

POST-NEWTONIAN METHODS AND APPLICATIONS

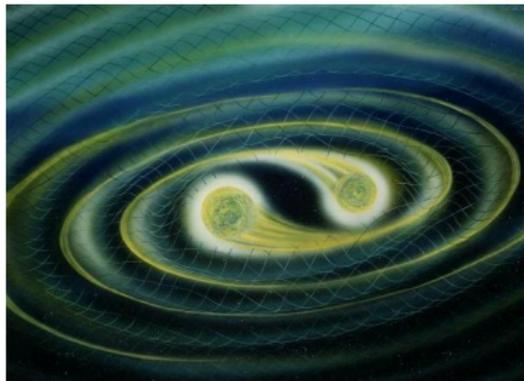
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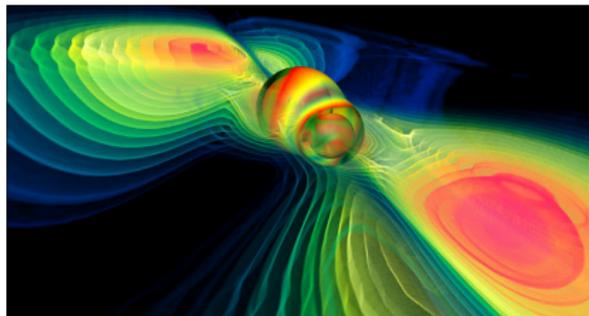
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ASTROPHYSICAL MOTIVATION

The inspiral and merger of compact binaries



Neutron stars spiral and coalesce



Black holes spiral and coalesce

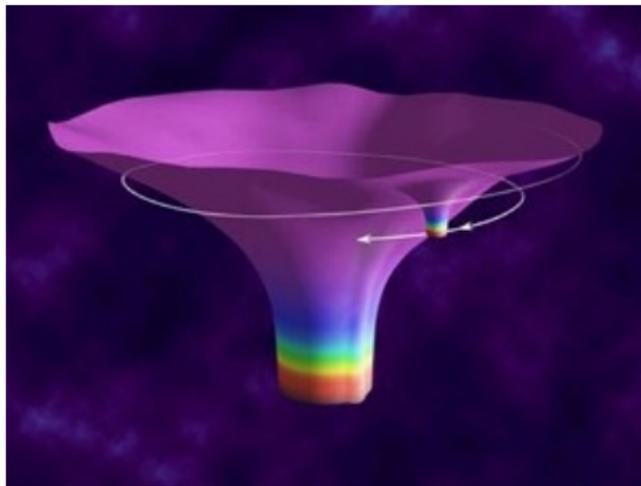
- 1 Neutron star ($M = 1.4 M_{\odot}$) events will be detected by ground-based detectors LIGO/VIRGO/GEO
- 2 Stellar size black hole ($5 M_{\odot} \lesssim M \lesssim 20 M_{\odot}$) events will also be detected by ground-based detectors
- 3 Supermassive black hole ($10^5 M_{\odot} \lesssim M \lesssim 10^8 M_{\odot}$) events will be detected by the space-based detector LISA

Supermassive black-hole coalescences as detected by LISA



When two galaxies collide their central supermassive black holes may form a bound binary system which will spiral and coalesce. LISA will be able to detect the gravitational waves emitted by such enormous events anywhere in the Universe

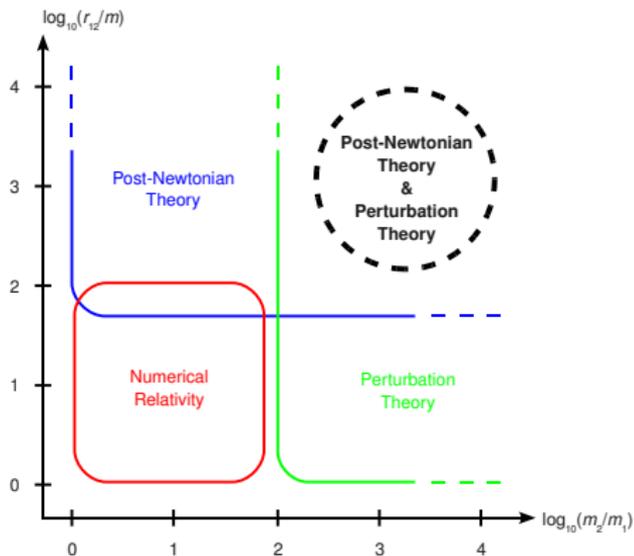
Extreme mass ratio inspirals (EMRI) for LISA



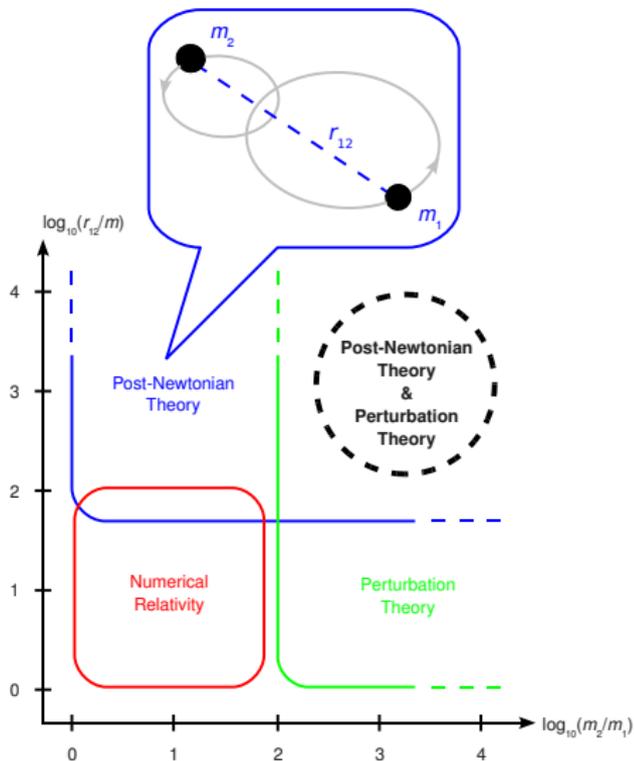
A neutron star or stellar-size black hole follows a highly relativistic orbit around a supermassive black hole. Testing general relativity in the strong field regime and verifying the nature of the central object (is it a Kerr black hole?) are important goals of LISA.

