YITP Workshop on Quantum Information Physics@YITP

Dynamics of Entanglement Entropy From Einstein Equation

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Main Results

AdS(gravity) side: Einstein equation

⇕

CFT side: constraint equation for entanglement entropy

Motivations

(1)*Field theoretical motivation*

In a quantum field theory, excited states properties are not well studied , so we study the excited states properties in CFTs (critical point theory of quantum many body systems.)

We consider the weakly excited states.

To study universal properties , we need to study the physical observables that can be defined in any theory.

→ We study entanglement entropy for excited states.

(2) *Gravity Motivation*

In the AdS/CFT context,

Entanglement entropy ⟷ Minimal surface

So entanglement entropy is directly related to the bulk metric.

On the other hand ,

excited states ⟷ deformation of metric

The deformed metric also satisfy the Einstein equation.

From these , we can expect that there should be a counterpart of the bulk Einstein equation which constrains the behavior of entanglement entropy.

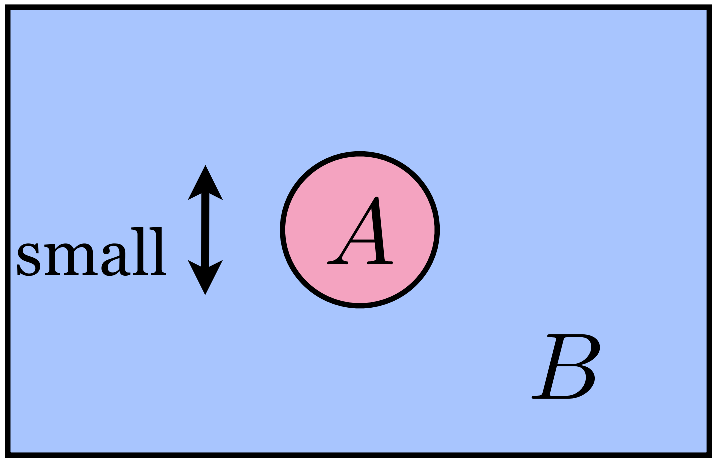
(3)*Thermodynamics Motivation*

It is known that first-law like relation holds for entanglement entropy in CFTsif the excitation is small, static ,and translational invariant.

→ Is this true for the time dependent excitation?

First law for entanglement entropy

First law like relation holds for entanglement entropy in conformal field theory if the excitation is sufficiently small and translational invariant :



$$\Delta E_A = T_{ent} \Delta S_A$$

[Bhattacharya-Nozaki-Ugajin-Takayanagi 12]

Energy in A

"Temperature":depend only on the geometry of A

ΔS_A is the difference between EE for excited states and ground states. For example, if the subsystem is a round ball, then

