_1i_2}(y)(\frac{d^2g^{j_1j_2}(x)}{dx^{i_1}dx^{i_2}})\frac{d^2}{dx^{j_1}dx^{j_2}}\right.\\
&\quad+2g^{i_1i_2}(y)(\frac{dA^j(x)}{dx^{i_1}})\frac{d^2}{dx^{i_2}dx^j}+g^{i_1i_2}(y)(\frac{dA^j(x)}{dx^{i_1}dx^{i_2}})\frac{d}{dx^j}\\
&\quad\left.2g^{i_1i_2}(y)(\frac{dB(x)}{dx^{i_1}})\frac{d}{dx^{i_2}}+g^{i_1i_2}(y)(\frac{d^2B(x)}{dx^{i_1}dx^{i_2}})\right).
\end{aligned} \tag{13}$$

Therefore, the trace is computed to be, with the help of the formulae (10),

$$Tr(\frac{1}{2}[X,Y]e^X) = \int d^dx\langle x|\frac{1}{2}[X,Y]e^X|x\rangle$$
$$= \int d^dx\frac{e(x)}{(2\pi\tau)^{\frac{d}{2}}}\left\{\tau\left(-\frac{1}{4}g^{i_1i_2}(x)g_{j_1j_2}(x)(\frac{d^2g^{j_1j_2}(x)}{dx^{i_1}dx^{i_2}}) - \frac{1}{2}(\frac{dA^i(x)}{dx^i})\right) + \frac{\tau^2}{2}g^{i_1i_2}(x)(\frac{d^2B(x)}{dx^{i_1}dx^{i_2}})\right\}. \tag{14}$$

Computation of $\frac{1}{6}[X,[X,Y]]e^X$
We compute the operator $[X,[X,Y]]$ as

$$\begin{aligned}
[X,[X,Y]] = \tau^3 \Bigg(& 4g^{i_1i_2}(y)g^{k_1k_2}(y)(\frac{d^2g^{j_1j_2}(x)}{dx^{i_1}dx^{k_1}})\frac{d^4}{dx^{i_2}dx^{k_2}dx^{j_1}dx^{j_2}}\\
&+4g^{i_1i_2}(y)g^{k_1k_2}(y)(\frac{d^3g^{j_1j_2}(x)}{dx^{i_1}dx^{i_2}dx^{k_1}})\frac{d^3}{dx^{k_2}dx^{j_1}dx^{j_2}} + g^{i_1i_2}(y)g^{k_1k_2}(y)(\frac{d^4g^{j_1j_2}(x)}{dx^{i_1}dx^{i_2}dx^{k_1}dx^{k_2}})\frac{d^2}{dx^{j_1}dx^{j_2}}\\
&+4g^{i_1i_2}(y)g^{k_1k_2}(y)(\frac{d^2A^j(x)}{dx^{i_1}dx^{k_1}})\frac{d^3}{dx^{i_2}dx^{k_2}dx^j}\\
&+4g^{i_1i_2}(y)g^{k_1k_2}(y)(\frac{d^3A^j(x)}{dx^{i_1}dx^{i_2}dx^{k_1}})\frac{d^2}{dx^{i_2}dx^j} + g^{i_1i_2}(y)g^{k_1k_2}(y)(\frac{d^4A^j(x)}{dx^{i_1}dx^{i_2}dx^{k_1}dx^{k_2}})\frac{d}{dx^j}\\
&+4g^{i_1i_2}(y)g^{k_1k_2}(y)(\frac{d^2B(x)}{dx^{i_1}dx^{k_1}})\frac{d^2}{dx^{i_2}dx^{k_2}} + 4g^{i_1i_2}(y)g^{k_1k_2}(y)(\frac{d^3B(x)}{dx^{i_1}dx^{i_2}dx^{k_1}})\frac{d}{dx^{k_2}}\\
&+g^{i_1i_2}(y)g^{k_1k_2}(y)(\frac{d^4B(x)}{dx^{i_1}dx^{i_2}dx^{k_1}dx^{k_2}})\Bigg).
\end{aligned} \tag{15}$$