GRAVITATIONAL WAVE TEMPLATES FOR BINARY INSPIRAL

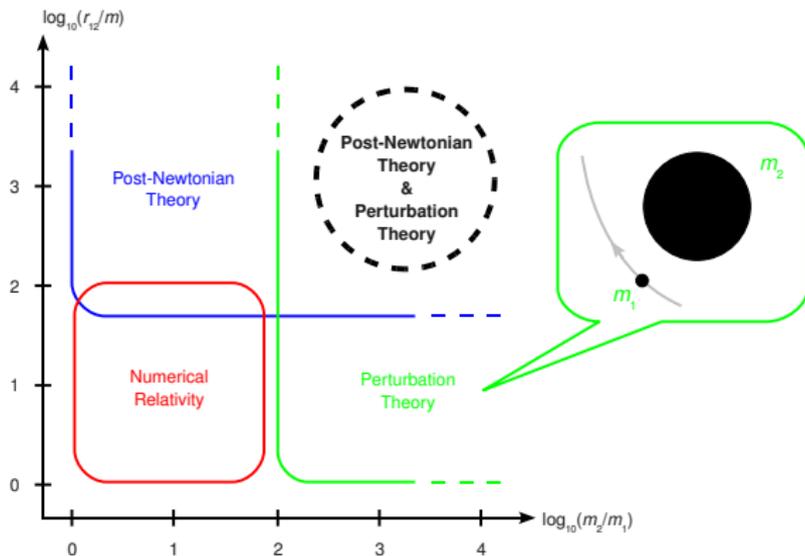
Methods to compute gravitational-wave templates



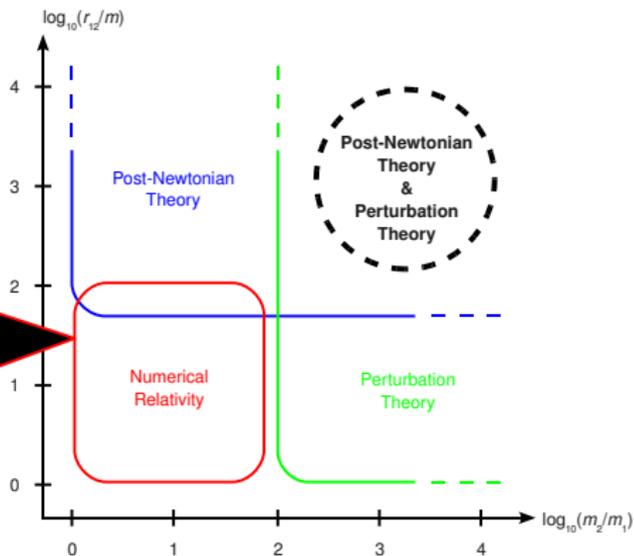
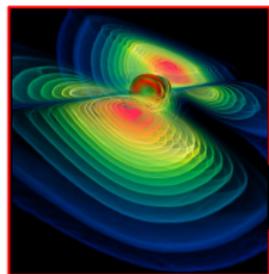
Methods to compute gravitational-wave templates



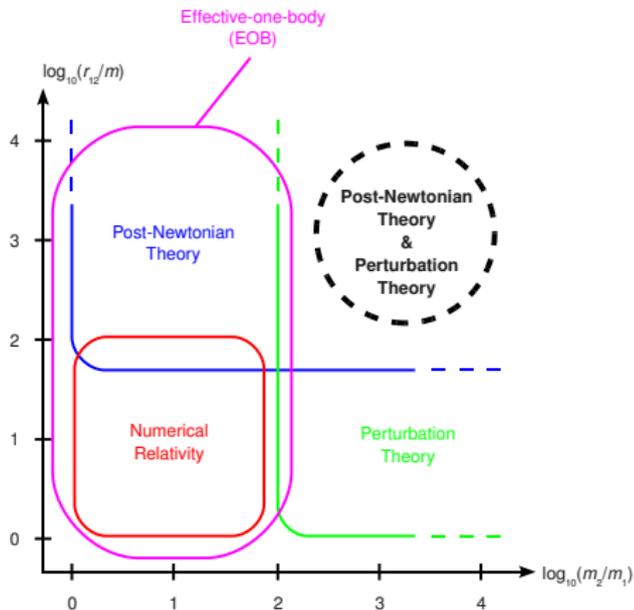
Methods to compute gravitational-wave templates



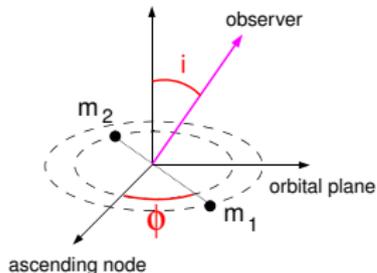
Methods to compute gravitational-wave templates



Methods to compute gravitational-wave templates



PN templates for inspiralling compact binaries



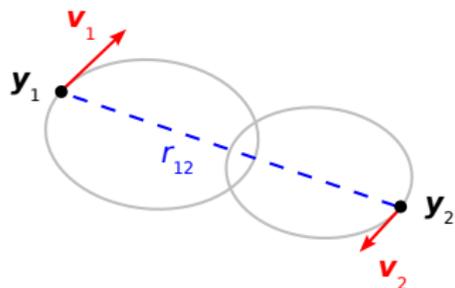
The orbital phase $\phi(t)$ should be monitored in LIGO/VIRGO detectors with precision

$$\delta\phi \sim \pi$$

$$\phi(t) = \phi_0 - \underbrace{\frac{1}{32\eta} \left(\frac{GM\omega}{c^3} \right)^{-5/3}}_{\substack{\text{result of the quadrupole formalism} \\ \text{(sufficient for the binary pulsar)}}} \left\{ \underbrace{1 + \frac{1\text{PN}}{c^2} + \frac{1.5\text{PN}}{c^3} + \dots + \frac{3\text{PN}}{c^6} + \dots}_{\text{needs to be computed with high PN precision}} \right\}$$

Detailed data analysis (using the sensitivity noise curve of LIGO/VIRGO detectors) show that the required precision is at least **2PN for detection and 3PN for parameter estimation**