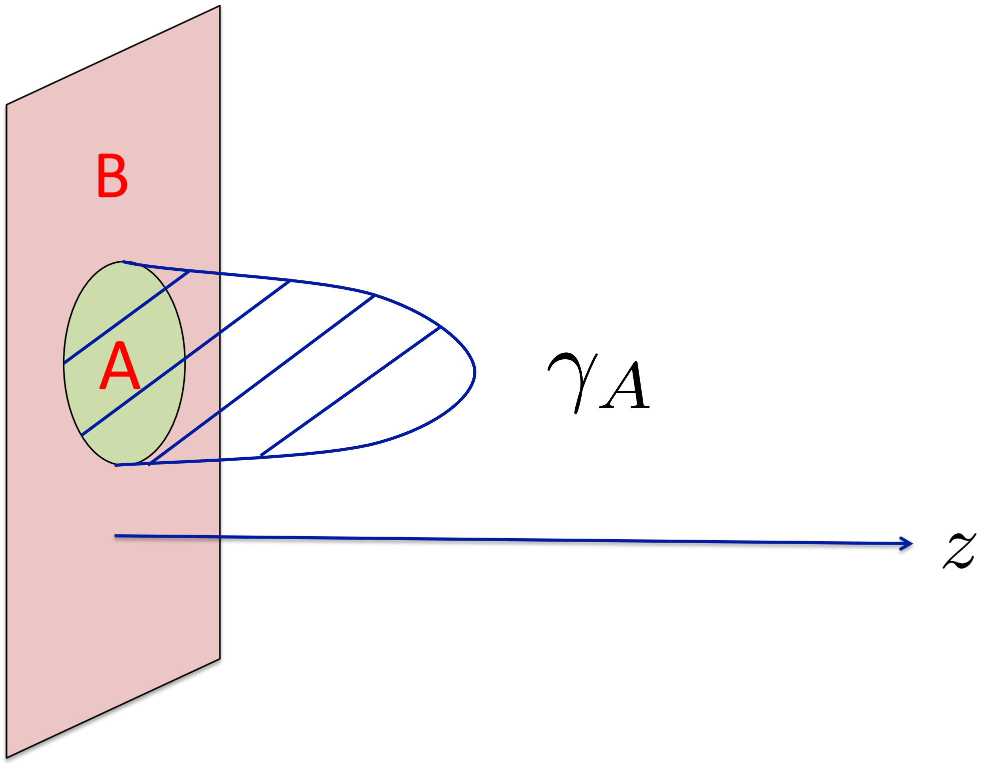
$$T_{ent} = \frac{2\pi l}{d+1}$$

Holographic Entanglement Entropy

In the AdS/CFT correspondence , EE in a CFT_d corresponds to the minimal surface in the bulk:

$$S_A = \frac{Area(\gamma_A)}{4G_N}$$

γ_A :minimal surface



The minimal surface shares the boundary with the subsystem A.

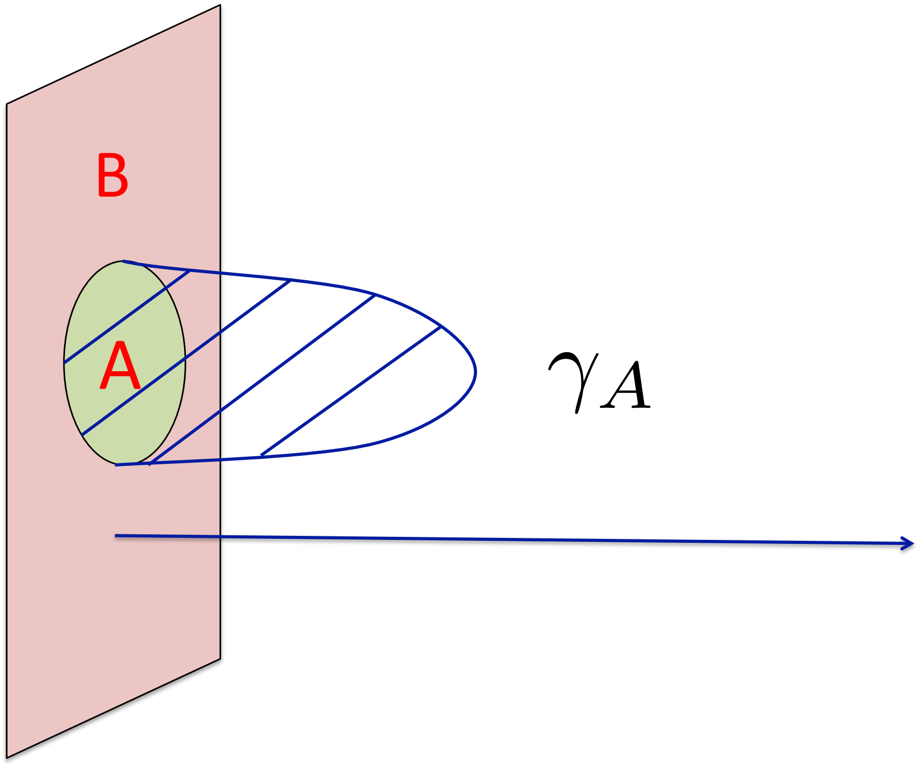
Entanglement entropy is a nonlocal observable , and this is reflected to the fact that the minimal surface extends to the bulk.

So naively, we think that we can detect the bulk using the minimal surface or entanglement entropy in the boundary CFT viewpoint .

