Beyond the “entropic boundary law” with entanglement renormalization
Tensor Network Methods
(DMRG, PEPS, TERG, MERA)

Potentially offer general formalism to efficiently describe many-body wave functions

- ground states of large systems in D=1,2 (maybe 3) spatial dimensions
- strong or weak interactions, frustrated interactions etc
- different particle statistics (e.g. spins/bosons, fermions, or even anyons),

  Only limited by the amount of entanglement in the state!

As Numerical Methods:

- Given Hamiltonian $H$ what are properties $x,y,z$ of the ground state?

Conceptual Aspects:

- Framework for describing many-body systems - entanglement structure!
Outline

• **Entanglement and tensor network methods**
  • Scaling of entanglement entropy in ground states
  • Scaling of entanglement entropy in tensor network ansatz
    - physical geometry vs holographic geometry
  • Comparison of entropy scaling:
    - ground states vs tensor network ansatz

• **Introduction of branching MERA**
  • Decoupling a many-body theory
  • Holographic trees
  • Scaling of entropy in the branching MERA
  • Example: $S_L \propto \log L$ entropy scaling in 2D fermions
  • Example: $S_L \propto (\log L)^2$ entropy scaling in 1D fermions
Entanglement entropy scaling in 1D systems

1D Gapped

• Boundary law: \( S_L = \text{const.} \)

• All sites of the block contribute to entanglement entropy!

• Logarithmic Correction: \( S_L \propto \log(L) \)

1D Critical

• Entanglement is localised near boundary

\( L \)

as opposed to bulk: \( S_L = L \)
Entanglement entropy scaling in 2D systems

**2D Gapped**
- Boundary law:
  \[ S_L = L \]

2D Critical
- Boundary law:
  \[ S_L = L \]
- Logarithmic violation:
  \[ S_L = L \log L \]
Scaling of entanglement entropy for free fermions

1D
\[ S_L \]
\begin{array}{ccc}
\text{Gap.} & \text{Crit.} \\
\text{const.} & \log(L) \\
\end{array}

2D
\[ S_L \]
\begin{array}{ccc}
\text{Gap.} & \text{Crit.I} & \text{Crit.II} \\
L & L & L \log(L) \\
\end{array}

\text{Can Tensor Network methods reproduce the appropriate entanglement entropy?}

1D
Vidal, Latorre, Rico, Kitaev, PRL 2003
Srednicki, PRL 1993

2D
Wolf, PRL 2006.
Gioev, Klich, PRL 2006.
Barthel, Chung, Schollwock, PRA 2006.
Li, Ding, Yu, Haas, PRB 2006.
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Tensor Networks in Physical Geometry

• Entanglement entropy scaling?

1D: MPS

\[ S_L = \text{const.} \]

2D: PEPS

\[ S_L = L \]

• Tensor networks based upon physical geometry produce boundary law for scaling of entropy:

\[ S_L \approx L^{D-1} \]

• Entropy scales as boundary in physical geometry:

\[ S(A) \sim |\partial A| \]
Entanglement Renormalization and the MERA

MERA (multi-scale entanglement renormalization ansatz)

- Reproduce the pattern of entanglement in the ground state

Coarse-graining transformation:
Entanglement Renormalization

Holographic geometry
AdS/CFT Correspondance

1D MERA
Computation of entanglement entropy

- $D=1$, MERA

- Entanglement entropy as boundary in holographic geometry:

$$S(A) \sim |\partial \Omega_A|$$

Computation of entanglement entropy

- $D=1$, MERA

Entanglement entropy as boundary in holographic geometry:

$$S(A) \sim |\partial \Omega_A|$$

MERA for D=1 spatial dimensions

- Entanglement entropy as **boundary** in **holographic geometry**:

\[
S(A) \sim \left| \partial \Omega_A \right|
\]

- Tensor network (MERA) based on **holographic geometry** can produce violations of boundary law, logarithmic violation:

\[
S(L) \approx \log L
\]
MERA for $D=2$ spatial dimensions

$D=1$ spatial dimensions

$D=2$ spatial dimensions

Disentanglers

Isometries
MERA for D=2 spatial dimensions

- Entanglement entropy as boundary in **holographic** geometry:

\[
S(A) \sim \left| \partial \Omega_A \right|
\]

Contributions to entropy:
- const.
- $L/8$
- $L/4$
- $L/2$
- $L$

Entanglement entropy as boundary in holographic geometry:

\[
S_L \approx L\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) \approx L \frac{\log L}{\log L}
\]

Boundary law for entropy scaling!
Entanglement entropy as boundary in holographic geometry:

\[
S(A) \sim \left| \partial \Omega_A \right|
\]

1D contributions to entropy:
- \(L\): constant
- \(\log L\): 1

2D contributions to entropy:
- \(L/8\)
- \(L/4\)
- \(L/2\)
- \(L\)

\[
S_L \approx (1 + 1 + \cdots + 1) \approx \log L
\]

\[
S_L \approx L \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots \right) \approx L
\]

Vidal, quant-ph/0610099
left out of PRL 101, 110501 (2008) !!!
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Entanglement entropy and Tensor Networks

- Can Tensor Network methods reproduce the proper entanglement entropy?

