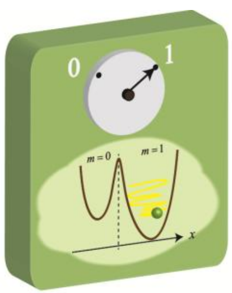
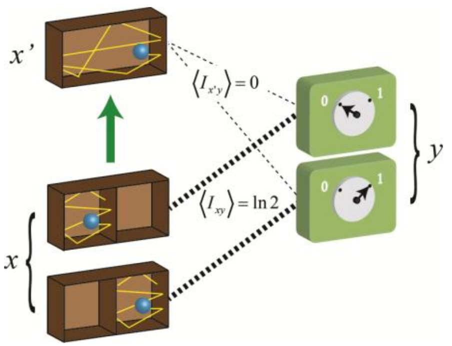
Quantum-information thermodynamics





Takahiro Sagawa

Department of Basic Science, University of Tokyo

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Collaborators on Information Thermodynamics

- Masahito Ueda (Univ. Tokyo)

- Shoichi Toyabe (LMU Munich)
- Eiro Muneyuki (Chuo Univ.)
- Masaki Sano (Univ. Tokyo)

- Sosuke Ito (Univ. Tokyo)
- Naoto Shiraishi (Univ. Tokyo)

- Sang Wook Kim (Pusan National Univ.)
- Jung Jun Park (National Univ. Singapore)
- Kang-Hwan Kim (KAIST)
- Simone De Liberato (Univ. Paris VII)

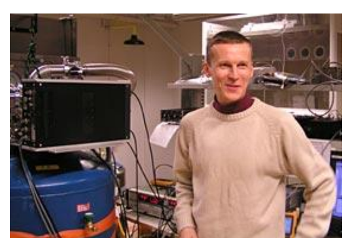
- Juan M. R. Parrondo (Univ. Madrid)
- Jordan M. Horowitz (Univ. Massachusetts)

- Jukka Pekola (Aalto Univ.)
- Jonne Koski (Aalto Univ.)
- Ville Maisi (Aalto Univ.)









Outline

- Introduction
- Quantum entropy and information
- Second law with quantum feedback
- Comprehensive framework of quantum-information thermodynamics
- Paradox of Maxwell’s demon

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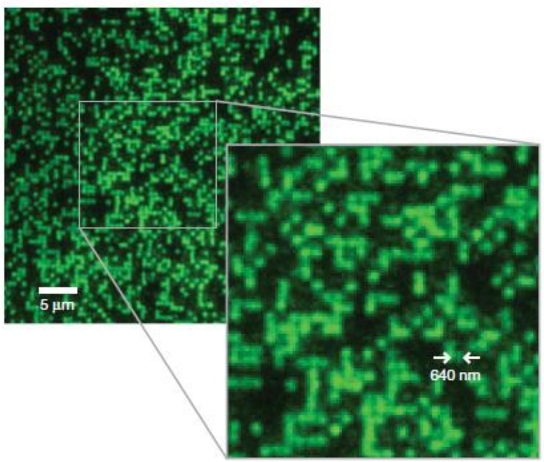
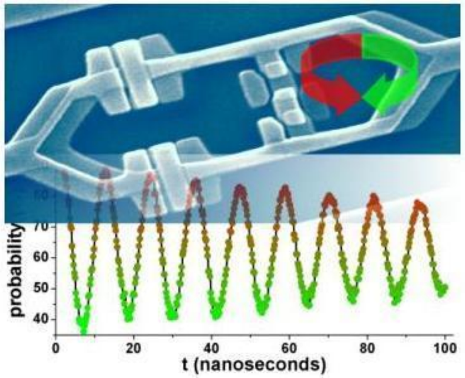
Thermodynamics in the Fluctuating World

Thermodynamics of small systems

➡ Thermodynamic quantities are fluctuating!