How to calculate Holographic EE for excited states

$$S_A = \frac{Area(\gamma_A)}{4G_N}$$

γ_A :minimal surface



Excited states $\longrightarrow$ bulk metric is changed from the AdS metric

Induced metric : $G_{\alpha\beta} = G^{(0)}_{\alpha\beta} + \varepsilon G^{(1)}_{\alpha\beta} + \mathcal{O}(\varepsilon^2)$

Minimal surface : $\gamma_A = \gamma^{(0)}_A + \varepsilon \gamma^{(1)}_A + \mathcal{O}(\varepsilon^2)$

Beause $\gamma^{(0)}_A$ is a minimal surface, In the first order of ε,

$$\Delta S_A = \frac{1}{8G_N} \int_{\gamma^{(0)}_A} \sqrt{G^{(0)}} G^{(1)}_{\alpha\beta} G^{\alpha\beta(0)}$$

We choose the subsystem A to be a round ball .

Case of AdS3/CFT2

First, we consider the case the bulk theory is pure Einstein gravity:

$$S = \frac{1}{16\pi G_N} \int \left(R + \frac{6}{L^2} \right)$$

We expand perturbatively the metric around the AdS solution
in the GF coordinate:

$$ds^2 = L^2 \frac{dz^2 + g_{\mu\nu}(z,x)dx^\mu dx^\nu}{z^2} \ , \quad g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}$$

Then, we get the EOM in the first order of ε

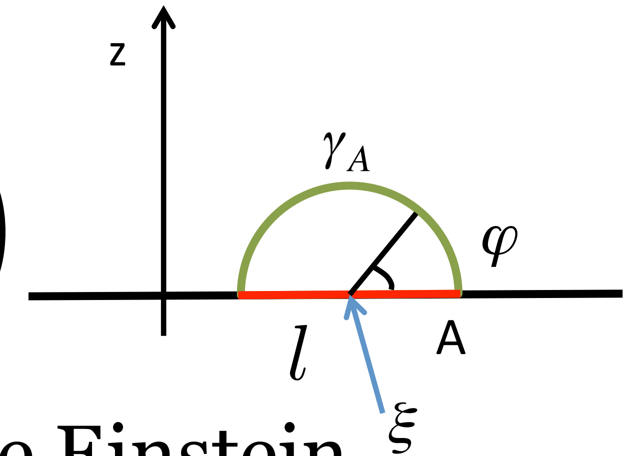
$$\Rightarrow \quad (\partial_t^2 - \partial_x^2) H(t,x) = 0$$

where

$$h_{tt} = h_{xx} = z^2 H(t,x), \quad \partial_t h_{tx} = z^2 \partial_x H(t,x), \quad \partial_x h_{tx} = z^2 \partial_t H(t.x)$$

Using $H(t,x)$, we can write the variation of EE as

$$\Delta S_A(\xi,l,t)=\frac{Ll^2}{32G_N}\int d\varphi\cos^3\varphi H\left(t,\xi+\frac{l}{2}\sin\varphi\right)$$



If we use the wave equation for $H(t,x)$ derived from the Einstein equation , we can get the following equations:

$$(\partial_t^2-\partial_\xi^2)\Delta S_A(\xi,l,t)=0$$

$$\left[\partial_l^2-\frac{1}{4}\partial_\xi^2-\frac{2}{l^2}\right]\Delta S_A(\xi,l,t)=0$$

This is the counterpart of perturbative Einstein eq.

Derivation of first law from Einstein eq

We consider the small subsystem limit $l \to 0$ (don't assume the translational invariance).

In this limit , HEE is written as follows:

$$\Delta S_A(\xi, l, t) \simeq \frac{Ll^2}{24G_N} H(t,\xi)$$

On the other hand, from the formula of Holographic energy momentum tensor we can find the following relations:

$$T_{tt}^{\mathrm{CFT}} = \frac{L}{8\pi G_N} H(t,\xi)$$

$$\Delta E_A = \int dl\ T_{tt}^{\mathrm{CFT}} \simeq l \cdot T_{tt}^{\mathrm{CFT}} = \frac{Ll}{8\pi G_N} H(t,\xi)$$

From these relations , we can get the first-law like relation:

$$\Delta E_A = T_{ent} \Delta S_A, \quad T_{ent} = \frac{3}{\pi l}$$

Case of AdS_4/CFT_3

We consider the case the bulk theory is a pure Einstein gravity.