Therefore, the trace is computed as

$$Tr(\frac{1}{6}[X,[X,Y]]e^X) = \int d^dx\langle x|\frac{1}{6}[X,[X,Y]]e^X|x\rangle$$
$$\begin{aligned}
= \int d^dx\frac{e(x)}{(2\pi\tau)^{\frac{d}{2}}}\Bigg\{ & \tau\left(\frac{1}{6}g^{i_1i_2}(x)g_{j_1j_2}(x)(\frac{d^2g^{j_1j_2}(x)}{dx^{i_1}dx^{i_2}}) + \frac{1}{3}(\frac{d^2g^{ij}(x)}{dx^idx^j})\right)\\
&-\tau^2\left(\frac{1}{12}g^{i_1i_2}(x)g^{j_1j_2}(x)g^{k_1k_2}(x)(\frac{d^4g^{j_1j_2}(x)}{dx^{i_1}dx^{i_2}dx^{k_1}dx^{k_2}}) + \frac{1}{3}g^{i_1i_2}(x)(\frac{d^3A^j(x)}{dx^{i_1i_2j}}) + \frac{1}{3}g^{i_1i_2}(x)(\frac{d^2B(x)}{dx^{i_1}dx^{i_2}})\right)\\
&+\frac{\tau^3}{6}(g^{i_1i_2}(x)g^{j_1j_2}(x))(\frac{d^4B(x)}{dx^{i_1}dx^{i_2}dx^{j_1}dx^{j_2}})\Bigg\}.
\end{aligned} \tag{16}$$

Computation of $\frac{1}{2}Y^2e^X$
The next job is the computation of the term $\frac{1}{2}Y^2$:

$$\begin{aligned}
Y^2 &= \left((g^{i_1i_2}(x)-g^{i_1i_2}(y))\frac{d^2}{dx^{i_1}dx^{i_2}} + A^i(x)\frac{d}{dx^i} + B(x)\right)\left((g^{j_1j_2}(x)-g^{j_1j_2}(y))\frac{d^2}{dx^{j_1}dx^{j_2}} + A^j(x)\frac{d}{dx^j} + B(x)\right)\\
&= \tau^2\Bigg((g^{i_1i_2}(x)-g^{i_1i_2}(y))(g^{j_1j_2}(x)-g^{j_1j_2}(y))\frac{d^4}{dx^{i_1}dx^{i_2}dx^{j_1}dx^{j_2}}\\
&\quad+2(g^{i_1i_2}(x)-g^{i_1i_2}(y))(\frac{dg^{j_1j_2}(x)}{dx^{i_1}})\frac{d^3}{dx^{i_2}dx^{j_1}dx^{j_2}} + (g^{i_1i_2}(x)-g^{i_1i_2}(y))(\frac{d^2g^{j_1j_2}(x)}{dx^{i_1}dx^{i_2}})\frac{d^2}{dx^{j_1}dx^{j_2}}\\
&\quad+2(g^{i_1i_2}(x)-g^{i_1i_2}(y))A^j(x)\frac{d^3}{dx^{i_1}dx^{i_2}dx^j} + 2(g^{i_1i_2}(x)-g^{i_1i_2}(y))(\frac{dA^j(x)}{dx^{i_1}})\frac{d^2}{dx^{i_2}dx^j}
\end{aligned}$$

$$+(g^{i_1i_2}(x)-g^{i_1i_2}(y))(\frac{d^2A^j(x)}{dx^{i_1}dx^{i_2}})\frac{d}{dx^j}+(g^{i_1i_2}(x)-g^{i_1i_2}(y))B(x)\frac{d^2}{dx^{i_1}dx^{i_2}}$$
$$+2(g^{i_1i_2}(x)-g^{i_1i_2}(y))(\frac{dB(x)}{dx^{i_1}})\frac{d}{dx^{i_2}}+(g^{i_1i_2}(x)-g^{i_1i_2}(y))(\frac{d^2B(x)}{dx^{i_1}dx^{i_2}})$$
$$+A^i(x)(\frac{dg^{j_1j_2}(x)}{dx^i})\frac{d^2}{dx^{j_1}dx^{j_2}}+A^i(x)A^j(x)\frac{d^2}{dx^idx^j}+A^i(x)B(x)\frac{d}{dx^i}+A^i(x)(\frac{dB(x)}{dx^i})$$
$$(g^{i_1i_2}(x)-g^{i_1i_2}(y))B(x)\frac{d^2}{dx^{j_1}dx^{j_2}}+B(x)A^i(x)\frac{d}{dx^i}+B(x)B(x)\Big). \tag{17}$$