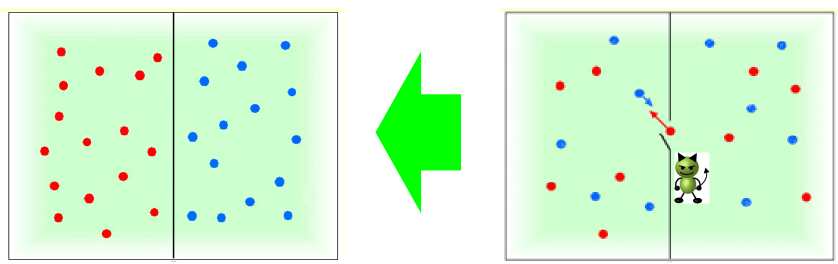


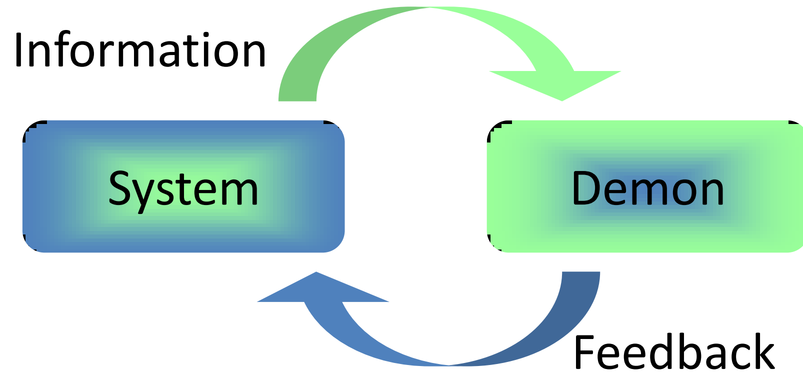


✓ **Second law of thermodynamics**

✓ **Nonequilibrium thermodynamics**

Information Thermodynamics



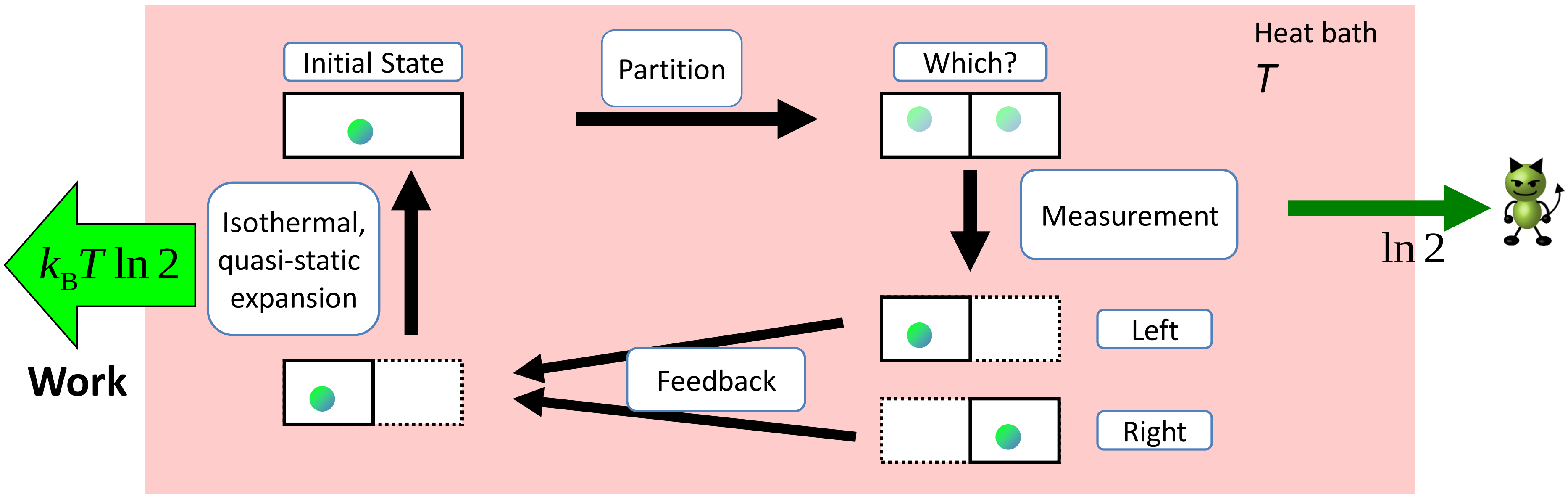


Information processing at the level of thermal fluctuations

- ✓ Foundation of the second law of thermodynamics
- ✓ Application to nanomachines and nanodevices

L. Szilard, Z. Phys. **53**, 840 (1929)

Szilard Engine (1929)



Free energy: $F = E - TS$

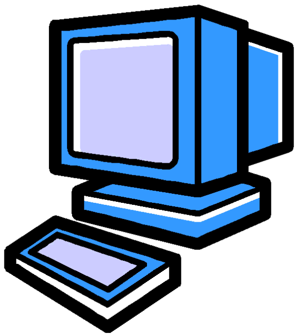
Increase

Decrease by feedback

Can control physical entropy by using information

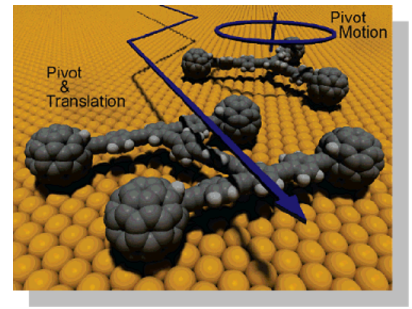
Information Heat Engine

Controller



Information

Feedback



Small system

$k_B T \ln 2$

$k_B T \ln 2$

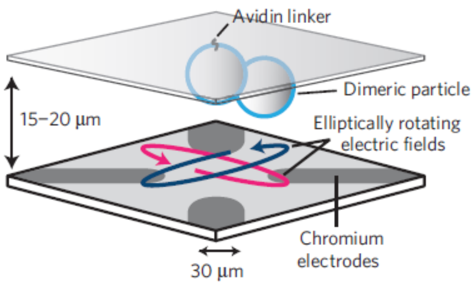
✓ **Can increase the system's free energy even if there is no energy flow between the system and the controller**

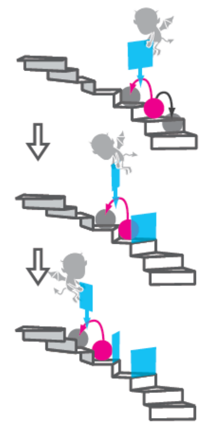
Experimental Realizations (yet classical)

- With a colloidal particle
  Toyabe, TS, Ueda, Muneyuki, & Sano, Nature Physics (2010)

Efficiency: 30%

Validation of $\left\langle e^{-\beta(W-\Delta F)} \right\rangle = \gamma$

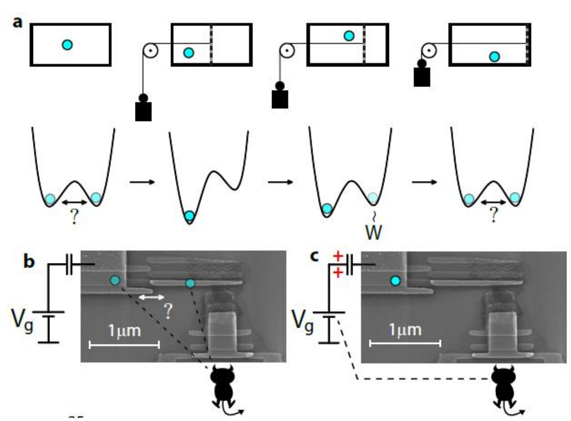




- With a single electron
  Koski, Maisi, TS, & Pekola, PRL (2014)

Efficiency: 75%

Validation of $\left\langle e^{-\beta(W-\Delta F)-I} \right\rangle = 1$



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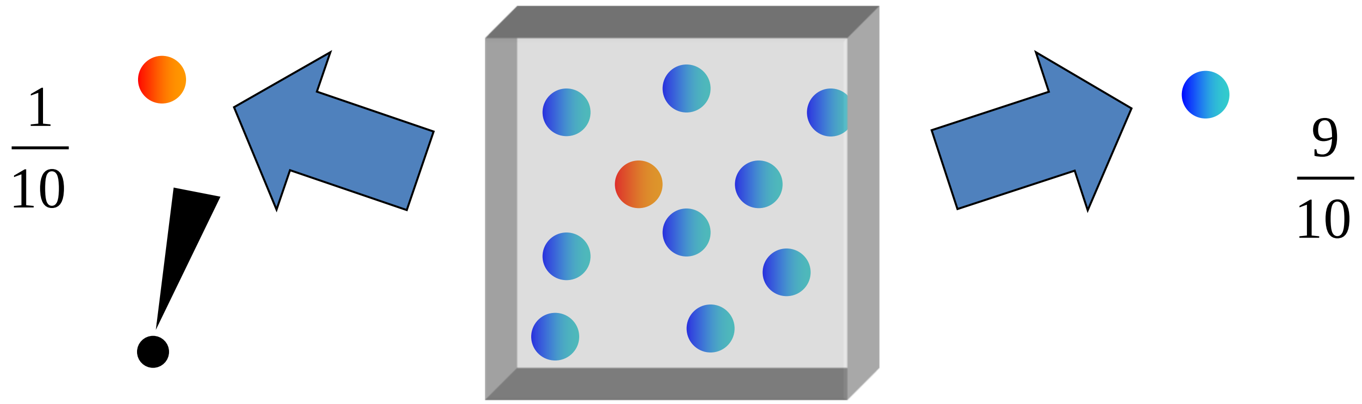
Classical Entropy



Information content with event k: $\ln \frac{1}{p_k}$

Average

Shannon entropy: $H = \sum_k p_k \ln \frac{1}{p_k}$

Quantum Entropy

Von Neumann entropy: $$S(\rho) = -\mathrm{tr}[\rho \ln \rho]$$

ρ : density operator

$$\rho = \sum_i q_i |\varphi_i\rangle\langle\varphi_i| \quad \Rightarrow \quad S(\rho) = -\sum_i q_i \ln q_i$$

with an orthonormal basis

Characterizes the randomness of the classical mixture in the density operator

Von Neumann and Thermodynamic Entropies

Canonical distribution: $$\rho_{\text{can}} = e^{\beta(F-E)}$$ E: Hamiltonian

Free energy:

$$F \equiv -k_B T \ln \operatorname{tr}\left[e^{-\beta E}\right]$$

Average energy:

$$\langle E \rangle_{\text{can}} \equiv \operatorname{tr}\left[\rho_{\text{can}} E\right]$$

$$\Rightarrow \quad \langle E \rangle_{\text{can}} - F = -k_B T \operatorname{tr}\left[\rho_{\text{can}} \ln \rho_{\text{can}}\right]$$

