# Celebrate the $60^{th}$ Birthdays of Professors

Takashi Nakamura

and

Kei-ichi Maeda

From Rong-Gen Cai

# Connection between Thermodynamics and Gravitational Dynamics

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(JGRG20, Kyoto, Sept 21, 2010)

## Gravity is one of four fundamental interactions in Nature ?

However, a recent paper by E. Verlinde
" On the Origin of Gravity and the Laws of Newton"
arXiv: 1001.0785

See also a paper by T. Padmanabhan, arXiv: 0912.3165
" Equipartition of energy in the horizon degrees of freedom and the emergence of gravity"

T. Padmanabhan:
**Thermodynamical Aspects of Gravity: New insights**
**arXiv:0911.5004, Rept. Prog. Phys. 73:046901,2010.**
A.D. Sakharov: **Vacuum quantum fluctuations in curved space and the theory of gravitation**
**Sov.Phys.Dokl.12:1040-1041,1968**

From Verlinde’s Blog

## What is new in Verlinde’s paper?

1) Space is emergent; 2) gravity is an entropic force

## Logic of the paper:

1) Microscopic theory without space or laws of Newton
➔ thermodynamics ➔ entropic force ➔ inertia

2) Thermodynamics + Holographic Principle
➔ Gravity

According to Verlinde,
“that gravity is an entropic force is more than just saying that it has something to do with thermodynamics”

RAPID COMMUNICATIONS

PHYSICAL REVIEW D **81**, 061501(R) (2010)

# Friedmann equations from entropic force

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In this paper, by use of the holographic principle together with the equipartition law of energy and the Unruh temperature, we derive the Friedmann equations of a Friedmann-Robertson-Walker universe.

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PHYSICAL REVIEW D **81**, 084012 (2010)

# Notes on entropy force in general spherically symmetric spacetimes

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In a recent paper [arXiv:1001.0785], Verlinde has shown that the Newton gravity appears as an entropy force. In this paper we show how gravity appears as entropy force in Einstein's equation of gravitational field in a general spherically symmetric spacetime. We mainly focus on the trapping horizon of the spacetime. We find that when matter fields are absent, the change of entropy associated with the trapping horizon indeed can be identified with an entropy force. When matter fields are present, we see that heat flux of matter fields also leads to the change of entropy. Applying arguments made by Verlinde and Smolin, respectively, to the trapping horizon, we find that the entropy force is given by the surface gravity of the horizon. The cases in the untrapped region of the spacetime are also discussed.

DOI: 10.1103/PhysRevD.81.084012 PACS numbers: 04.20.Cv, 04.50.−h, 04.70.Dy

Some pieces of evidence for the deep relation between

**Thermodynamics and Gravity**

# Thermodynamics of black hole：

$$T = \frac{\hbar k}{2\pi}$$

$$S = \frac{A}{4G\hbar}$$

（S.Hawking, 1974, J.Bekenstein, 1973)

First law：dM =TdS
Questions: why?   (T. Jacobson, 1995)

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1) The usual logic:  spacetime => horizon => thermodynamics

2) The new route:  thermodynamics =>  gravity ?

**Why does GR know that a black hole has a temperature proportional to its surface gravity and an entropy proportional to its horizon area?**

**T. Jacobson is the first to ask this question.**

**T. Jacobson, Phys. Rev. Lett. 75 (1995) 1260**

*Thermodynamics of Spacetime: The Einstein Equation of State*

## From the first law of thermodynamics to Einstein equations

The Einstein equation is derived from the proportionality of entropy and horizon area together with the fundamental relation $\delta Q = TdS$ connecting heat, entropy, and temperature. The key idea is to demand that this relation hold for all the local Rindler causal horizons through each spacetime point, with $\delta Q$ and $T$ interpreted as the energy flux and Unruh temperature seen by an accelerated observer just inside the horizon. This requires that gravitational lensing by matter energy distorts the causal structure of spacetime in just such a way that the Einstein equation holds. Viewed in this way, the Einstein equation is an equation of state. This perspective suggests that it may be no more appropriate to canonically quantize the Einstein equation than it would be to quantize the wave equation for sound in air.