**Tensor networks in physical geometry:**

1D
- **MPS:** $S_L = \text{const.}$

2D
- **PEPS:** $S_L = L$

**Gap.**

1D $S_L$
- \text{const.}
- $\log(L)$

2D $S_L$
- $L$
- $L$
- $L \log(L)$

**Crit.**

1D $S_L$
- $\log(L)$

2D $S_L$
- $L \log(L)$

- First and second order phase transitions
- Frustrated Magnets
- Interacting Fermions

- All on large (or infinite) 2D lattices!
Entanglement entropy and Tensor Networks

- Can Tensor Network methods reproduce the proper entanglement entropy?

Tensor networks in holographic geometry:

1D MERA: \( S_L = \log L \)

2D MERA: \( S_L = L \)

1D \( S_L \):
- Gap: \( \text{const.} \)
- Crit.: \( \log(L) \)

2D \( S_L \):
- Gap: \( L \)
- Crit.I: \( L \)
- Crit.II: \( L \log(L) \)

- Natural description of scale-invariant 1D systems
- First and second order phase transitions
- Frustrated Magnets
- Interacting Fermions

All on large (or infinite) 2D lattices!
Entanglement entropy and Tensor Networks

- Certain types of critical 2D phases cannot be properly addressed (large N simulations) with current tensor-network techniques
- The entanglement structure of these systems is not properly understood

- Fermi liquids
- Spin Bose Metals

$\log(L)$
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Entanglement entropy and Tensor Networks

- Entanglement entropy in MERA as boundary in \textit{holographic} geometry: \( S(A) \sim |\partial \Omega_A| \)

- By considering \textit{exotic} holographic geometries we obtain a more general class of MERA that reproduces more entanglement entropy

- Potentially a good ansatz for 2D systems with a 1D Fermi surface

- Can reproduce other violations to boundary law for entropy scaling
  - e.g. 1D quantum system with entropy: \( S_L \propto (\log L)^2 \)

- Provides a framework for understanding entanglement structure
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Low energy decoupling and holographic branching

MOTIVATION:

at low energies, sometimes sets of degrees of freedom decouple

Examples:

• 1D system: spin-charge separation

• 2D systems with 1D Fermi surface (or 1D Bose surface)
  
  ▪ free fermions
  
  ▪ Fermi liquids
  
  ▪ spin Bose metal
    e.g. Block, Sheng, Motrunich, Fisher, arXiv:1009.1179.
• simplified diagrammatic representation for the MERA
• Entanglement renormalization in the presence of decoupling

branching MERA
Entanglement entropy?

- given by boundary in exotic holographic geometry:

\[ S(A) \sim |\partial \Omega_A| \]
• Holographic branching can increase size of $\Omega_A$

branching MERA can reproduce more entanglement entropy!
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Holographic tree

\[ \mathcal{L}^{(3)} \]
\[ \mathcal{L}^{(2)} \]
\[ \mathcal{L}^{(1)} \]
\[ \mathcal{L}^{(0)} \]

\[ U^{(3)} \]
\[ U^{(2)} \]
\[ U^{(1)} \]
\[ U^{(0)} \]

Collection of independent theories

Low energies

Original theory
Holographic Trees

- regular b-ary holographic trees:

\[ b = 1 \quad b = 2 \quad b = 3 \]
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Holographic trees and entanglement entropy

$b=2$ branching MERA in $D=1$ spatial dimensions

\[ S_L \approx L \]

entropic “bulk” law (!)
Holographic tree and entanglement entropy

\[ S_L \approx \frac{L + L + \cdots + L}{\log L} \]

logarithmic violation (!!)

\[ S_L \approx L \log L \]

\( b=2 \) branching MERA in \( D=2 \) spatial dimensions
Is the (b=2) branching MERA a good ansatz for $S_L = L \log L$ phase??

Example: free fermions in 2D

$$H = \sum_{\langle x,y \rangle} (a_x^\dagger a_y + h.c.)$$

critical model type II
(1D Fermi surface)

$$S_L \approx L \log L$$

Branching MERA

Evenbly, Vidal, in preparation

Guifre Vidal
Scaling of entanglement: free fermions vs branching MERA

• Free Fermions:

<table>
<thead>
<tr>
<th>Spatial dimension</th>
<th>Γ=0</th>
<th>Γ=1</th>
<th>Γ=2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1D</td>
<td>$\log(L)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2D</td>
<td>$L$</td>
<td>$L\log(L)$</td>
<td></td>
</tr>
<tr>
<td>3D</td>
<td>$L^2$</td>
<td>$L^2$</td>
<td>$L^2\log(L)$</td>
</tr>
</tbody>
</table>