Equations of motion of compact binaries



The equations of motion are written in Newtonian-like form (with $t = x^0/c$ playing the role of Newton's "absolute time")

$$\frac{d\mathbf{v}_1}{dt} = \mathbf{A}_1^N + \frac{1}{c^2} \mathbf{A}_1^{1\text{PN}} + \frac{1}{c^4} \mathbf{A}_1^{2\text{PN}} + \underbrace{\frac{1}{c^5} \mathbf{A}_1^{2.5\text{PN}}}_{\text{radiation reaction}} + \overbrace{\frac{1}{c^6} \mathbf{A}_1^{3\text{PN}}}^{\text{very difficult term to compute}} + \underbrace{\frac{1}{c^7} \mathbf{A}_1^{3.5\text{PN}}}_{\text{radiation reaction}} + \mathcal{O}\left(\frac{1}{c^8}\right)$$

1PN	[Lorentz & Droste 1917; Einstein, Infeld & Hoffmann 1938]
2PN	[Ohta <i>et al.</i> 1973; Damour & Deruelle 1981, 1982]
2.5PN	[Damour 1983; LB, Faye & Ponsot 1998]
3PN	[Jaranowski & Schäfer 1999; LB & Faye 2000, 2001; Itoh & Futamase 2003]
3.5PN	[Pati & Will 2002; Nisanke & LB 2005]

Two equivalent PN wave generation formalisms

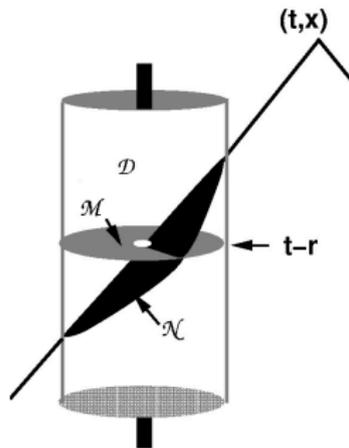
The field equations are integrated in the exterior of an extended PN source by means of a multipolar expansion

BD multipole moments [LB & Damour 1989; LB 1995, 1998]

$$M_L^{\mu\nu}(t) = \text{Finite Part}_{B=0} \int d^3x x_L \bar{\tau}^{\mu\nu}(\mathbf{x}, t)$$

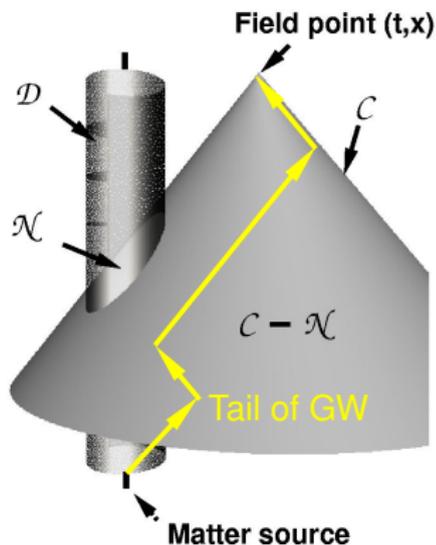
WW multipole moments [Will & Wiseman 1996]

$$W_L^{\mu\nu}(t) = \int_{\mathcal{M}} d^3x x_L \bar{\tau}^{\mu\nu}(\mathbf{x}, t)$$



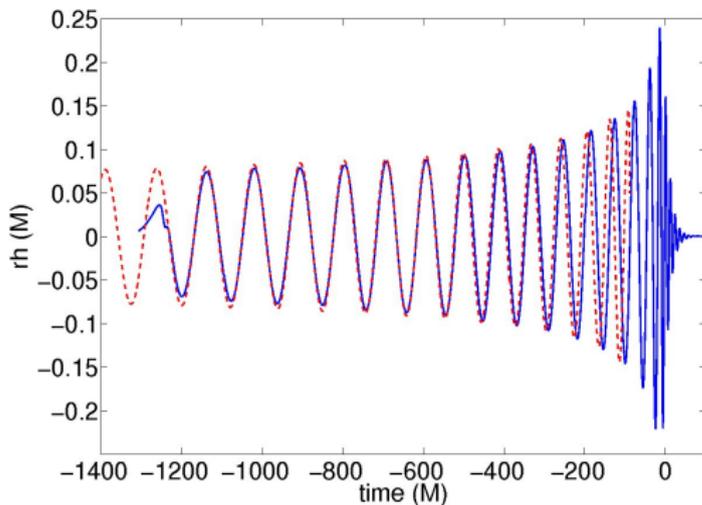
These formalisms solved the long-standing problem of divergencies in the PN expansion for general extended sources

Tails are an important part of the GW signal



- Tails are produced by backscatter of GWs on the curvature induced by the matter source's total mass M
- They appear at 1.5PN order beyond the “Newtonian” approximation given by the Einstein quadrupole formula

The compact binary inspiral waveform

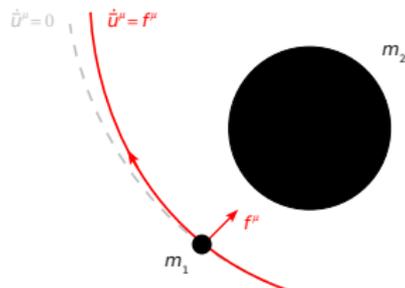


- Current precision of the PN inspiral waveform is 3.5PN [LB, Damour, Iyer, Will & Wiseman 1995; LB, Faye, Iyer & Siddhartha 2008]
- In the test-mass limit there is agreement with results from BH perturbation theory [Tagoshi & Nakamura 1994; Tagoshi & Sasaki 1994, Tanaka, Tagoshi & Sasaki 1996]
- The PN waveform is accurately matched to the numerical merger waveform

GRAVITATIONAL SELF-FORCE THEORY

General problem of the self-force

- A particle is moving on a background space-time
- Its own stress-energy tensor modifies the background gravitational field
- Because of the “back-reaction” the motion of the particle deviates from a background geodesic hence the appearance of a **self force**



The self acceleration of the particle is proportional to its mass

$$\frac{D\bar{u}^\mu}{d\tau} = f^\mu = \mathcal{O}\left(\frac{m_1}{m_2}\right)$$

The gravitational self force includes both dissipative (**radiation reaction**) and conservative effects.