My talk will go on along this line to see the relationship between the first law of thermodynamics and Einstein equations

# Remainder of My Talk:

1. Rindler horizon:

------From first law to Einstein's Equation

2. Black hole horizon:

-------Einstein's equation = first law

3. Apparent horizon of FRW universe:

------From first law to Friedmann equation

4. Conclusion

# 1. Rindler Horizon: from the first law to Einstein equations

(T. Jacobson, PRL 75, 1260 (1995))

i) Rindler horizon;
ii) Unruh temperature;
iii) Jacobson's observation;

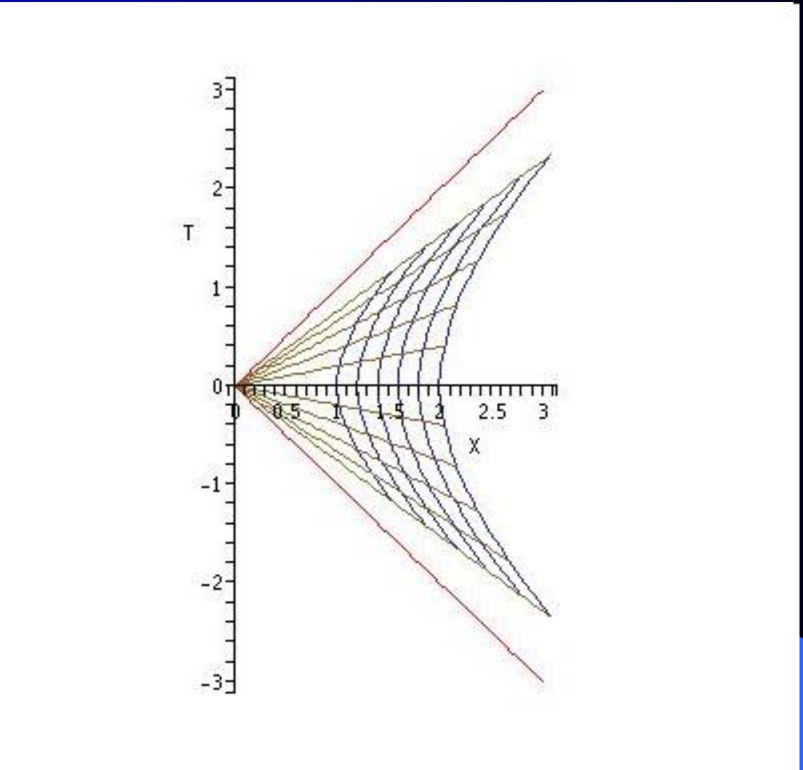
## i) Rindler horizon

The Rindler coordinate chart describes a part of flat spacetime, or say, Minkowski vacuum. This chart was introduced by Wolfgang Rindler. The Rindler coordinate system or frame describes a uniformly accelerating frame of reference in Minkowski space. In special relativity, a uniformly accelerating particle undergoes a hyperbolic motion; For each such a particle a Rindler frame can be chosen in which it is at rest.