Cf. $$\langle E \rangle_{\text{can}} - F = T S_{\text{therm}}$$

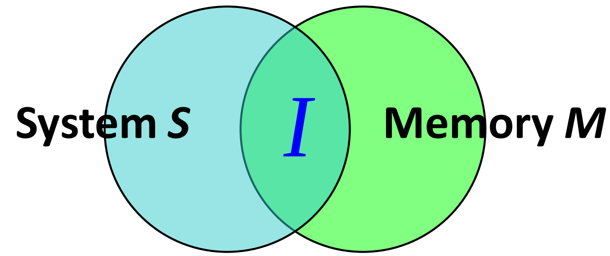
Thermodynamic entropy

The von Neumann entropy is consistent with thermodynamic entropy in the canonical distribution

Mutual Information

System S → **Memory M** (measurement device)

Measurement with stochastic errors

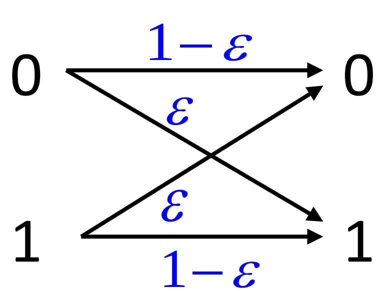


$$I(S:M) \equiv H(S) + H(M) - H(SM)$$

$$0 \leq I \leq H(M)$$

No information

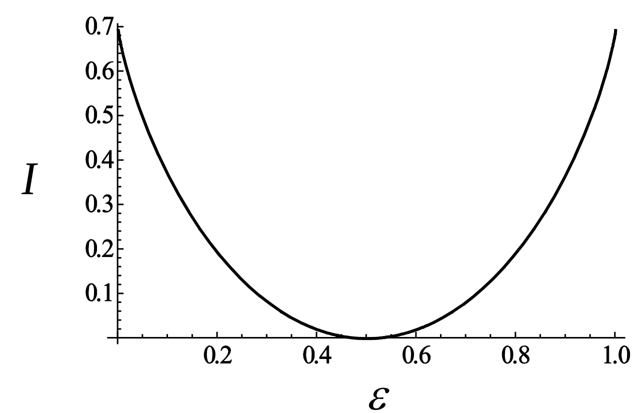
No error



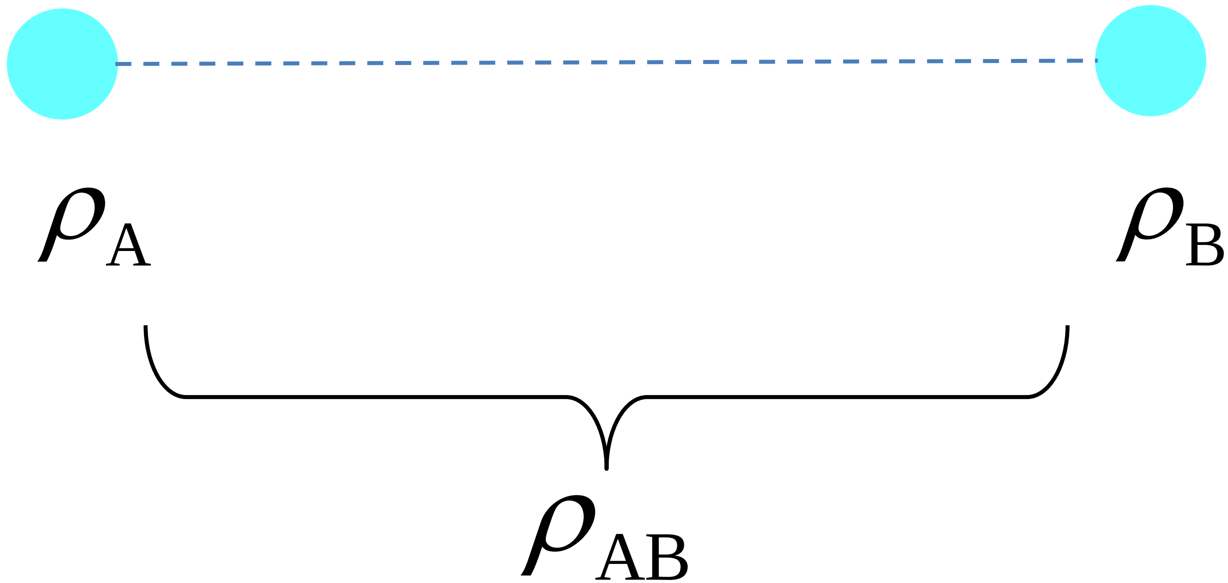
Ex. Binary symmetric channel

Correlation between S and M

$$I = \ln 2 + \varepsilon \ln \varepsilon + (1-\varepsilon)\ln(1-\varepsilon)$$



Quantum Mutual Information



$$I_{A:B}(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$$

$$I_{A:B}(\rho_{AB}) \geq 0$$

Quantum Measurement

Projection measurement (error-free)

Observable $A = \sum_k \alpha_k P_k$

Projection operators $\{P_k\}$

Probability $p_k = \mathrm{tr}(\rho P_k)$

Post-measurement state $\frac{1}{p_k} P_k \rho P_k$

General measurement

Kraus operators $\{M_{k,i}\}$

k : measurement outcome

POVM : $\{E_k\}$

$E_k = \sum_i M_{ki}^\dagger M_{ki}$ $\sum_k E_k = I$

Probability $p_k = \mathrm{tr}(\rho E_k)$

Post-measurement state $\frac{1}{p_k} \sum_i M_{ki} \rho M_{ki}^\dagger$

QC-mutual Information (1)

Information flow from **Q**uantum system to **C**lassical outcome by quantum measurement

$$I_{\mathrm{QC}} \equiv S(\rho) - \sum_k p_k S(\rho_k)$$

ρ : measured density operator

$p_k = \mathrm{tr}[\rho M_k^\dagger M_k]$: probability of obtaining outcome k

$\rho_k = \frac{1}{p_k} M_k \rho M_k^\dagger$: post-measurement state with outcome k

Assumed a single Kraus operator for each outcome

H. J. Groenewold, Int. J. Theor. Phys. **4**, 327 (1971).
M. Ozawa, J. Math. Phys. **27**, 759 (1986).
TS and M. Ueda, PRL **100**, 080403 (2008).