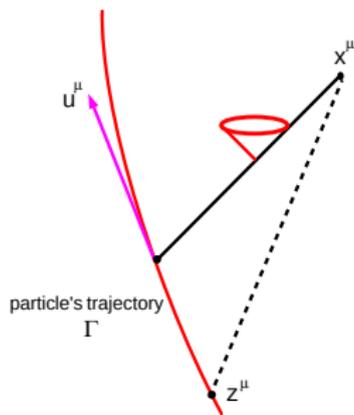
Self-force in perturbation theory

The space-time metric $g_{\mu\nu}$ is decomposed as a background metric $\bar{g}_{\mu\nu}$ plus

$h_{\mu\nu} =$ linearized perturbation of the background space-time

The field equation in an harmonic gauge reads

$$\square h^{\mu\nu} + 2R_{\rho\sigma}^{\mu\nu} h^{\rho\sigma} = -16\pi T^{\mu\nu}$$



The retarded solution is

$$h^{\mu\nu}(x) = 4m_1 \int_{\Gamma} G_{\text{ret}}^{\mu\nu}{}_{\rho\sigma}(x, z) \bar{u}^{\rho} \bar{u}^{\sigma} + \mathcal{O}(m_1^2)$$

The symmetric Green function is defined by the prescription

$$G_S = \frac{1}{2} \left[G_{\text{ret}} + G_{\text{adv}} - H \right]$$

where H is homogeneous solution of the wave equation

- G_S is **symmetric under a time reversal** hence corresponds to stationary waves at infinity and does not produce a reaction force on the particle
- It has the same **divergent behavior as G_{ret}** on the particle's worldline
- It is non zero only when x and z are related by a **space-like interval**

The radiative Green function responsible for the self force is

$$G_R(x, z) = G_{\text{ret}}(x, z) - G_S(x, z) = \frac{1}{2} \left[G_{\text{ret}} - G_{\text{adv}} + H \right]$$

- 1 The metric perturbation is decomposed as

$$h_{\mu\nu} = h_S^{\mu\nu} + h_R^{\mu\nu}$$

where the particular solution $h_S^{\mu\nu}$ (symmetric in a time reversal) **diverges** on the particle's location, but where the homogeneous solution $h_R^{\mu\nu}$ is **regular**

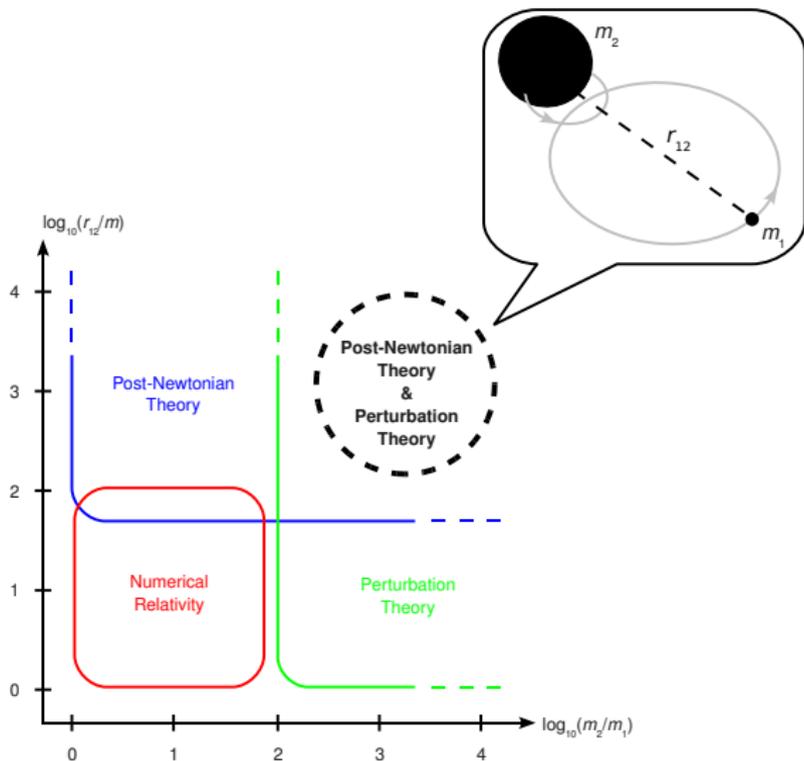
- 2 The self-force f^μ is computed from the geodesic motion with respect to

$$g_{\mu\nu}^{\text{SF}} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

- 3 The divergence on the particle's trajectory due to G_S can be **renormalized** in a redefinition of the particle's mass
- 4 The result is the so-called **MiSaTaQuWa** expression of the self-force

POST-NEWTONIAN VERSUS SELF-FORCE PREDICTIONS

Common regime of validity of SF and PN



Why and how comparing PN and SF predictions?

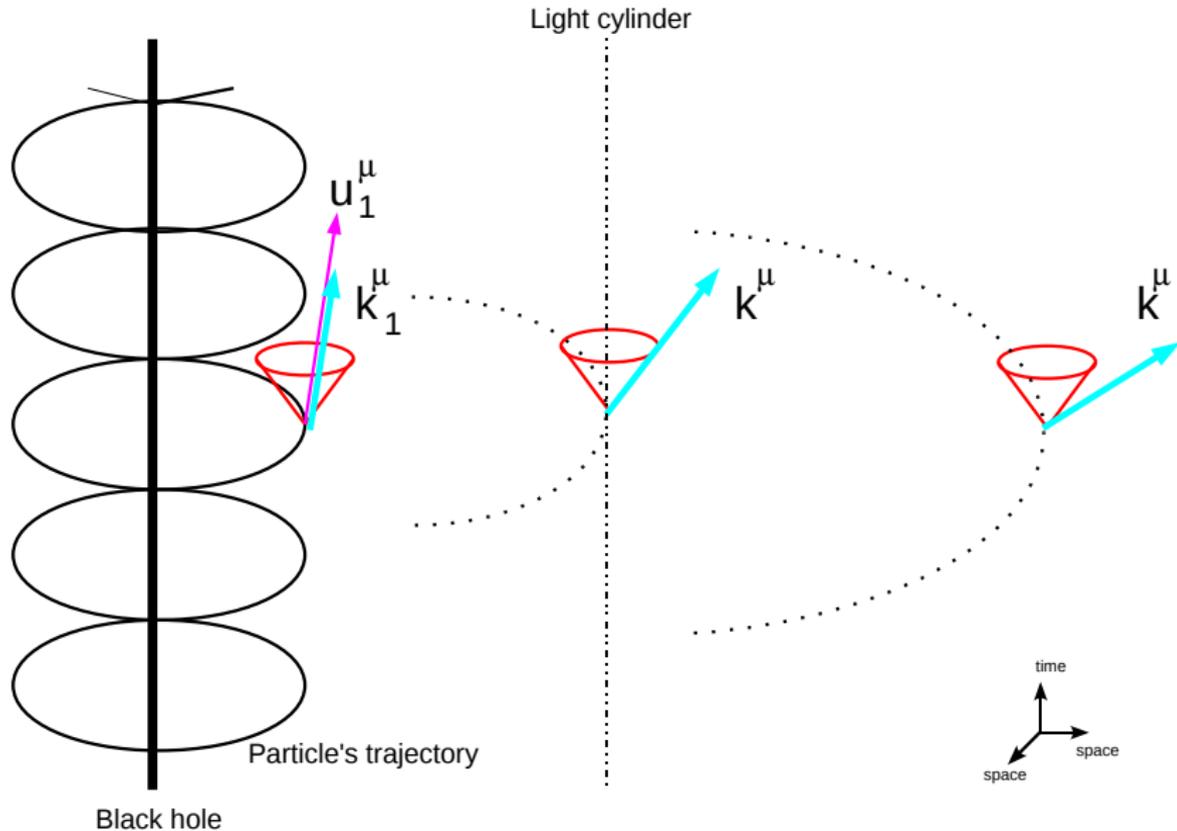
Both the PN and SF approaches use a self-field regularization for point particles followed by a renormalization. However, the prescription are very different