Consider a Minkowski space in Cartesian chart

$$ds^2 = -dT^2 + dX^2 + dY^2 + dZ^2,$$

$$-\infty < T,\ X,\ Y,\ Z < \infty$$




The following region is often called Rindler wedge

$$0 < X < \infty, \quad -X < T < X$$

Defining a coordinate transformation

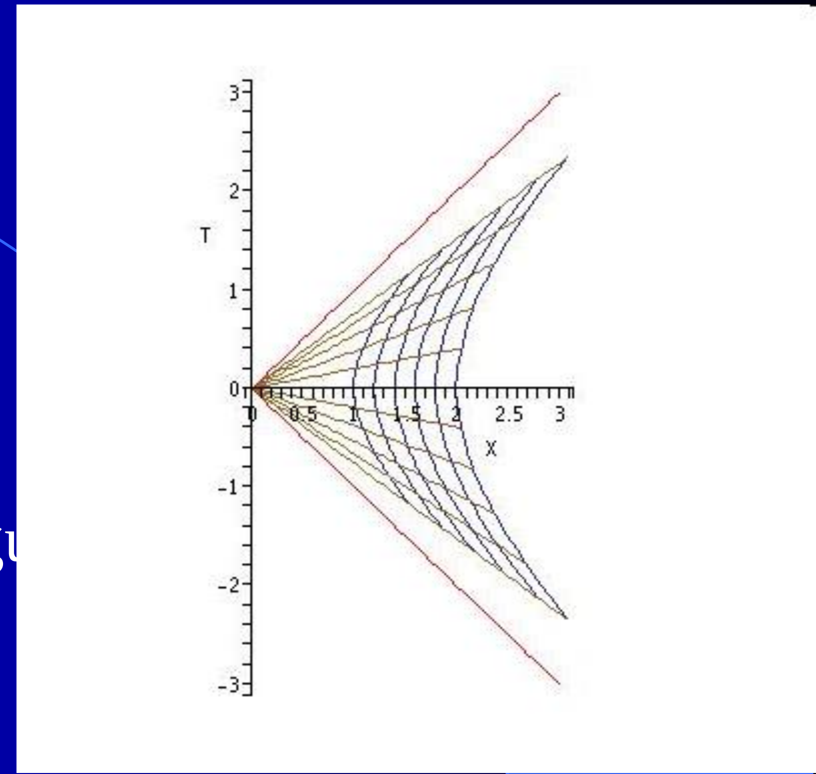
$$T = x\sinh(t), \quad X = x\cosh(t), \quad Y = y, \quad Z = z$$

Then in the Rindler chart, the Minkowski space turns to be:

$$ds^2 = -x^2dt^2 + dx^2 + dy^2 + dz^2$$

$$0 < x < \infty, \quad -\infty < t,\ y,\ z < \infty$$

$$ds^2 = -x^2 dt^2 + dx^2 + dy^2 + dz^2$$




The Rindler coordinate chart has a coordinate singularity at $x = 0$.
The acceleration of the Rindler observers diverges there. As we can see from the figu illustrating the Rindler wedge, the locus $x = 0$ in the Rindler chart corresponds to the locus

$$X^2 - T^2 = 0, \quad X > 0$$

in the Cartesian chart, which consists of two null half-planes.

$$t = \operatorname{arctanh}(T/X), \quad x = \sqrt{X^2 - T^2}, \quad y = Y, \quad z = Z$$

ii) Unruh temperature:

for a uniformly accelerating observer with acceleration *a*

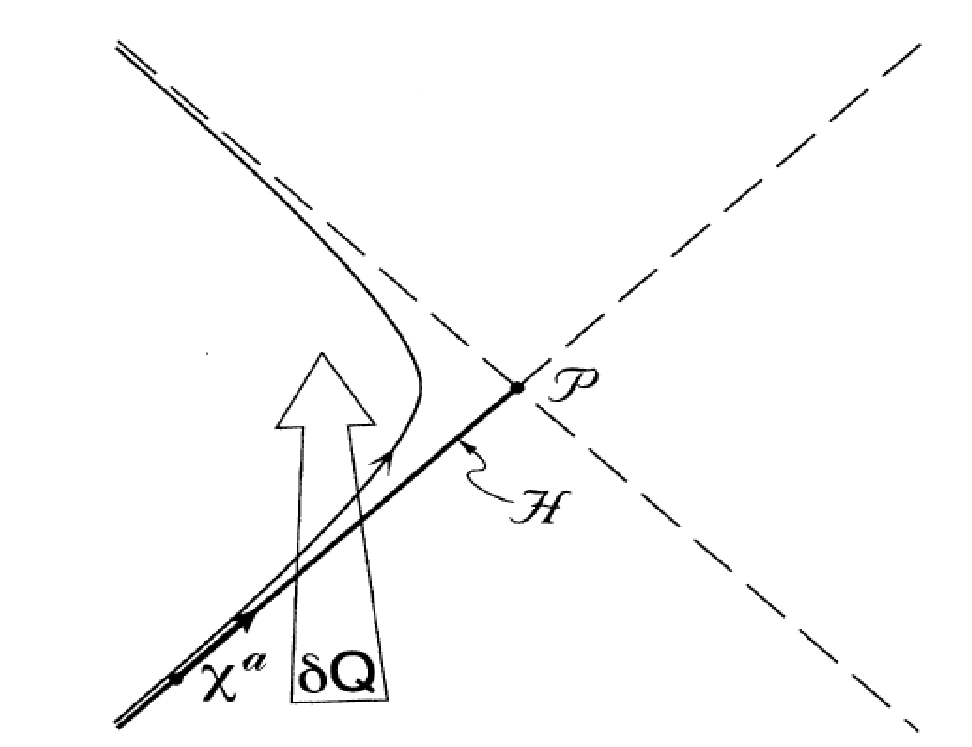
$$T = \frac{\hbar a}{2\pi c k}$$

$$a = 1/x_0$$

# iii) Jacobson's observation

> Consider any event *P* and introduce a local inertial frame (LIF) around it with Riemann normal coordinates.

➢ Transform the LIF to a local Rindler frame (LRF) by accelerating along an axis with acceleration κ.




Spacetime diagram showing the heat flux δQ across the local Rindler horizon *H* of a 2-surface element *P*. The hyperbola is a uniformly accelerated worldline, and χ^a is the approximate boost Killing vector on *H*.

Local Rindler horizon:

$$T = \hbar\kappa/2\pi$$

Heat flux across the horizon

$$\delta Q = \int_{\mathcal{H}} T_{ab}\chi^a \, d\Sigma^b .$$

$$\chi^a = -\kappa\lambda k^a \text{ and } d\Sigma^a = k^a d\lambda d\mathcal{A},$$

where k^a is the tangent vector to the horizon generators for an fine parameter lambda which vanishes at *P;* d*A* is the area element on a cross section of the horizon.

$$\delta Q = -\kappa \int_{\mathcal{H}} \lambda T_{ab} k^a k^b \, d\lambda \, d\mathcal{A} \, .$$

a) On the other hand, that the causal horizons should be associated with entropy is suggested by the observation that they hide information!

b) Now we assume that the entropy is proportional to the horizon area, so that the entropy variation associated with a piece of the horizon satisfies

$$dS = \eta \delta \mathcal{A}$$

$$\delta \mathcal{A} = \int_{\mathcal{H}} \theta \, d\lambda \, d\mathcal{A}$$

Using the Raychaudhuri equation:

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma^2 - R_{ab}k^a k^b,$$

Assuming the vanishing of theta and sigma at *P*, to the leading order,

$$\theta = -\lambda R_{ab}k^a k^b$$

$$\Rightarrow \quad \delta A = -\int_{\mathcal{H}} \lambda R_{ab}k^a k^b \, d\lambda \, d\mathcal{A} .$$

Now the Clausius relation

$$\delta Q = T dS \quad \Rightarrow \quad (2\pi/\hbar\eta)T_{ab} = R_{ab} + f g_{ab}$$

$$(2\pi/\hbar\eta)T_{ab} = R_{ab} + f g_{ab}$$

By local conservation of energy-momentum tensor, one has

$$f = -R/2 + \Lambda$$

⇒

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = \frac{2\pi}{\hbar\eta} T_{ab} .$$

⇐ Einstein equations

The constant of proportionality eta between the entropy and the area determine the Newtonian constant as $G = (4\hbar\eta)^{-1}$

# What does it tell us:

Classical General relativity ⟷ Thermodynamics of Spacetime

?