QC-mutual Information (2)

$$0 \leq I_{\mathrm{QC}} \leq H$$

$$H = -\sum_k p_k \ln p_k$$

No information

Error-free & classical

Any POVM element is identity operator

Any POVM element is projection and commutable with measured state

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Classical measurement, I_{QC} reduces to the classical mutual information

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If the measured state is a pure state: $I_{\mathrm{QC}} = 0$

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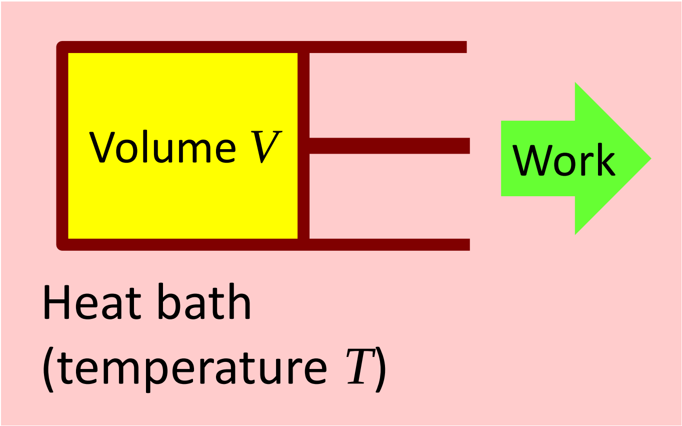
The Second Law of Thermodynamics (without Feedback)



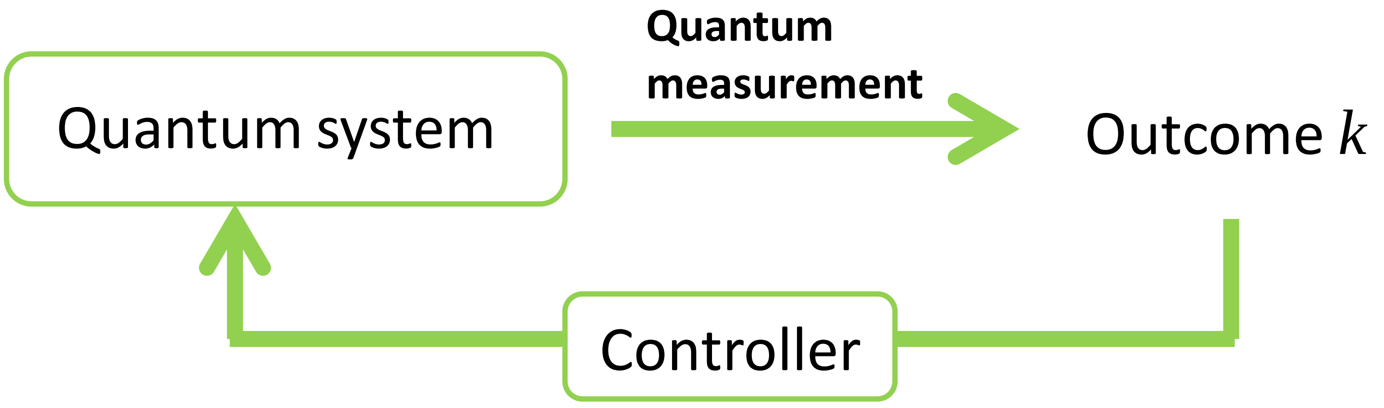
We extract the work by changing external parameters (volume of the gas, frequency of optical tweezers, …).

$$W_{\mathrm{ext}} \leq -\Delta F$$ (the equality is achieved in the quasi-static process)

With a cycle, $\Delta F = 0$ holds, and therefore

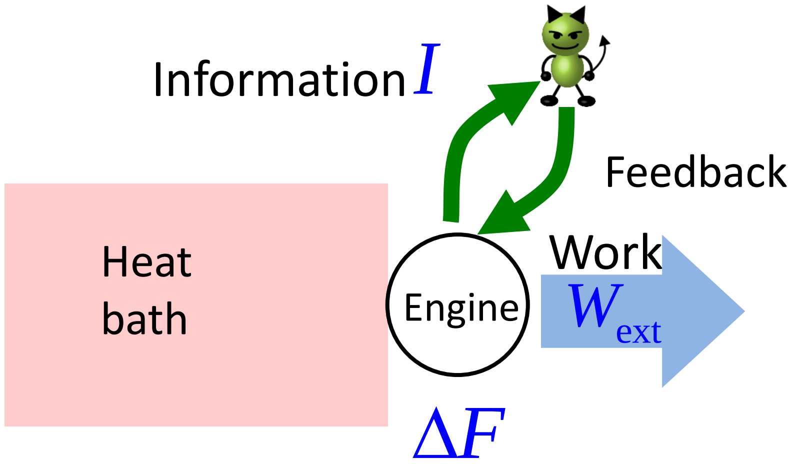
$$W_{\mathrm{ext}} \leq 0$$ (Kelvin's principle)

Quantum Feedback Control



Control protocol can depend on outcome k after measurement

Generalized Second Law with Feedback



TS and M. Ueda, PRL **100**, 080403 (2008)

$$W_{\mathrm{ext}} \leq -\Delta F + k_B T I_{\mathrm{QC}}$$

The upper bound of the work extracted by the demon is bounded by the QC-mutual information.

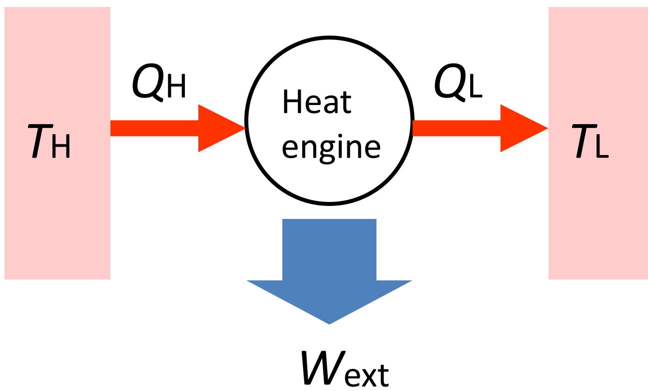
The equality can be achieved:

K. Jacobs, PRA **80**, 012322 (2009)
J. M. Horowitz & J. M. R. Parrondo, EPL **95**, 10005 (2011)
D. Abreu & U. Seifert, EPL **94**, 10001 (2011)
J. M. Horowitz & J. M. R. Parrondo, New J. Phys. **13**, 123019 (2011)
T. Sagawa & M. Ueda, PRE **85**, 021104 (2012)
M. Bauer, D. Abreu & U. Seifert, J. Phys. A: Math. Theor. **45**, 162001 (2012)

Information Heat Engine

Conventional heat engine:

Heat → Work



Heat efficiency

$$e \equiv \frac{W_{ext}}{Q_H} \le 1 - \frac{T_L}{T_H}$$

Carnot cycle

Information heat engine:

Mutual information → Work and Free energy

$$W_{ext} + \Delta F \le k_B T I_{QC}$$

Szilard engine

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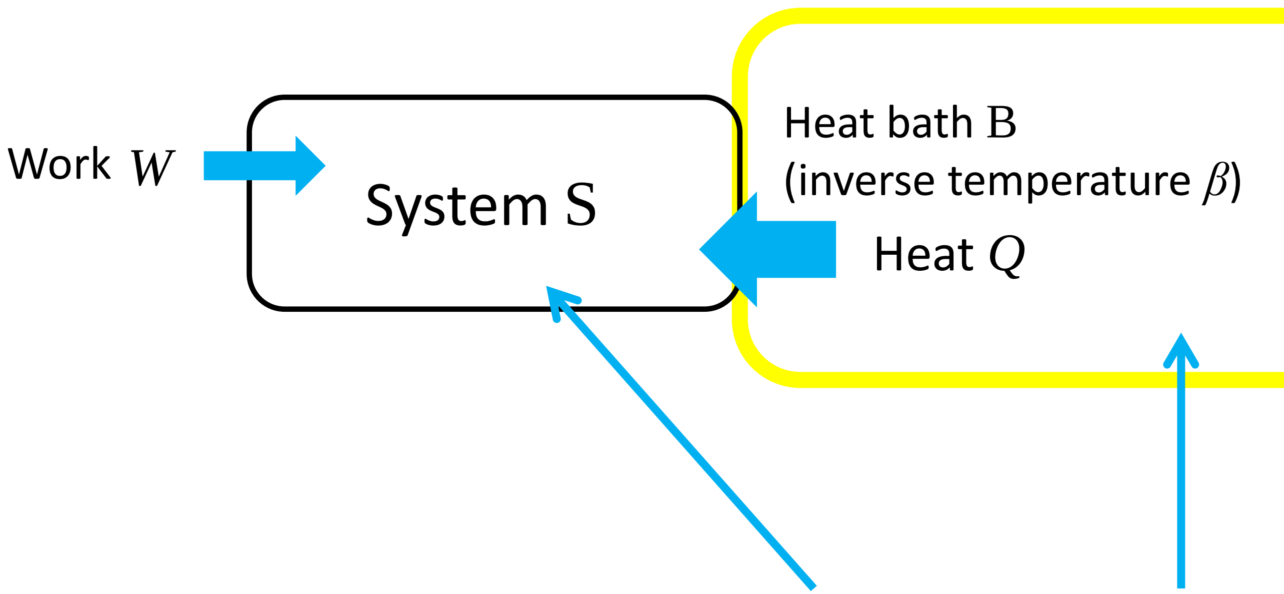
Entropy Production in Nonequilibrium Dynamics



Entropy production in the total system:

$$\Delta S_{\mathrm{SB}} = \underline{\Delta S_{\mathrm{S}}} - \beta Q$$

Change in the von Neumann entropy of S

If the initial and the final states are canonical distributions: $\Delta S_{\mathrm{SB}} = \beta(W - \Delta F)$

Free-energy difference

Second Law of Thermodynamics

$$\Delta S_{\mathrm{SB}} \geq 0$$

Holds true for nonequilibrium initial and final states

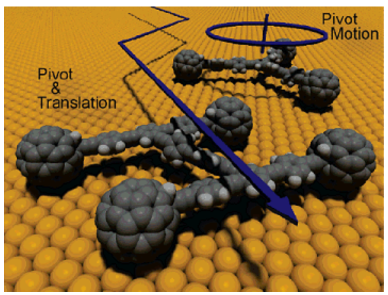
Entropy production in the whole universe is nonnegative!

A lot of “derivations” have been known
(Positivity of the relative entropy, Its monotonicity,
Fluctuation theorem & Jarzynski equality, …)

If the initial state is the canonical distribution: $W \geq \Delta F$

System and Memory

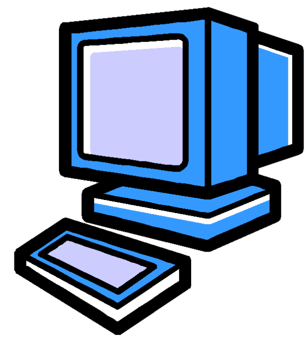
System
(Working engine)



Information

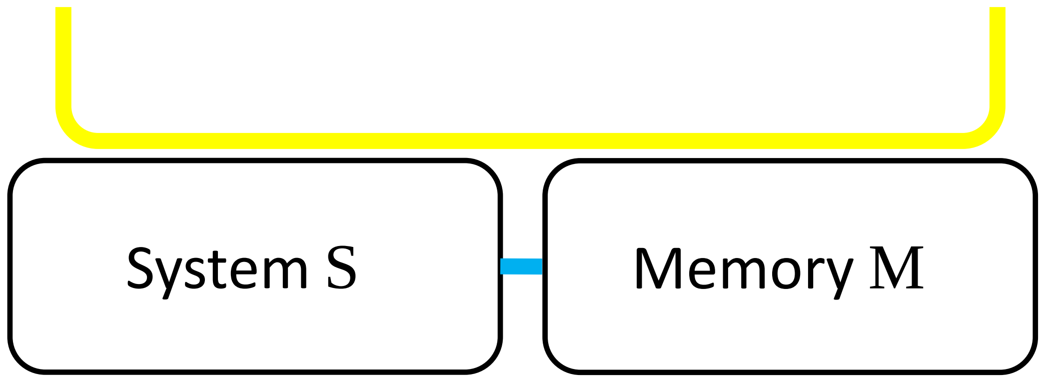
Feedback

Memory
(Controller)



Consider the role of memory explicitly

Entropy Change in System and Memory



$$S(\rho_{\mathrm{SM}}) = S(\rho_{\mathrm{S}}) + S(\rho_{\mathrm{M}}) - I_{\mathrm{S:M}}(\rho_{\mathrm{SM}})$$

$$\Rightarrow \quad \Delta S_{\mathrm{SM}} = \Delta S_{\mathrm{S}} + \Delta S_{\mathrm{M}} - \Delta I_{\mathrm{S:M}}$$

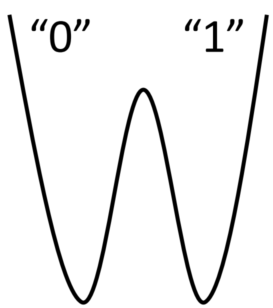
$$\Rightarrow \quad \Delta S_{\mathrm{SMB}} = \Delta S_{\mathrm{S}} + \Delta S_{\mathrm{M}} - \Delta I_{\mathrm{S:M}} - \beta Q$$

Memory Structure

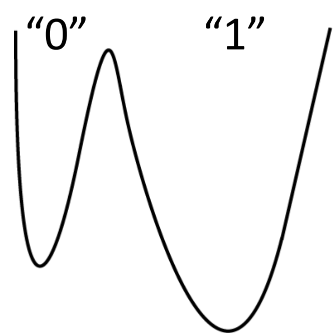
$$\mathbf{H}_{\mathrm{M}} = \bigoplus_{k} \mathbf{H}_{\mathrm{M}}^{(k)}$$

Total Hilbert space of memory is the direct sum of subspaces corresponding to outcome k

Symmetric memory



Asymmetric memory



Measurement Process

Initial state: $\rho_{\mathrm{SM}} = \rho_{\mathrm{S}} \otimes \rho_{\mathrm{M}}^{(0)}$ $\rho_{\mathrm{M}}^{(0)}$ is on $\mathbf{H}_{\mathrm{M}}^{(0)}$

CPTP map of measurement process: $\mathrm{E}^{\mathrm{meas}}$

Post-measurement state:

Assume

$$\rho'_{\mathrm{SM}} \equiv \mathrm{E}^{\mathrm{meas}}(\rho_{\mathrm{SM}}) = \sum_k p_k \rho'^{(k)}_{\mathrm{S}} \otimes \rho'^{(k)}_{\mathrm{M}}$$

p_k : probability of obtaining outcome k

$\rho'^{(k)}_{\mathrm{S}} = \frac{1}{p_k} M_k \rho_{\mathrm{S}} M_k^\dagger$:post-measurement state of S

$\rho'^{(k)}_{\mathrm{M}}$ is on $\mathbf{H}_{\mathrm{M}}^{(k)}$ **(projection postulate)**

Entropy Change during Measurement

Before measurement: $S(\rho_{\mathrm{SM}}) = S(\rho_{\mathrm{S}}) + S(\rho_{\mathrm{M}})$

After measurement:

$$S(\rho'_{\mathrm{SM}}) = S(\rho'_{\mathrm{S}}) + S(\rho'_{\mathrm{M}}) - I_{\mathrm{S:M}}(\rho'_{\mathrm{SM}})$$

$$\Delta S_{\mathrm{SM}}^{\mathrm{meas}} = \underline{\Delta S_{\mathrm{S}}^{\mathrm{meas}}} + \Delta S_{\mathrm{M}}^{\mathrm{meas}} - I_{\mathrm{S:M}}(\rho'_{\mathrm{SM}})$$

Back-action of measurement

Mutual information: $I_{\mathrm{S:M}}(\rho'_{\mathrm{SM}}) = S(\rho'_{\mathrm{S}}) - \sum_k S(\rho'^{(k)}_{\mathrm{S}})$

Second Law for Measurement

$$\Delta S_{\mathrm{SMB}}^{\mathrm{meas}} = \Delta S_{\mathrm{S}}^{\mathrm{meas}} + \Delta S_{\mathrm{M}}^{\mathrm{meas}} - I_{\mathrm{S:M}}(\rho'_{\mathrm{SM}}) - \beta Q_{\mathrm{M}}$$

$$\Delta S_{\mathrm{SMB}}^{\mathrm{meas}} \geq 0$$

Assume:
heat is absorbed only by memory during measurement

$$\Leftrightarrow \quad \Delta S_{\mathrm{M}}^{\mathrm{meas}} - \beta Q_{\mathrm{M}} \geq \Delta S_{\mathrm{S}}^{\mathrm{meas}} + I_{\mathrm{S:M}}(\rho'_{\mathrm{SM}})$$

$$\Leftrightarrow \quad \Delta S_{\mathrm{M}}^{\mathrm{meas}} - \beta Q_{\mathrm{M}} \geq \Delta S_{\mathrm{S}}^{\mathrm{meas}} + \underline{S(\rho'_{\mathrm{S}}) - \sum_k p_k S(\rho'^{(k)}_{\mathrm{S}})}$$

QC-mutual information

$$\Leftrightarrow \quad \Delta S_{\mathrm{M}}^{\mathrm{meas}} - \beta Q_{\mathrm{M}} \geq I_{\mathrm{QC}}$$

Energy Cost for Measurement

$$\Delta S_{\mathrm{M}}^{\mathrm{meas}} - \beta Q_{\mathrm{M}} \geq I_{\mathrm{QC}}$$

⇔ $$W_{\mathrm{M}} \geq \underbrace{\Delta E_{\mathrm{M}} - k_{\mathrm{B}} T \Delta S_{\mathrm{M}}^{\mathrm{meas}}} + \underline{k_{\mathrm{B}} T I_{\mathrm{QC}}}$$

Same bound as that without measurement

Additional energy cost to obtain information

Information is not free!

$\Delta F = 0$ (symmetric memory) & $H = I_{\mathrm{QC}}$ (classical and error-free measurement)

⇒ $$W_{\mathrm{M}} \geq 0$$

TS and M. Ueda, PRL **102**, 250602 (2009); **106**, 189901(E) (2011).

Feedback Process

Assume

CPTP map of feedback process: $\mathrm{E}^{\mathrm{fb}} = \sum_{\mathrm{k}} \mathrm{E}_{\mathrm{S}}^{\mathrm{fb}\,(k)} \otimes \mathrm{P}_{\mathrm{M}}^{(k)}$

Projection super-operator onto $\mathbf{H}_{\mathrm{M}}^{(k)}$

Use only classical outcome for feedback

Post-feedback state:

$$\rho''_{\mathrm{SM}} \equiv \mathrm{E}^{\mathrm{fb}}(\rho'_{\mathrm{SM}}) = \sum_{k} p_k \rho''^{(k)}_{\mathrm{S}} \otimes \rho'^{(k)}_{\mathrm{M}}$$

Unchanged

$$\rho''^{(k)}_{\mathrm{S}} = \mathrm{E}_{\mathrm{S}}^{\mathrm{fb}\,(k)}[\rho'^{(k)}_{\mathrm{S}}]$$

Entropy Change during Feedback

Before feedback: $S(\rho'_{\mathrm{SM}}) = S(\rho'_{\mathrm{S}}) + S(\rho'_{\mathrm{M}}) - I_{\mathrm{S:M}}(\rho'_{\mathrm{SM}})$

After feedback: $S(\rho''_{\mathrm{SM}}) = S(\rho''_{\mathrm{S}}) + \underline{S(\rho'_{\mathrm{M}})} - I_{\mathrm{S:M}}(\rho''_{\mathrm{SM}})$

Unchanged

$$\Delta S_{\mathrm{SM}}^{\mathrm{fb}} = \Delta S_{\mathrm{S}}^{\mathrm{fb}} + I_{\mathrm{S:M}}(\rho'_{\mathrm{SM}}) - I_{\mathrm{S:M}}(\rho''_{\mathrm{SM}})$$

$$I_{\mathrm{S:M}}(\rho'_{\mathrm{SM}}) = S(\rho'_{\mathrm{S}}) - \sum_k S(\rho'^{(k)}_{\mathrm{S}})$$

Second law for Feedback

$$\Delta S_{\mathrm{SMB}}^{\mathrm{fb}} = \Delta S_{\mathrm{S}}^{\mathrm{fb}} + I_{\mathrm{S:M}}(\rho'_{\mathrm{SM}}) - I_{\mathrm{S:M}}(\rho''_{\mathrm{SM}}) - \beta Q_{\mathrm{S}}$$

$$\Delta S_{\mathrm{SMB}}^{\mathrm{fb}} \geq 0 \iff \Delta S_{\mathrm{S}}^{\mathrm{fb}} - \beta Q_{\mathrm{S}} \geq -I_{\mathrm{S:M}}(\rho'_{\mathrm{SM}}) + I_{\mathrm{S:M}}(\rho''_{\mathrm{SM}})$$

Entropy decrease by feedback

$$\Rightarrow \quad \Delta S_{\mathrm{S}}^{\mathrm{fb}} - \beta Q_{\mathrm{S}} \geq -I_{\mathrm{S:M}}(\rho'_{\mathrm{SM}})$$

Entire Entropy Change in Engine

By adding $\Delta S_{\mathrm{S}}^{\mathrm{meas}}$ to the both-hand sides of $\Delta S_{\mathrm{S}}^{\mathrm{fb}} - \beta Q_{\mathrm{S}} \geq -I_{\mathrm{S:M}}(\rho'_{\mathrm{SM}})$