- 1 SF theory is based on a prescription for the Green function G_R that is at once **regular and causal**
- 2 PN theory uses **dimensional regularization** and it was shown that subtle issues appear at the 3PN order due to the appearance of **poles** $\propto (d-3)^{-1}$

How can we make a meaningful comparison?

- 1 To restrict attention to the **conservative part** of the dynamics
- 2 To find a **gauge-invariant observable** computable in both formalisms

Circular orbits admit a helical Killing vector



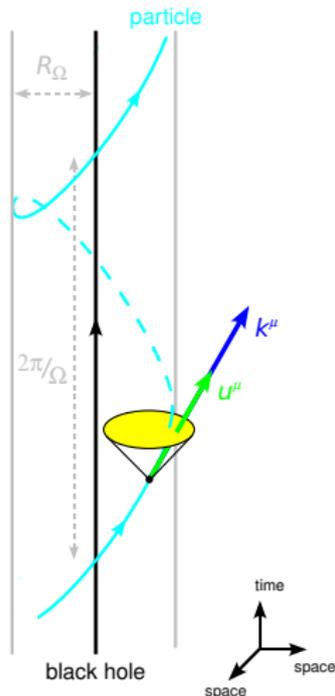
- 1 For exactly circular orbits the geometry admits a helical Killing vector with

$$k^\mu \partial_\mu = \partial_t + \Omega \partial_\varphi \quad (\text{asymptotically})$$

- 2 The four-velocity of the particle is necessarily tangent to the Killing vector hence

$$u_1^\mu = u_1^T k_1^\mu$$

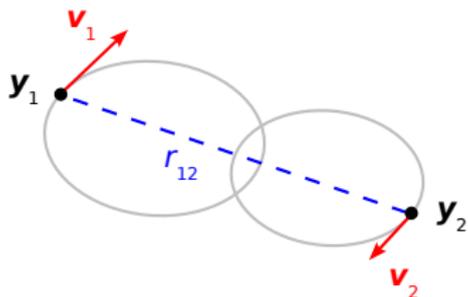
- 3 The relation $u_1^T(\Omega)$ is well-defined in both PN and SF approaches and is gauge-invariant



Post-Newtonian calculation

In a coordinate system such that $k^\mu \partial_\mu = \partial_t + \Omega \partial_\varphi$ everywhere this invariant quantity reduces to the zero component of the particle's four-velocity,

$$u_1^t = \left(- \underbrace{\text{Reg}_1 [g_{\mu\nu}]}_{\text{regularized metric}} \frac{v_1^\mu v_1^\nu}{c^2} \right)^{-1/2}$$



One needs a self-field regularization

- Hadamard regularization will yield an ambiguity at 3PN order
- **Dimensional regularization** will be free of any ambiguity at 3PN order

- The 3PN result is expressed in terms of $x = \left(\frac{GM\Omega}{c^3}\right)^{3/2}$ as

$$u^T = 1 + A_0 x + A_1 x^2 + A_2 x^3 + \underbrace{A_3 x^4}_{3\text{PN}} + o(x^4)$$

- The coefficients depend on mass ratios $\eta = m_1 m_2 / M^2$, $\Delta = (m_1 - m_2) / M$

$$A_3 = \frac{2835}{256} + \frac{2835}{256} \Delta - \left[\frac{2183}{48} - \frac{41}{64} \pi^2 \right] \eta - \left[\frac{12199}{384} - \frac{41}{64} \pi^2 \right] \Delta \eta + \text{other terms}$$

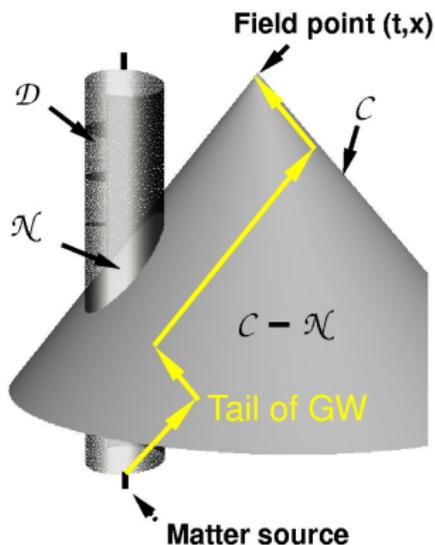
- We find that the poles $\propto \varepsilon^{-1}$ cancel out

- Logarithmic contributions start occurring at 4PN order

$$\begin{aligned}
 u^T &= 1 + A_0 x + A_1 x^2 + A_2 x^3 + A_3 x^4 \\
 &+ \underbrace{[A_4 + B_4 \ln x] x^5}_{4\text{PN}} + \underbrace{[A_5 + B_5 \ln x] x^6}_{5\text{PN}} + o(x^6)
 \end{aligned}$$