Quantum gravity Theory ⟷ Statistical Physics of Spacetime

## Further Remarks:

i) For f(R) gravity and scalar-tensor gravity, a non-equilibrium thermodynamic setup has to be employed
[ C. Eling,R. Guedens and T. Jacobson, PRL 96: 121301 (2006)
M. Akbar and RGC, PLB 635: 7 (2006)
RGC and L.M. Cao, PRD75:064008 (2007) ]

ii) Assuming the nonvanishing of the shear, even for Einstein gravity, a non-equilibrium thermodynamic setup is needed, $dS = dQ/T + dS_i$
$dS_i$ is proportional to the squared shear of the horizon,
$\eta /s=1/4 \pi$ .[C. Eling, JHEP 0811: 048 (2008); G. Chiro and S. Liberati, arXiv: 0909.4194 ]

iii) For any diffeomorphism invariant theory, given Wald's entropy formula, by the Clausius relation, it is possible to derive the gravitational field equations
[ R. Brustein and M. Hadad, PRL103: 101301 (2009); 0903.0823
M. Parikh and S. Sarkar, arXiv: 0903.1176
T. Padmanabhan, arXiv: 0903.1254]

# 2. Even horizon of black holes

## (i) Einstein general relativity

(T. Padmanabhan CQG 19, 5387 (2003)

RGC and N. Ohta, PR**D81, 084061 (2010**), arXiv:0910.2307)

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu},$$

Consider a generic static, spherically symmetric spacetime

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + b^2(r)(d\theta^2 + \sin^2\theta d\varphi^2),$$

Suppose there exists a nondegenerated horizon at r_+, temperature

$$T = \frac{1}{4\pi}f'(r)|_{r=r_+} \equiv \frac{1}{4\pi}f'(r_+),$$

# Einstein equations

$$
\begin{aligned}
G^t_t &= \frac{1}{b^2}(-1 + fb'^2 + b(f'b' + 2fb'')), \\
G^r_r &= \frac{1}{b^2}(-1 + bf'b' + fb'^2).
\end{aligned}
$$

at the black hole horizon, there f(r)=0

$$
G^t_t|_{r=r_+} = G^r_r|_{r=r_+} = \frac{1}{b^2}(-1 + bf'b')|_{r=r_+}.
$$

The t-t component of Einstein equations at the horizon

$$
-1 + bf'b' = 8\pi G b^2 P,
$$

where $P = T^r_r|_{r=r_+}$ is the radial pressure of matter at the horizon

Now we multiply displacement dr_+ on both sides of the equation

We will arrive

$$\frac{1}{2G} b f' b' dr_+ - \frac{1}{2G} dr_+ = 4\pi b^2 P dr_+.$$

$$\Rightarrow \quad Td\left(\frac{4\pi b^2}{4G}\right) - d\left(\frac{r_+}{2G}\right) = PdV,$$

where $dV = 4\pi b^2 dr_+$

This equation can be rewritten as

$$TdS - dE = PdV,$$

What means by this ?

with identifications

$$S = \frac{4\pi b^2}{4G} = \frac{A}{4G}, \quad E = \frac{r_+}{2G}.$$

## (ii) Black holes in Horava-Lifshitz gravity

(RGC and N. Ohta, arXiv: 0910.2307)

It is convenient to use ADM formalism

$$ds^2 = -N^2dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt),$$

For a spacelike hypersurface with a fixed time, its extrinsic curvature

$$K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i),$$

The action of Horava-Lifshitz theory (with detailed balance)

$$\begin{aligned}
I &= \int dt d^3x (\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_m), \\
\mathcal{L}_0 &= \sqrt{g}N\left\{\frac{2}{\kappa^2}(K_{ij}K^{ij} - \lambda K^2) + \frac{\kappa^2\mu^2(\Lambda R - 3\Lambda^2)}{8(1-3\lambda)}\right\}, \\
\mathcal{L}_1 &= \sqrt{g}N\left\{\frac{\kappa^2\mu^2(1-4\lambda)}{32(1-3\lambda)}R^2 - \frac{\kappa^2}{2\omega^4}\left(C_{ij} - \frac{\mu\omega^2}{2}R_{ij}\right)\left(C^{ij} - \frac{\mu\omega^2}{2}R^{ij}\right)\right\}.
\end{aligned}$$