The trace is thus

$$Tr(\frac{1}{2}Y^2e^X)=\int d^dx\langle x|\frac{1}{2}Y^2e^X|x\rangle$$
$$=\int d^dx\frac{e(x)}{(2\pi\tau)^{\frac{d}{2}}}\Big\{\tau\Big(-\frac{1}{4}A^i(x)g_{j_1j_2}(x)(\frac{dg^{j_1j_2}(x)}{dx^i})-\frac{1}{4}A^i(x)A^j(x)g_{ij}(x)\Big)$$
$$+\tau^2(\frac{1}{2}A^i(x)(\frac{dB(x)}{dx^i})+\frac{1}{2}B(x)B(x))\Big\}. \tag{18}$$

Computation of $\frac{1}{8}[X,Y]^2e^X$

We next compute the commutator $[X,Y]^2$, however, from now on, the computation becomes more complicated than before, and we give only the trace:

$$Tr(\frac{1}{8}[X,Y]^2e^X)=\int d^dx\langle x|\frac{1}{8}[X,Y]^2e^X|x\rangle$$
$$=\int d^dx\frac{e(x)}{(2\pi\tau)^{\frac{d}{2}}}\Big\{\tau\Big(-\frac{1}{16}g^{ik}(x)g_{j_1j_2}(x)g_{l_1l_2}(x)(\frac{dg^{j_1j_2}(x)}{dx^i})(\frac{dg^{l_1l_2}(x)}{dx^k})$$
$$-\frac{1}{8}g^{ik}(x)g_{j_1l_1}(x)g_{j_2l_2}(x)(\frac{dg^{j_1j_2}(x)}{dx^i})(\frac{dg^{l_1l_2}(x)}{dx^k})-\frac{1}{4}(\frac{dg^{j_1j_2}(x)}{dx^{j_1}})(\frac{dg^{l_1l_2}(x)}{dx^{j_2}})g_{l_1l_2}(x)$$
$$-\frac{1}{4}g_{j_2l_2}(x)(\frac{dg^{l_1l_2}(x)}{dx^{j_1}})(\frac{dg^{j_1j_2}(x)}{dx^{l_1}})-\frac{1}{4}g_{ij}(x)(\frac{dg^{ip}(x)}{dx^p})(\frac{dg^{jq}(x)}{dx^q})\Big)+\mathcal{O}(\tau^2)\Big\}. \tag{19}$$

Computation of $\frac{1}{3}Y[X,Y]e^X$

$$Tr(\frac{1}{3}Y[X,Y]e^X)=\int d^dx\langle x|\frac{1}{3}Y[X,Y]e^X|x\rangle$$
$$=\int d^dx\frac{e(x)}{(2\pi\tau)^{\frac{d}{2}}}\Big\{\tau\Big(\frac{1}{6}A^i(x)(\frac{dg^{j_1j_2}(x)}{dx^i})g_{j_1j_2}(x)+\frac{1}{3}A^i(x)g_{ij}(x)(\frac{dg^{j_1j_2}(x)}{dx^{j_2}})\Big)$$
$$-\tau^2\Big(\frac{1}{6}g^{k_1k_2}(x)g_{ij}(x)A^i(x)(\frac{d^2A^j(x)}{dx^{k_1}dx^{k_2}})+\frac{1}{3}A^i(x)(\frac{dB(x)}{dx^i})$$
$$+\frac{1}{6}g^{k_1k_2}(x)g_{j_1j_2}(x)(\frac{d^2g^{j_1j_2}(x)}{dx^{k_1}dx^{k_2}})B(x)+\frac{1}{6}g^{k_1k_2}(x)g_{j_1j_2}(x)A^i(x)(\frac{d^3g^{j_1j_2}(x)}{dx^idx^{k_1}dx^{k_2}})$$
$$+\frac{1}{3}A^i(x)(\frac{d^2A^j(x)}{dx^idx^j})+\frac{1}{3}B(x)(\frac{dA^i(x)}{dx^i})\Big)$$
$$+\frac{\tau^3}{3}B(x)g^{k_1k_2}(x)(\frac{d^2B(x)}{dx^{k_1}dx^{k^2}})\Big\}. \tag{20}$$