- The 4PN and 5PN logarithmic contributions B_4 and B_5 are associated with gravitational wave tails and read

$$\begin{aligned}
 B_4 &= -\frac{32}{5}\eta(1 + \Delta) + \frac{64}{15}\eta^2 \\
 B_5 &= \frac{478}{105}\eta(1 + \Delta) + \frac{1684}{21}\eta^2 + \text{other terms}
 \end{aligned}$$



$$F_{\text{radiation reaction}}^i = -\frac{2}{5c^5} \rho x^j \left[Q_{ij}^{(5)}(t) + \frac{4GM}{c^3} \int_{-\infty}^t dt' \underbrace{\ln\left(\frac{t-t'}{2r}\right)}_{\text{logarithms appear at 4PN order}} Q_{ij}^{(7)}(t') \right]$$

logarithms appear at 4PN order

High-order PN prediction for the self-force

- We re-expand in the small mass-ratio limit $q = m_1/m_2 \ll 1$ so that

$$u^T = u_{\text{Schw}}^T + \underbrace{q u_{\text{SF}}^T}_{\text{self-force}} + \underbrace{q^2 u_{\text{PSF}}^T}_{\text{post-self-force}} + \mathcal{O}(q^3)$$

- Posing $y = \left(\frac{Gm_2\Omega}{c^3}\right)^{3/2}$ we find

$$\begin{aligned} u_{\text{SF}}^T &= -y - 2y^2 - 5y^3 + \overbrace{\left(-\frac{121}{3} + \frac{41}{32}\pi^2\right)}^{3\text{PN}} y^4 \\ &+ \underbrace{\left(a_4 + \frac{64}{5} \ln y\right)}_{4\text{PN}} y^5 + \underbrace{\left(a_5 - \frac{956}{105} \ln y\right)}_{5\text{PN}} y^6 + o(y^6) \end{aligned}$$

High-order PN fit to the numerical self-force

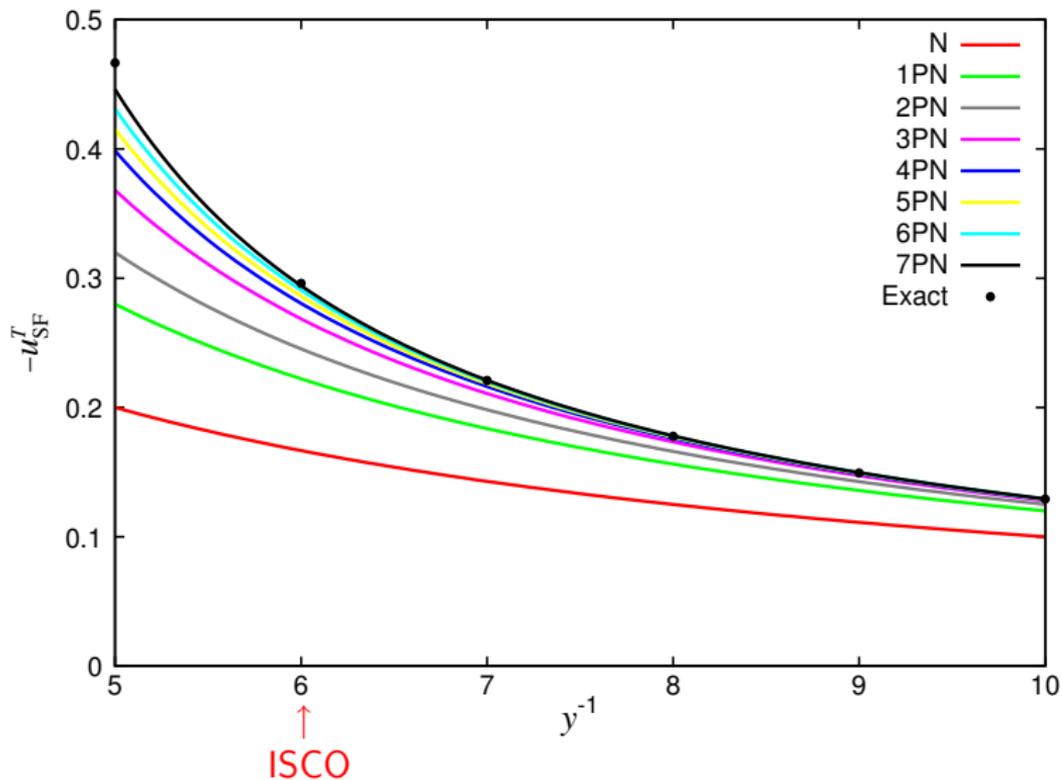
- Post-Newtonian coefficients are fitted up to 7PN order

PN coefficient	SF value
a_4	-114.34747(5)
a_5	-245.53(1)
a_6	-695(2)
b_6	+339.3(5)
a_7	-5837(16)

- The 3PN prediction agrees with the SF value with 7 significant digits

3PN value	SF fit
$a_3 = -\frac{121}{3} + \frac{41}{32}\pi^2 = -27.6879026\dots$	$-27.6879034 \pm 0.0000004$

Comparison between PN and SF predictions

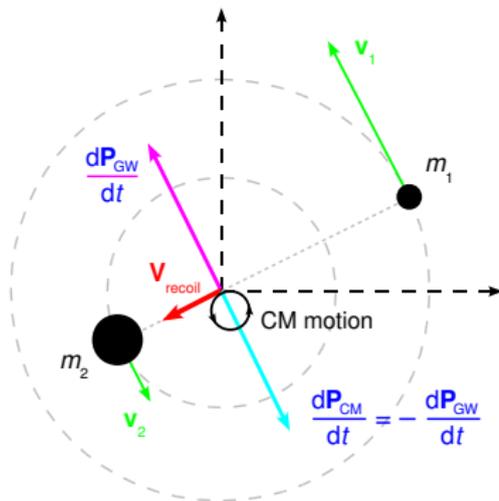


GRAVITATIONAL RECOIL OF BINARY BLACK HOLES

Gravitational recoil of BH binaries

The linear momentum ejection is in the direction of the lighter mass' velocity

[Wiseman 1993]