The speed of light, Newtonian constant and cosmological constant

$$c = \frac{\kappa^2\mu}{4}\sqrt{\frac{\Lambda}{1-3\lambda}}, \quad G = \frac{\kappa^2 c}{32\pi}, \quad \tilde{\Lambda} = \frac{3}{2}\Lambda,$$

We consider the black hole solution in a form

$$ds^2 = -\tilde{N}^2(r)f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_k^2,$$

The action reduces to

$$\begin{aligned} I &= \frac{\kappa^2\mu^2\Lambda\Omega_k}{8(1-3\lambda)}\int dt dr \tilde{N}\Big\{-3\Lambda r^2 - 2(f-k) - 2r(f-k)' \\ &\quad + \frac{(\lambda-1)f'^2}{2\Lambda} + \frac{(2\lambda-1)(f-k)^2}{\Lambda r^2} - \frac{2\lambda(f-k)}{\Lambda r}f' + \alpha r^2\mathcal{L}_m\Big\}, \end{aligned}$$

$\alpha = 8(1-3\lambda)/\kappa^2\mu^2\Lambda.$ Consider the case with $\lambda = 1$

$$I = \frac{\kappa^2\mu^2\sqrt{-\Lambda}\Omega_k}{16}\int dt dx \tilde{N}\left\{\left(x^3 - 2x(f-k) + \frac{(f-k)^2}{x}\right)' + x^2(\frac{\alpha}{-\Lambda})\mathcal{L}_m\right\}.$$

where $x = \sqrt{-\Lambda}r$

Varying the action with respect to ¥tilde N yields

$$-\frac{\kappa^2\mu^2\sqrt{-\Lambda}\Omega_k}{16}\left(3x^2 - 2(f-k) - \frac{(f-k)^2}{x^2} - 2xf' + \frac{2(f-k)f'}{x}\right) = x^2\frac{\Omega_k}{(-\Lambda)^{3/2}}\frac{\delta(\tilde{N}\mathcal{L}_m)}{\delta\tilde{N}}.$$

At the horizon, where f(r)=0,

$$-\frac{\kappa^2\mu^2\sqrt{-\Lambda}\Omega_k}{16}\left(3x_+^2 + 2k - \frac{k^2}{x_+^2} - 2x_+f' - \frac{2kf'}{x_+}\right) = x_+^2\frac{\Omega_k}{(-\Lambda)^{3/2}}\left.\frac{\delta(\tilde{N}\mathcal{L}_m)}{\delta\tilde{N}}\right|_{x=x_+}$$

Multiply dx_+ on both sides of the equation

$$\frac{\kappa^2\mu^2\sqrt{-\Lambda}\Omega_k}{16}\left(2(x_+ + \frac{k}{x_+})f'dx_+ - (3x_+^2 + 2k - \frac{k^2}{x_+^2})dx_+\right) = x_+^2\frac{\Omega_k}{(-\Lambda)^{3/2}}Pdx_+,$$

where $P = \frac{\delta(\tilde{N}\mathcal{L}_m)}{\delta\tilde{N}}\Big|_{x=x_+}$.

Now consider the Hawking temperature of the black hole

$$T = \frac{1}{4\pi}\tilde{N}(r)\left.\frac{df}{dr}\right|_{r=r_+} = \frac{\sqrt{-\Lambda}}{4\pi}\tilde{N}(x)f'|_{x=x_+}.$$

then

$$TdS - dE = PdV,$$

$$V = \frac{\Omega_k}{3}r_+^3$$

$$S = \frac{\pi\kappa^2\mu^2\Omega_k}{4}\left(x_+^2 + 2k\ln x_+\right) + S_0,$$

$$E = \frac{\kappa^2\mu^2\sqrt{-\Lambda}\Omega_k}{16x_+}\left(x_+^2 + k\right)^2,$$