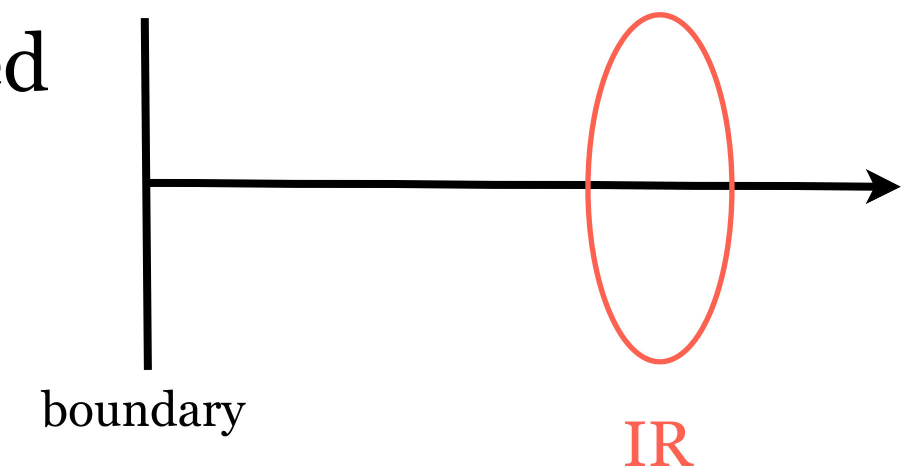
The equation for EE that is the counterpart of Einstein eq becomes as follow:

$$\left[\frac{\partial^2}{\partial l^2} - \frac{1}{l}\frac{\partial}{\partial l} - \frac{3}{l^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right]\Delta S_A = 0$$

[Bhattacharya-Takayanagi 13]

This equation contains no time derivatives.

➡ The time evolution of EE is determined by the IR boundary condition.



If we take the limit of $l \to 0$, we can find the first-law like relation.

The meaning of the equation

Roughly speaking , the differential equation is hyperbolic PDE:

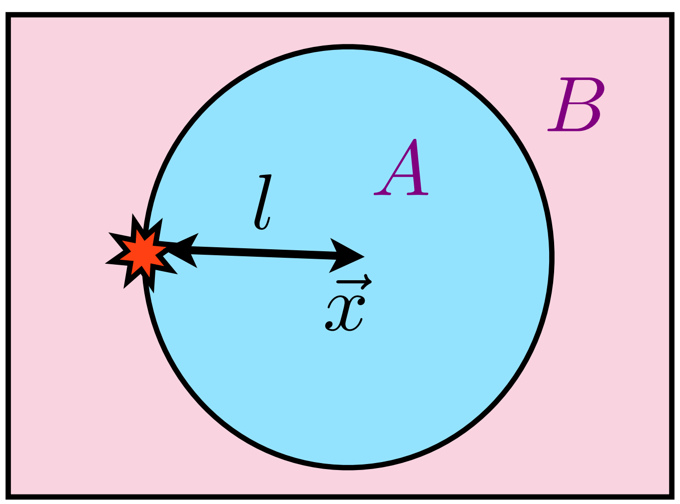
$$(\partial_l^2 - \partial_{\vec{x}}^2)\Delta S_A(t, \vec{x}, l) \approx 0$$

$$\Delta S_A(t, \vec{x}, l) \approx f(l - |\vec{x}|) + g(l + |\vec{x}|)$$

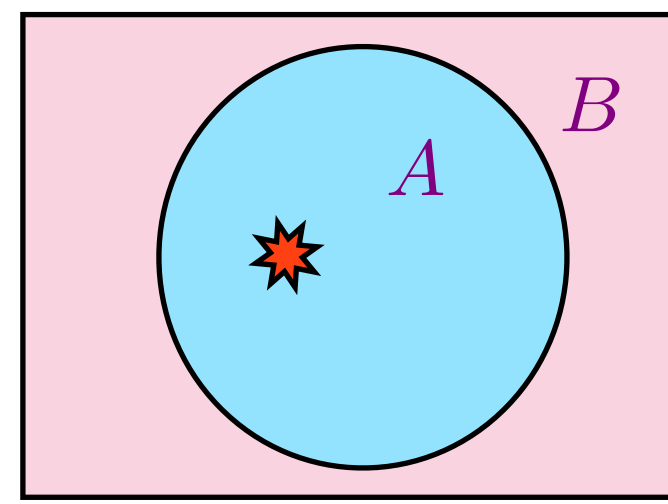
Consider the case of local excitation.