In the Newtonian approximation [with $f(\eta) \equiv \eta^2 \sqrt{1 - 4\eta}$]

$$\begin{aligned} V_{\text{recoil}} &= 20 \text{ km/s} \left(\frac{6M}{r} \right)^4 \frac{f(\eta)}{f_{\text{max}}} \\ &= 1500 \text{ km/s} \left(\frac{2M}{r} \right)^4 \frac{f(\eta)}{f_{\text{max}}} \quad \text{[Fitchett 1983]} \end{aligned}$$

$$\left(\frac{dP^i}{dt}\right)^{\text{GW}} = \frac{464}{105} f(\eta) x^{11/2} \left[\overbrace{1 + \left(-\frac{452}{87} - \frac{1139}{522}\eta\right)x}^{\text{1PN}} + \overbrace{\frac{309}{58}\pi x^{3/2}}^{\text{tail}} + \underbrace{\left(-\frac{71345}{22968} + \frac{36761}{2088}\eta + \frac{147101}{68904}\eta^2\right)x^2}_{\text{2PN}} \right] \hat{\lambda}^i$$

- The recoil of the center-of-mass follows from integrating

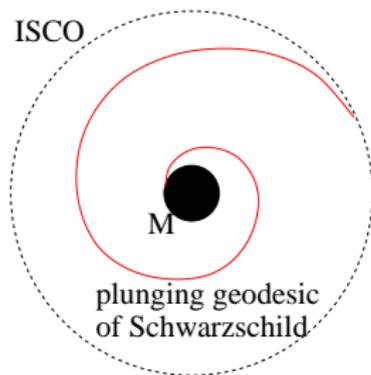
$$\frac{dP_{\text{recoil}}^i}{dt} = - \left(\frac{dP^i}{dt}\right)^{\text{GW}}$$

- We find a maximum recoil velocity of 22 km/s at the ISCO

Estimating the recoil during the plunge

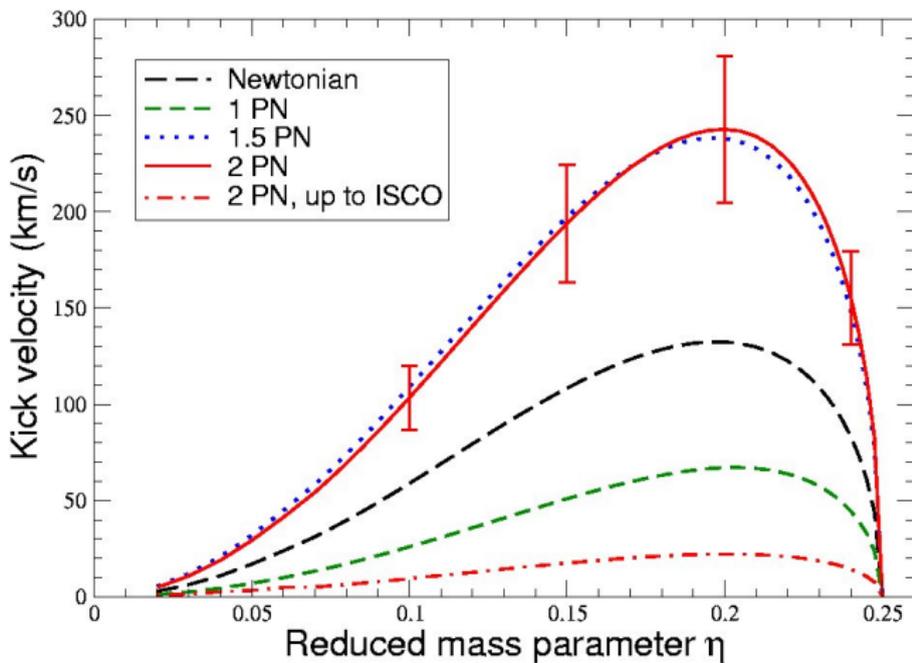
- 1 The plunge is approximated by that of a test particle of mass μ moving on a geodesic of the Schwarzschild metric of a BH of mass M
- 2 The 2PN linear momentum flux is integrated on that orbit ($y \equiv M/r$)

$$\Delta V_{\text{plunge}}^i = L \int_{\text{ISCO}}^{\text{horizon}} \left(\frac{1}{M \omega} \frac{dP^i}{dt} \right) \frac{dy}{\sqrt{E^2 - (1 - 2y)(1 + L^2 y^2)}}$$

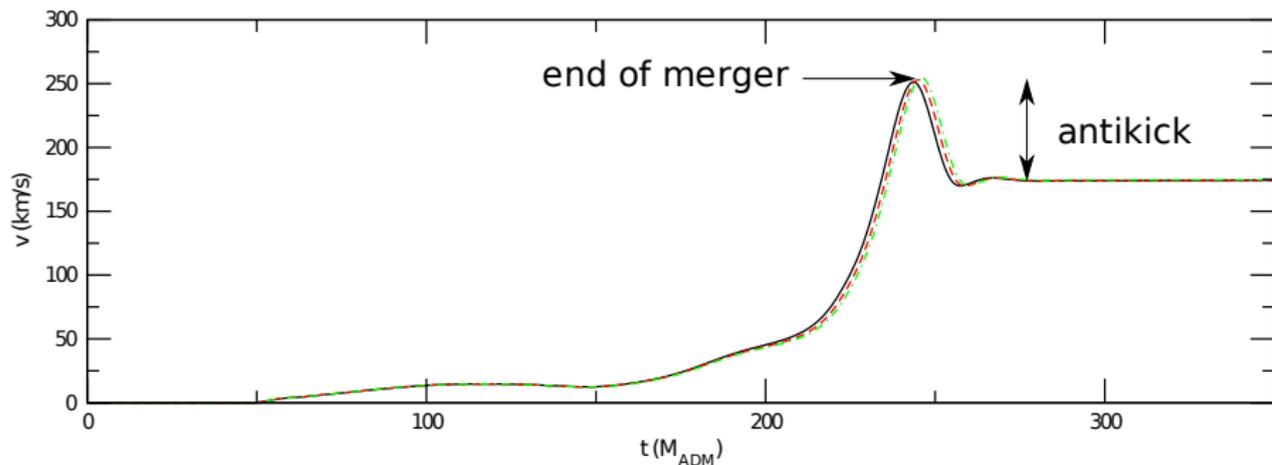


- E and L are the constant energy and angular momentum of the Schwarzschild plunging orbit
- Method similar to the EOB approach