(RGC, L.M. Cao and N. Ohta, PRD**80, 024003 (2009); PLB 679, 504 (2009)**)

## a) With the "soften" broken of the "detailed balance"

$$I = \int dt d^3x (\mathcal{L}_0 + (1-\epsilon^2)\mathcal{L}_1 + \mathcal{L}_m)$$

In this case

$$I = \frac{\kappa^2\mu^2\sqrt{-\Lambda}\Omega_k}{16}\int dt dx \tilde{N}\left\{\left(x^3 - 2x(f-k) + (1-\epsilon^2)\frac{(f-k)^2}{x}\right)' + x^2(\frac{\alpha}{-\Lambda})\mathcal{L}_m\right\}.$$

→

$$-\frac{\kappa^2\mu^2\sqrt{-\Lambda}\Omega_k}{16}\left(3x_+^2 + 2k - (1-\epsilon^2)\frac{k^2}{x_+^2} - 2x_+ f' - (1-\epsilon^2)\frac{2kf'}{x_+}\right) = x_+^2\frac{\Omega_k}{(-\Lambda)^{3/2}}\frac{\delta(\tilde{N}\mathcal{L}_m)}{\delta\tilde{N}}\Big|_{x=x}$$

Once again:

$$TdS - dE = PdV,$$

$$S = \frac{\pi\kappa^2\mu^2\Omega_k}{4}\left(x_+^2 + 2k(1-\epsilon^2)\ln x_+\right) + S_0,$$
$$E = \frac{\kappa^2\mu^2\sqrt{-\Lambda}\Omega_k}{16x_+}\left(x_+^4 + 2kx_+ + (1-\epsilon^2)k^2\right).$$

## b) Including z=4 terms (RGC, Y. Liu and Y.W. Sun, JHEP **0906, 010(2009)**

$$\mathcal{L}_1 = -\sqrt{g}N\frac{\kappa^2}{8}\Big\{\frac{4}{\omega^4}C^{ij}C_{ij} - \frac{4\mu}{\omega^2}C^{ij}R_{ij} - \frac{4}{\omega^2 M}C^{ij}L_{ij} + \mu^2 G_{ij}G^{ij} + \frac{2\mu}{M}G^{ij}L_{ij} + \frac{2\mu}{M}\Lambda L + \frac{1}{M^2}L^{ij}L_{ij} - \tilde{\lambda}\Big(\frac{L^2}{M^2} - \frac{\mu L}{M}(R-6\Lambda) + \frac{\mu^2}{4}R^2\Big)\Big\}, \quad (3.20)$$

where

$$G^{ij} = R^{ij} - \frac{1}{2}g^{ij}R$$

$$L^{ij} = (1+2\beta)(g^{ij}\nabla^2 - \nabla^i\nabla^j)R + \nabla^2 G^{ij} + 2\beta R(R^{ij} - \frac{1}{4}g^{ij}R) + 2(R^{imjn} - \frac{1}{4}g^{ij}R^{mn})R_{mn},$$

$$L \equiv g^{ij}L_{ij} = \left(\frac{3}{2}+4\beta\right)\nabla^2 R + \frac{\beta}{2}R^2 + \frac{1}{2}R_{ij}R^{ij}, \quad (3.21)$$

# Equation of motion:

$$-\frac{\kappa^2\mu^2\sqrt{-\Lambda}\Omega_k}{16}\Big(x^3-2x(f-k)+\frac{(f-k)^2}{x} - 2\frac{\tilde\beta}{\tilde\mu}\big(\frac{(f-k)^3}{x^3}-\frac{(f-k)^2}{x}\big)+\frac{\tilde\beta^2}{\tilde\mu^2}\frac{(f-k)^4}{x^5}\Big)' = x^2\frac{\Omega_k}{(-\Lambda)^{3/2}}\frac{\delta(\tilde N\mathcal{L}_m)}{\delta\tilde N}$$