Computation of $\frac{1}{6}[X,Y]Ye^X$

$$Tr(\frac{1}{6}[X,Y]Ye^X) = \int d^dx \langle x|\frac{1}{6}[X,Y]Ye^X|x\rangle$$
$$= \int d^dx \frac{e(x)}{(2\pi\tau)^{\frac{d}{2}}} \Big\{ \tau \Big(\frac{1}{12} g^{k_1k_2}(x) g_{i_1i_2}(x) g_{j_1j_2}(x) (\frac{dg^{i_1i_2}(x)}{dx^{k_1}})(\frac{dg^{j_1j_2}(x)}{dx^{k_2}})$$
$$+\frac{1}{6} g^{k_1k_2}(x) g_{i_1j_1}(x) g_{i_2j_2}(x) (\frac{dg^{i_1i_2}(x)}{dx^{k_1}})(\frac{dg^{j_1j_2}(x)}{dx^{k_2}})$$
$$+\frac{1}{6} g_{i_1i_2}(x)(\frac{dg^{i_1i_2}(x)}{dx^{j_1}})(\frac{dg^{j_1j_2}(x)}{dx^{j_2}}) + \frac{1}{3} g_{i_2j_2}(x)(\frac{dg^{j_1j_2}(x)}{dx^{i_1}})(\frac{dg^{i_1i_2}(x)}{dx^{j_1}})$$
$$+\frac{1}{12} g_{j_1j_2}(x)(\frac{dg^{j_1j_2}(x)}{dx^i}) A^i(x) + \frac{1}{6}(\frac{dg^{j_1j_2}(x)}{dx^{j_1}}) g_{j_2i}(x) A^i(x) \Big) + \mathcal{O}(\tau^2) \Big\}. \quad (21)$$

Seeley de Witt coefficient of the second lowest order

Now that we have computed all of the contribution of the Seeley de Witt coefficient of the order $\mathcal{O}(\tau^{1-\frac{d}{2}})$, we sum all the results. Then, the trace is finally rewritten as

$$Tr(e^{-\tau D^2}) = \int d^dx \langle x|e^{-\tau D^2}|x\rangle = \int d^dx \frac{e(x)}{(2\pi\tau)^{\frac{d}{2}}}(a_0 + \tau a_1 + \cdots). \quad (22)$$

It goes without stating that the coefficient a_0 of the lowest order is $a_0 = 1$. Then, the subleading effect is