$$\Rightarrow \quad \Delta S_A \approx \delta(l - |\vec{x}|)$$

$\Delta S_A \neq 0$



$\Delta S_A = 0$



The differential equation put a constraint that ΔS_A is non-trivial only when the ∂A intersect with the excited region !

Case of Einstein-Scalar theory

We consider a gravity with matter(scalar field).

$$S = \frac{1}{16\pi G_N}\int (R - 2\Lambda) + \frac{1}{4}\int \left((\partial\phi)^2 + m^2\phi^2\right)$$

In this case , the differential equations for entanglement entropy is modified as follows:

- Case of AdS_3/CFT_2

$$(\partial_t^2 - \partial_\xi^2)\Delta S_A(\xi, l, t) = \langle O \rangle \langle O \rangle$$

$$\left[\partial_l^2 - \frac{1}{4}\partial_\xi - \frac{2}{l^2}\right]\Delta S_A(\xi, l, t) = \langle O \rangle \langle O \rangle$$

O :operator dual to the bulk scalar

dual ! $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}$

- Case of AdS_4/CFT_3

$$\left[\frac{\partial^2}{\partial l^2} - \frac{\partial}{\partial l} - \frac{3}{l^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y^2}\right]\Delta S_A = \langle O \rangle \langle O \rangle$$

First-law like relation also holds.

Conclusion

- We derive the equations for entanglement entropy dual to the bulk Einstein equation.
- We calculate the variation of entanglement entropy explicitly and confirm that the first-law like relation is satisfied if we take the limit subsystem is sufficiently small .

Future problem

- We assume that the theory is invariant under the conformal transformation.

  ⇒ If a theory doesn't have conformal invariance, are there relations?

- We linearize Einstein equations.

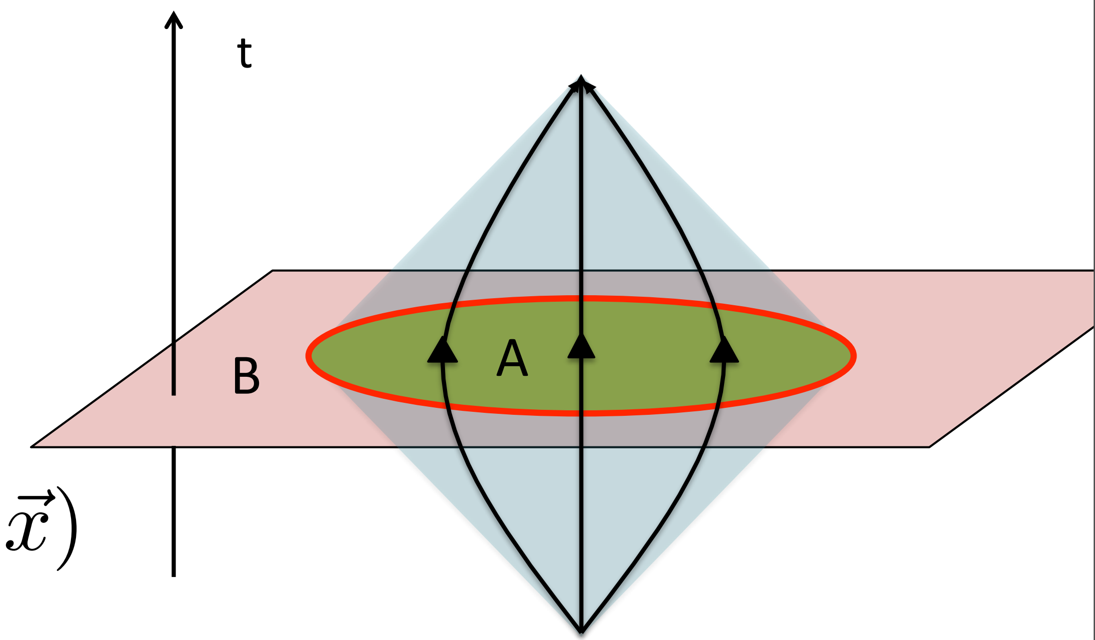
  ⇒ What is the nonlinear version?