Dimension of Fermi Surface, $\Gamma$

• Regular branching MERA:

<table>
<thead>
<tr>
<th>Spatial dimension</th>
<th>$b=1$</th>
<th>$b=2$</th>
<th>$b=4$</th>
<th>$b=8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1D</td>
<td>$\log(L)$</td>
<td>$L$</td>
<td></td>
<td></td>
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<td>$L^2$</td>
<td>$L^2$</td>
<td>$L^2\log(L)$</td>
<td>$L^3$</td>
</tr>
</tbody>
</table>

Branching Parameter, $b$

• Proposed relation between dimensionality of Fermi surface and branching parameter:

$$b = 2^\Gamma$$
Corrections to the Boundary Law for Entanglement Entropy

\[ S_L = L^{D-1} f(L) \]

**Boundary Law**

**Correction**

**Arbitrary Corrections!**

\[ f(L) = 1 \]
**Boundary Law**

\[ b < 2^{D-1} \]

\[ f(L) = \log L \]
**Logarithmic Violation**

\[ b = 2^{D-1} \]

\[ f(L) = L \]
**Bulk Law**

\[ b = 2^D \]

Sub-logarithmic

\[ e.g. \quad f(L) = \log(\log L) \]

Super-logarithmic

\[ e.g. \quad f(L) = (\log L)^\alpha \]

\[ f(L) = L^\alpha \]

Example: 1D branching MERA with \( S_L \approx (\log L)^2 \)
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Branching MERA beyond Regular Holographic Trees

1D branching MERA:

\[ S_L \approx (\log L)^2 \]

- Can we find a Hamiltonian that has this ground state entropy scaling?

Yes!

\[
H = \sum_{r=-\infty}^{\infty} \left( \sum_{d=-\infty}^{\infty} \sum_{d \neq 0} \frac{\phi(d)}{d^2} (\hat{a}_{r+d}^\dagger \hat{a}_r + \hat{a}_r^\dagger \hat{a}_{r+d}) \right) - \mu \sum_r \hat{a}_r^\dagger \hat{a}_r
\]

\[
\phi(d) \approx \cos \left( \log_2 |d| \right)
\]
### Branching MERA beyond Regular Holographic Trees

#### Holographic Tree:

#### Hamiltonian:

**Gapped Ising**

\[
H = \frac{1}{2} \sum_r (\hat{a}_r^{\dagger} \hat{a}_r + \hat{a}_r \hat{a}_r^{\dagger} + \text{h.c.}) - \lambda \sum_r \hat{a}_r^{\dagger} \hat{a}_r
\]

**Critical XX**

\[
H = \frac{1}{2} \sum_r (\hat{a}_r^{\dagger} \hat{a}_{r+1} + \hat{a}_r \hat{a}_{r+1}^{\dagger})
\]

#### Dispersion:

**E(k)**

- **Left**: Lower branch of the dispersion relation with an energy gap
- **Right**: Upper branch of the dispersion relation
Branching MERA beyond Regular Holographic Trees

Holographic Tree:

Hamiltonian:
\[
H = \sum_{r=-\infty}^{\infty} \left( \sum_{d=-\infty}^{\infty} \phi(d) \left( \hat{a}_r^\dagger \hat{a}_{r+d} + h.c. \right) \right) - \mu \sum_r \hat{a}_r^\dagger \hat{a}_r
\]

Dispersion:
\[
E(k) = \left| \sin \left( \frac{k}{2} \right) \cos \left( \pi \log_2 \left| \frac{\pi}{k} \right| \right) \right|
\]
Chemical Potential:

\[ \mu = 0.6 \]

\[ \mu = 0.2 \]

\[ \mu = 0 \]
Chemical Potential:

\[
\mu = 0.6 \quad \mu = 0.2 \quad \mu = 0
\]

Blocksize, \( L \), Block entropy, \( S \):

\[
a_1 (\log L)^2 + a_2 \log L + a_3
\]

\[
a_1 = 0.166 \approx 1/6 \quad a_2 = 0.835 \approx 5/6 \quad a_3 = 0.909
\]

\[
S_L \approx \frac{c}{3} \log L
\]
Phase Transition in D=1 Free Fermions

Chemical Potential:

\[ \mu < 0 \]

Crit:

\[ S_L \approx \log L \]

\[ \mu = 0 \]

Crit.II:

\[ S_L \approx (\log L)^2 \]

\[ \mu > 0 \]

Crit:

\[ S_L \approx \log L \]

Holographic Geometry:
Summary/conclusions

entanglement renormalization, when applied to a theory that decouples into a collection of independent theories at low energy, provides:

- a formalism to explicitly factorise the theory into several theories
- new notions of scale invariance, RG flow, RG fixed points,…

branching MERA:

- admits a holographic interpretation of entropy scaling
- an efficient ansatz for critical phases beyond reach of MPS/PEPS
- (further on…) basis of an algorithm to simulate highly entangled critical phases of matter