$$\boxed{\begin{aligned} a_1(x) &= B(x) - \frac{1}{2}(\frac{dA^i(x)}{dx^i}) + \frac{1}{3}(\frac{d^2g^{ij}(x)}{dx^idx^j}) - \frac{1}{12} g^{i_1i_2}(x) g_{j_1j_2}(x)(\frac{d^2g^{j_1j_2}(x)}{dx^{i_1}dx^{i_2}}) \\ &+ \frac{1}{12} g_{i_2j_2}(x)(\frac{dg^{j_1j_2}(x)}{dx^{i_1}})(\frac{dg^{i_1i_2}(x)}{dx^{j_1}}) - \frac{1}{4} A^i(x)A^j(x) g_{ij}(x) \\ &+ \frac{1}{2} A^i(x) g_{ij_1}(x)(\frac{dg^{j_1j_2}(x)}{dx^{j_2}}) \\ &+ \frac{1}{48} g^{k_1k_2}(x) g_{i_1i_2}(x) g_{j_1j_2}(x)(\frac{dg^{i_1i_2}(x)}{dx^{k_1}})(\frac{dg^{j_1j_2}(x)}{dx^{k_2}}) \\ &+ \frac{1}{24} g^{k_1k_2}(x) g_{i_1j_1}(x) g_{i_2j_2}(x)(\frac{dg^{i_1i_2}(x)}{dx^{k_1}})(\frac{dg^{j_1j_2}(x)}{dx^{k_2}}) \\ &- \frac{1}{12} g_{i_1i_2}(x)(\frac{dg^{i_1i_2}(x)}{dx^{j_1}})(\frac{dg^{j_1j_2}(x)}{dx^{j_2}}) - \frac{1}{4} g_{ij}(x)(\frac{dg^{ip}(x)}{dx^p})(\frac{dg^{jq}(x)}{dx^q}). \quad (23) \end{aligned}}$$

Consistency Check with respect to the covariant Laplace Beltrami operator

We now check the consistency of the result (23), by applying the above results to the covariant Laplace Beltrami operator

$$\begin{aligned} \Delta(x) &= \frac{1}{\sqrt{g(x)}} \Big(\frac{d}{dx^i} \sqrt{g(x)} g^{ij}(x) \frac{d}{dx^j} \Big) \\ &= g^{ij}(x) \frac{d}{dx^i}\frac{d}{dx^j} + \Big((\frac{dg^{ij}(x)}{dx^j}) - \frac{1}{2} g^{ij}(x)(\frac{d}{dx^j} g^{kl}(x)) g_{kl}(x) \Big) \frac{d}{dx^i}, \end{aligned} \quad (24)$$

where we have utilized the differentiation of the determinant

$$\delta g(x) = g(x) g^{ij}(x) \delta g_{ij}(x) = -g(x) g_{ij}(x) \delta g^{ij}(x). \quad (25)$$

Then, the problem corresponds to the case in which

$$A^i(x) = \left((\frac{dg^{ij}(x)}{dx^j}) - \frac{1}{2} g^{ij}(x) (\frac{d}{dx^j} g^{kl}(x)) g_{kl}(x) \right) \quad B(x) = 0. \tag{26}$$

In this case, we expect the coefficient $a_1(x)$ to be

$$\begin{aligned}
\frac{R(x)}{6} &= \frac{1}{6} g^{ij}(x)(-\partial_i \Gamma^k_{kj} + \partial_k \Gamma^k_{ij} - \Gamma^k_{il}\Gamma^l_{kj} + \Gamma_k \Gamma^k_{ij}) \\
&= \frac{1}{6} g^{ij}(x) g_{l_1 l_2}(x) (\frac{d^2 g^{l_1 l_2}(x)}{dx^i dx^j}) - \frac{1}{6} (\frac{d^2 g^{l_1 l_2}(x)}{dx^{l_1} dx^{l_2}}) + \frac{1}{6} (\frac{dg^{em}(x)}{dx^m})(\frac{dg^{l_1 l_2}(x)}{dx^e}) g_{l_1 l_2}(x) \\
&- \frac{5}{24} g^{ij}(x) g_{l_1 m_1}(x) g_{l_2 m_2}(x) (\frac{dg^{l_1 l_2}(x)}{dx^i})(\frac{dg^{m_1 m_2}(x)}{dx^j}) + \frac{1}{12} g_{l_1 l_2}(x) (\frac{dg^{m_2 l_1}}{dx^{m_1}})(\frac{dg^{m_1 l_2}(x)}{dx^{m_2}}) \\
&- \frac{1}{24} g^{ij}(x) g_{l_1 l_2}(x) g_{m_1 m_2}(x) (\frac{dg^{l_1 l_2}(x)}{dx^i})(\frac{dg^{m_1 m_2}(x)}{dx^j}).
\end{aligned} \tag{27}$$

as investigated in Di Francesco's textbook.

And when we substitute (26) into the Seeley de Will coefficient $a_1(x)$, we successfully obtain (27).