- The inverse of our results , or derivation of Einstein eqs from constraints for EE. ⇒ Already done by Raamsdonk et.al .

Gravitation from "Entanglement thermodynamics"

[Lashkari-McDerott-Raamsdonk 13]
[Faulkner-Guica-Hartman-Myers-Raamsdonk 13]

If the subsystem is a round ball, the first-law like relation holds also when the subsystem is not small:



$$\Delta H_A = \Delta S_A$$

[Blanco-Casini-Hung-Myers 13]

where

$$\Delta H_A = 2\pi \int_A \frac{R^2 - |\vec{x} - \vec{x}_0|^2}{2R} T_{tt}^{\mathrm{CFT}}(t_0, \vec{x})$$

generator of isometry of the causal development of the round ball A

R is the radius of subsystem A .

This is the integrated version of our results.

In the small size limit , we can reproduce the first-law like relation $\Delta E_A = T_{ent} \Delta S_A$.

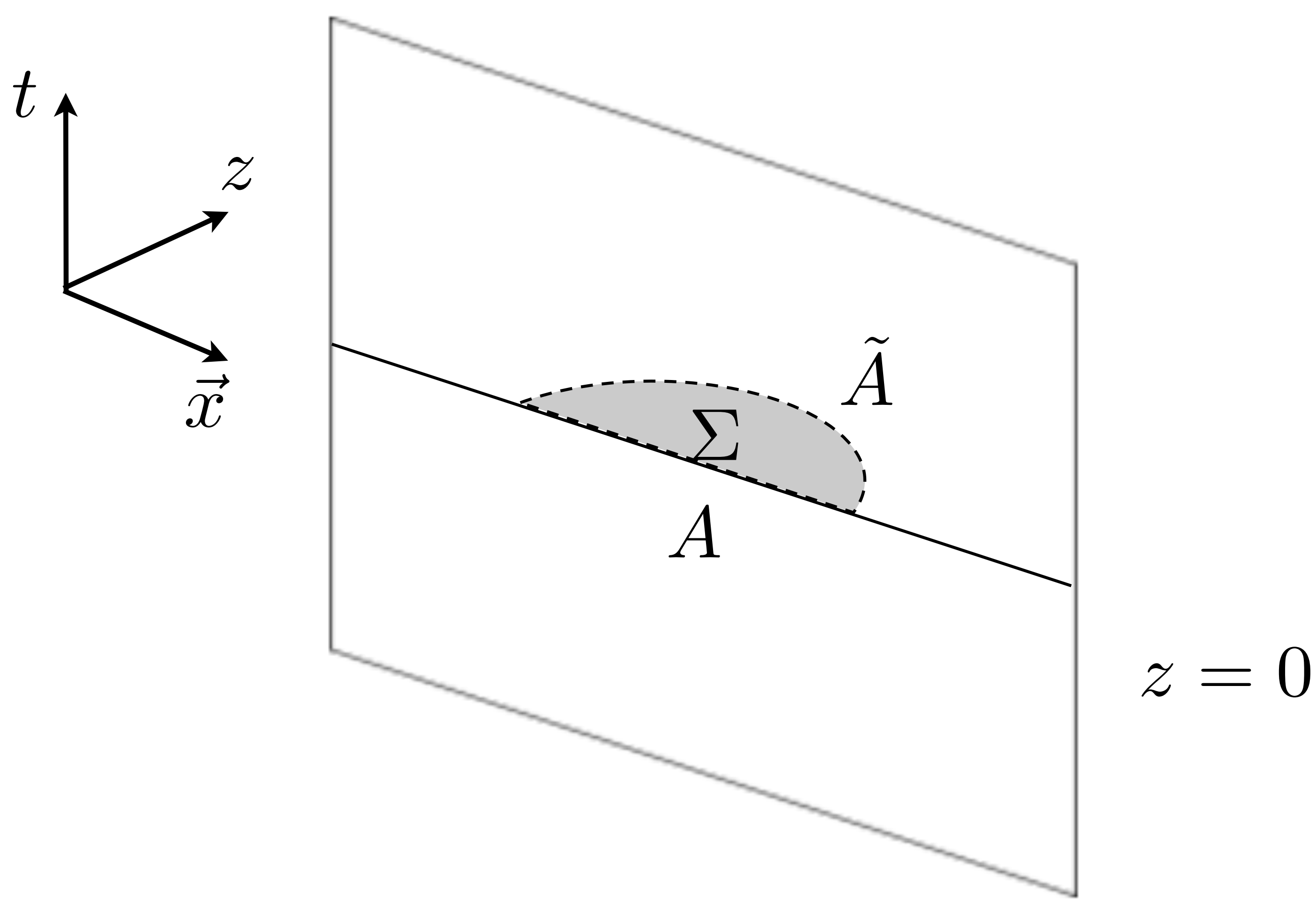
Einstein eq from first-law like relation