During measurement and feedback,

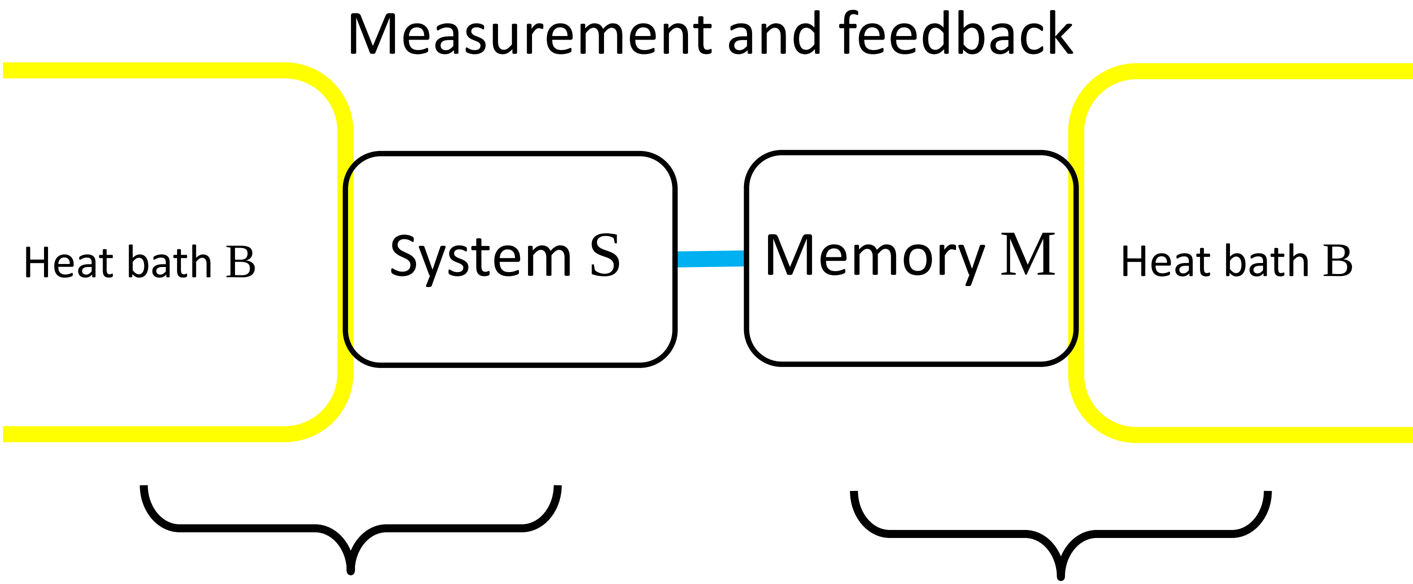
$$\Delta S_{\mathrm{S}} - \beta Q_{\mathrm{S}} \geq \Delta S_{\mathrm{S}}^{\mathrm{meas}} - I_{\mathrm{S:M}}(\rho'_{\mathrm{SM}})$$

QC-mutual information

$$\Leftrightarrow \quad \Delta S_{\mathrm{S}} - \beta Q_{\mathrm{S}} \geq -I_{\mathrm{QC}}$$

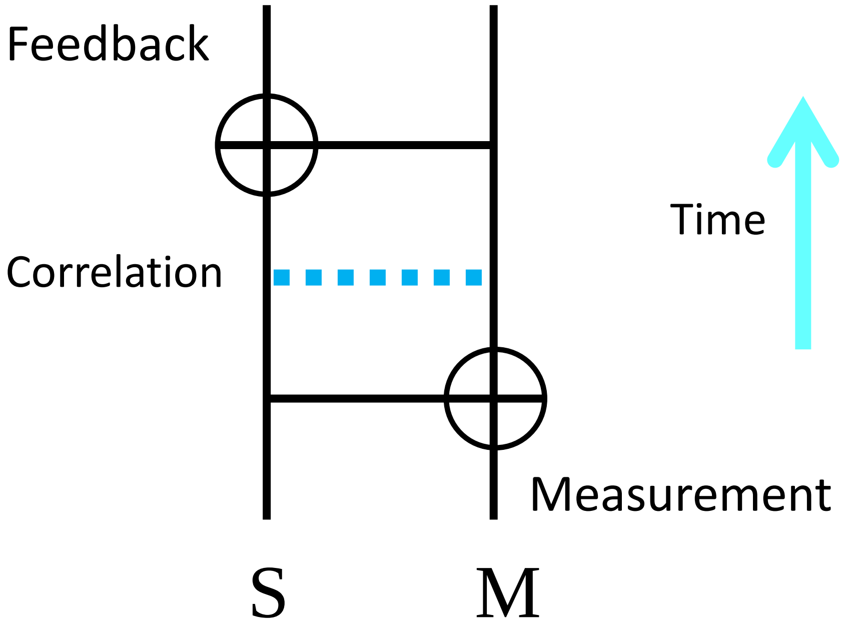
$$\Rightarrow \quad W_{\mathrm{ext}} \leq -\Delta F_{\mathrm{S}} + k_{\mathrm{B}} T I_{\mathrm{QC}}$$

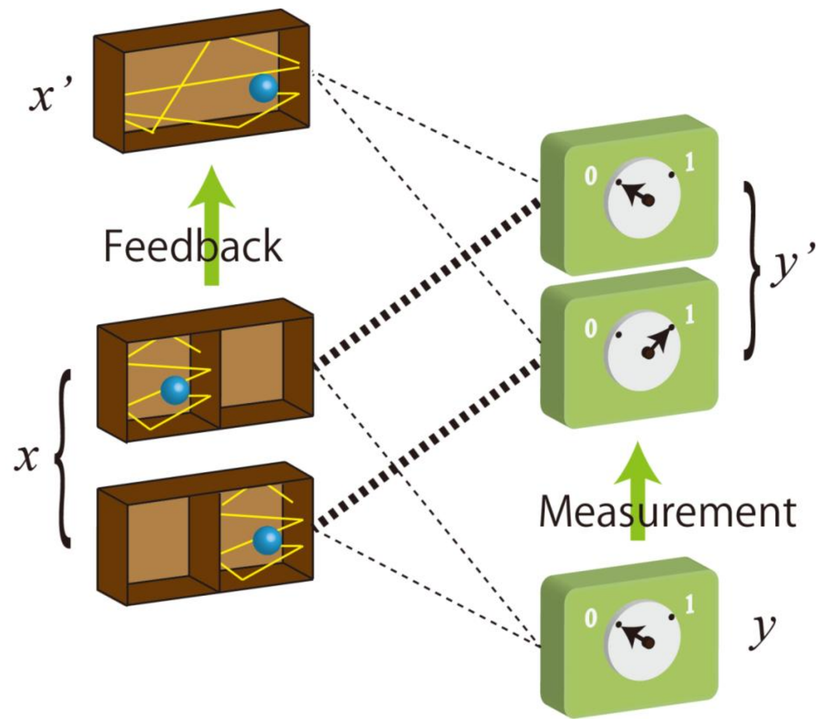
Generalized Second Law: Summary



$$\Delta S_{\mathrm{S}} - \beta Q_{\mathrm{S}} \geq -I_{\mathrm{QC}} \qquad \Delta S_{\mathrm{M}} - \beta Q_{\mathrm{M}} \geq +I_{\mathrm{QC}}$$