Recoil up to merger at $r = 2M$ [LB, Qusailah & Will 2005]



Comparison with numerical relativity



For a mass ratio $\eta = 0.19$:

- Kick at the maximum is 250 km/s in good agreement with BQW
- But final kick is 160 km/s

Close-limit expansion with PN initial conditions

[Le Tiec & LB 2009]

- 1 Start with the 2PN-accurate metric of two point-masses

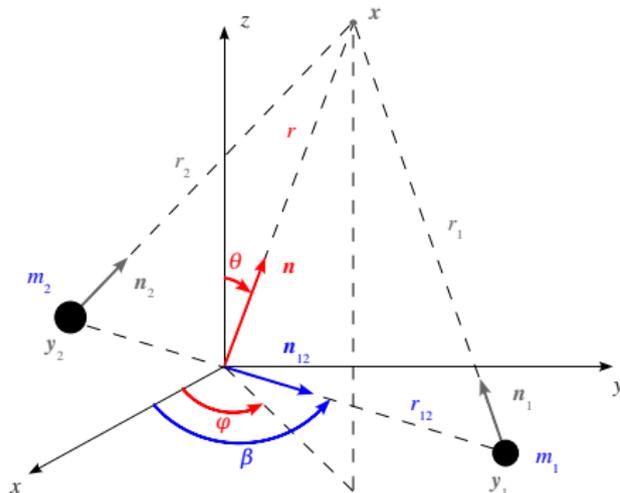
$$g_{00}^{2\text{PN}} = -1 + \frac{2Gm_1}{c^2 r_1} + \frac{2Gm_2}{c^2 r_2} + \dots$$

- 2 Expand it formally in CL form i.e.

$$\frac{r_{12}}{r} \rightarrow 0$$

- 3 Identify the perturbation from the Schwarzschild BH

$$g_{\mu\nu}^{2\text{PN}} = g_{\mu\nu}^{\text{Schw}} + h_{\mu\nu}$$



Numerical evolution of the perturbation

- 1 We recast the initial PN perturbation in Regge-Wheeler-Zerilli formalism

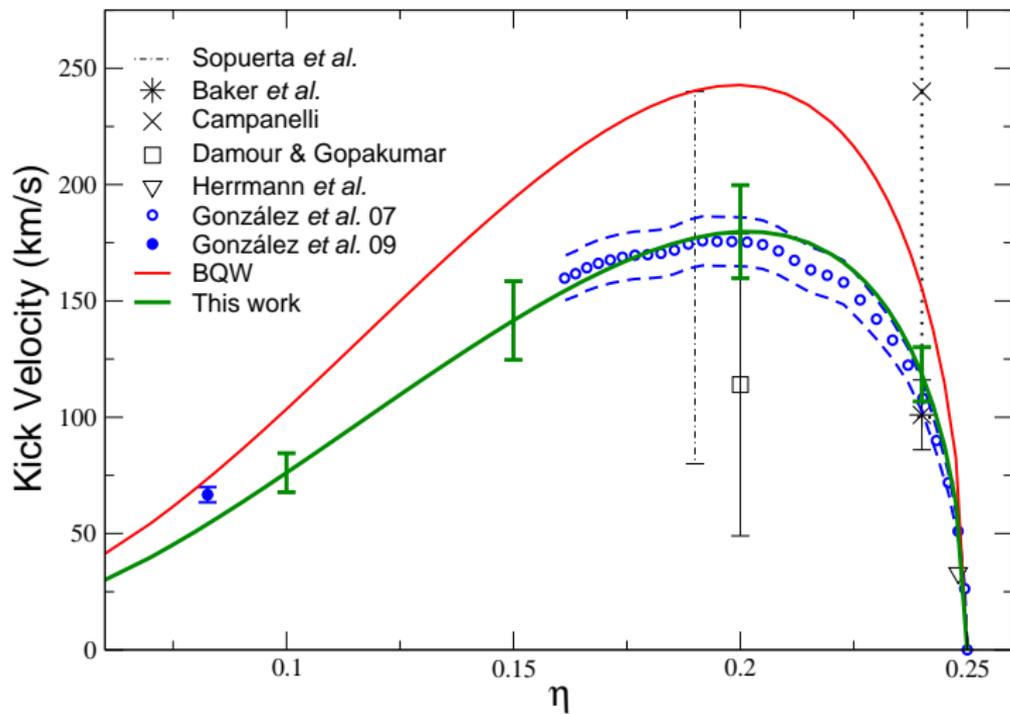
$$h_{\mu\nu} = \underbrace{h_{\mu\nu}^{(e)}}_{\text{polar modes}} + \underbrace{h_{\mu\nu}^{(o)}}_{\text{axial modes}}$$

- 2 Starting from these PN conditions the Regge-Wheeler and Zerilli master functions are evolved numerically

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V_\ell^{(e,o)} \right) \Psi_{\ell,m}^{(e,o)} = 0$$

- 3 The linear momentum flux is obtained in a standard way as

$$\frac{dP_x}{dt} + i \frac{dP_y}{dt} = -\frac{1}{8\pi} \sum_{\ell,m} \left[i a_{\ell,m} \dot{\Psi}_{\ell,m}^{(e)} \dot{\Psi}_{\ell,m+1}^{(o)} + b_{\ell,m} \left(\dot{\Psi}_{\ell,m}^{(e)} \dot{\Psi}_{\ell+1,m+1}^{(e)} + \dot{\Psi}_{\ell,m}^{(o)} \dot{\Psi}_{\ell+1,m+1}^{(o)} \right) \right]$$



The unreasonable effectiveness of the PN approximation¹

¹Clifford Will, adapting Wigner's "The unreasonable effectiveness of mathematics in the natural sciences"

- 1 PN theory has proved to be the appropriate tool to describe the inspiral phase of compact binaries up to the ISCO.
- 2 The 3.5PN templates should be sufficient for detection and analysis of neutron star binary inspirals in LIGO/VIRGO
- 3 For massive BH binaries the PN templates should be matched to the results of numerical relativity for the merger and ringdown phases
- 4 The PN approximation is now tested against different approaches such as the SF and performs very well. This provides a test of the self-field regularization scheme for point particles
- 5 A combination of semi-analytic approximations based on PN theory gives the correct result for the recoil (essentially generated in the strong field regime)