From the gravitational view point, $\Delta H_A = \Delta S_A$ means there is a relation between the two functionals of linearized metric:

$$\int_A f_E(h_{\mu\nu}) = \int_{\tilde{A}} f_S(h_{\mu\nu})$$

This is a nonlocal constraint, but the Einstein eq is a local constraint.

This achieved by the following way.



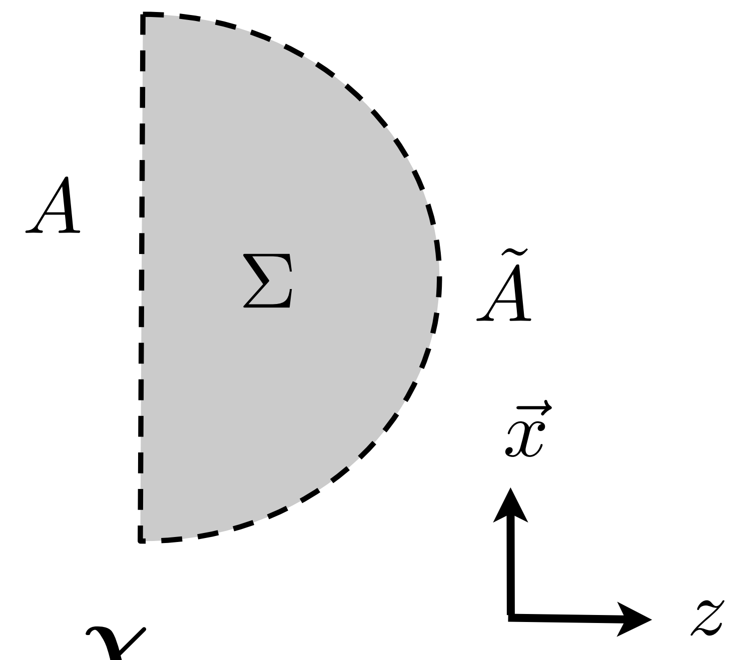
We denote Einstein eq by $G_{ab} = 0$.

We can find a $(d-1)$-form χ which satisfy the following properties:

$$\int_A \chi = \Delta H_A, \quad \int_{\tilde{A}} \chi = \Delta S_A$$

$$d\chi = -2f(x) G_{tt} \mathrm{vol}_\Sigma$$

tt component of Einstein eq



From the first-law like relation,

$$0 = \Delta S_A - \Delta E_A = \int_{\tilde{A}} \chi - \int_A \chi = \int_{\partial\Sigma} \chi$$

Then , from the stokes' theorem,

$$\int_{\partial\Sigma} \chi = \int_\Sigma d\chi = 0$$

Since we can choose A arbitrary ball, this equality folds for any Σ.

Then we can conclude that the integrand becomes 0 :

$$d\chi = 0 \Rightarrow \boxed{G_{tt} = 0}$$

tt component of Einstein eq

Other components can be shown the same way.