“Duality” between Measurement and Feedback





Time-reversal transformation

Swap system and memory

Measurement becomes feedback (and vice versa)

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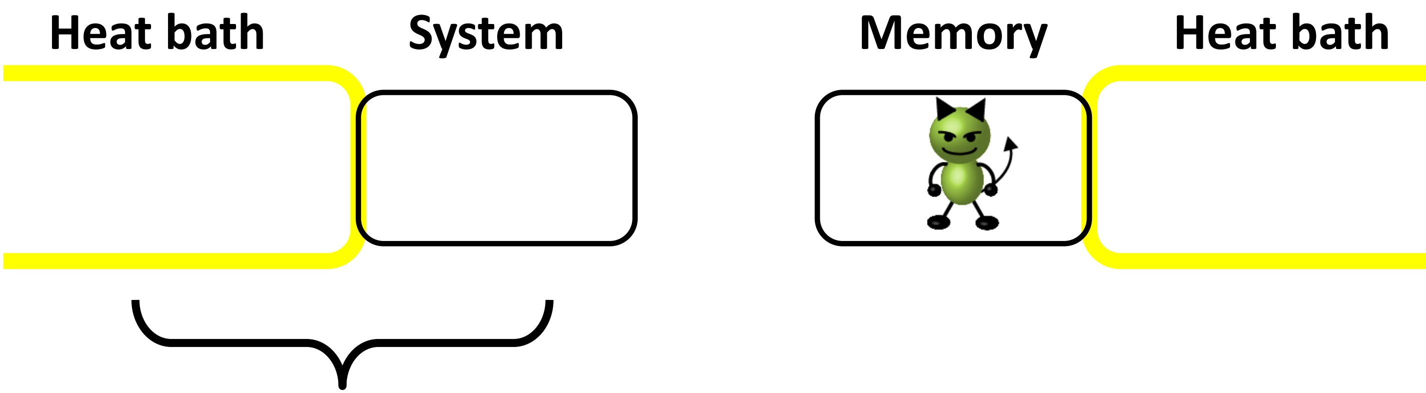
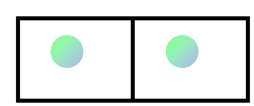
Problem



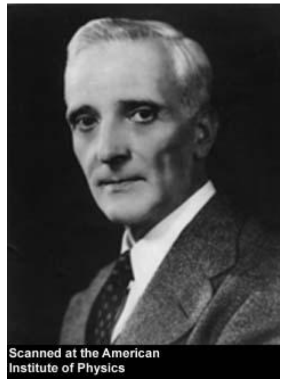
What compensates for the entropy decrease here?

For Szilard engine, $\Delta S_{\mathrm{S}} - \beta Q_{\mathrm{S}} = -\ln 2$



Conventional Arguments

Measurement process!



Brillouin

Erasure process!
(From Landauer principle)

Bennett
&
Landauer

Widely accepted since 1980’s…

Total Entropy Production

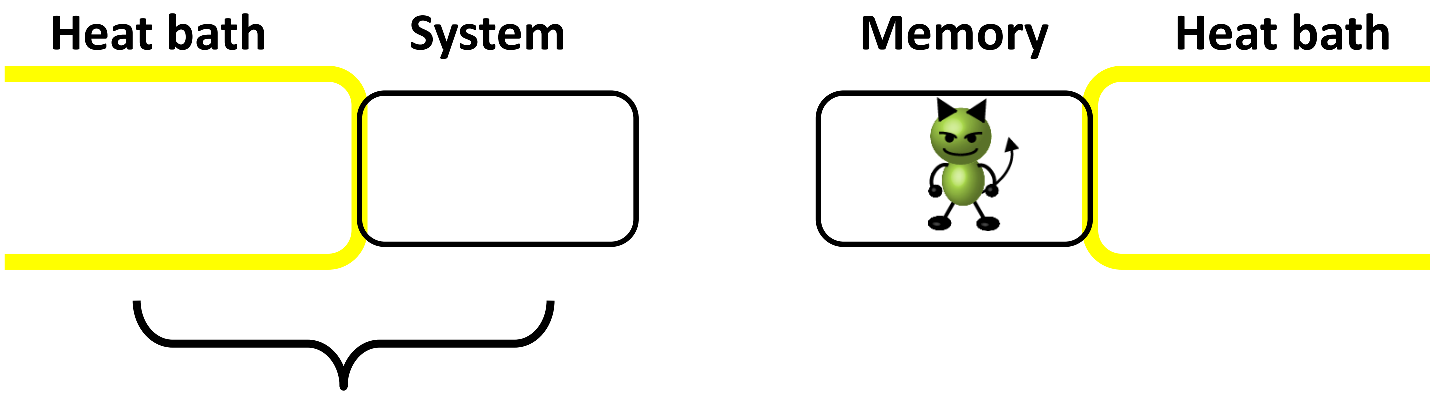
Generalized second laws have been derived from the second law for the total system:

$$\Delta S_{\mathrm{SMB}} \geq 0$$

$$\Leftrightarrow \quad \Delta S_{\mathrm{S}} + \Delta S_{\mathrm{M}} - \Delta I_{\mathrm{S:M}} - \beta Q \geq 0$$

If the quantum mutual information is taken into account, the total entropy production is **always nonnegative** for each process of measurement or feedback.

Revisit the Problem



What compensates for the entropy decrease here?

The quantum mutual information compensates for it.

For Szilard engine, $\Delta S_{\mathrm{S}} - \beta Q_{\mathrm{S}} = -\ln 2$

$$\Delta S_{\mathrm{SMB}} = \Delta S_{\mathrm{S}} - \beta Q_{\mathrm{S}} + I = -\ln 2 + \ln 2 = 0$$

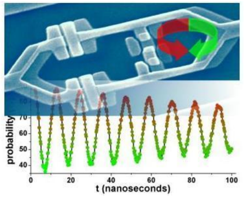
Resolution of the "Paradox"

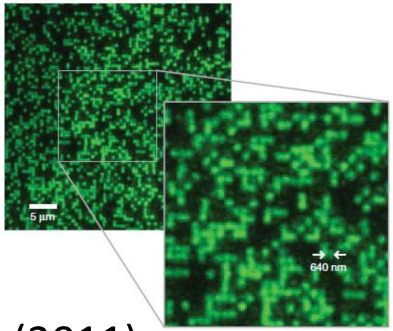
- Maxwell's demon is consistent with the second law for measurement and feedback processes **individually**
  - The quantum mutual information is the key

- We don't need the Landauer principle to understanding the consistency

Summary

- ✓ Unified theory of quantum-information thermodynamics
- ✓ Minimal energy cost for quantum information processing
- ✓ Paradox of Maxwell's demon





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Review:

J. M. R. Parrondo, J. M. Horowitz, & T. Sagawa, *Nature Physics*, Coming soon!

Thank you for your attentions!