$$TdS - dE = PdV,$$

$$E = \frac{\kappa^2\mu^2\sqrt{-\Lambda}\Omega_k}{16}\Big(x_+^3+2kx_+ +\frac{k^2}{x_+}+2\frac{\tilde\beta}{\tilde\mu}\big(\frac{k^3}{x_+^3}+\frac{k^2}{x}\big)+\frac{\tilde\beta^2}{\tilde\mu^2}\frac{k^4}{x_+^5}\Big)$$

$$S = \frac{\pi\kappa^2\mu^2\Omega_k}{4}\Big(x_+^2+2k\ln x_+ -3\frac{\tilde\beta k^2}{\tilde\mu x_+^2}-\frac{\tilde\beta^2k^3}{\tilde\mu^2x_+^4}+4\frac{\tilde\beta k}{\tilde\mu}\ln x_+\Big)+S_0$$

S ~ A + ln A + 1/A + 1/A^2+…

# Further remarks:

a) **This holds for Lovelock gravity** **(A. Paranjape, S. Sarkar and T. Padmanabhan, PRD74:104015,2006)**

b) **This holds in stationary black holes and evolving spherically symmetric horizons** **(D. Kothawala, S. Sarkar and T. Padmanabhan, PLB 652:338-342,2007)**

c) **This holds for charged BTZ black holes** **(M. Akbar and A. Siddiqui, PLB B656:217-220,2007 )**

d) **A non-equilibrium thermodynamic setting is needed for f(R) gravity** (M. Akbar and RGC, PLB**648:243,2007**)

# 3. Apparent horizon of FRW universe

i) From the first law to Friedmann equation

ii) Fridmann equation at the apparent horizon = first law

iii) Corrected entropy-area relation and modified Friedmann equatio

iv) Hawking radiation at the apparent horizon

# i) From the First Law to Friedmann Equations

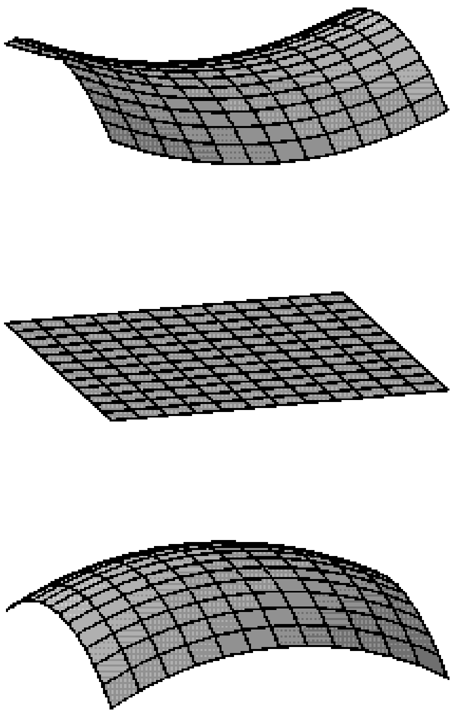
Friedmann-Robertson-Walker Universe:

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right)$$

1) $k = -1$
open

2) $k = 0$
flat

3) $k = 1$
closed



# Friedmann Equations:

$$H^2 + \frac{k}{a^2} = \frac{16\pi G}{n(n-1)}\rho.$$

$$\dot{H} - \frac{k}{a^2} = -\frac{8\pi G}{n-1}(\rho + p).$$

**for (n+1)-dimensions**

the Hubble parameter, $H \equiv \dot{a}/a.$

**Our goal :**

$$-dE = TdS.$$

$$H^2 + \frac{k}{a^2} = \frac{16\pi G}{n(n-1)}\rho.$$

$$\dot{H} - \frac{k}{a^2} = -\frac{8\pi G}{n-1}(\rho + p).$$

**Some related works:**


(1) A. Frolov and L. Kofman, JCAP 0305 (2003) 009

(2) Ulf H. Daniesson, PRD 71 (2005) 023516

(3) R. Bousso, PRD 71 (2005) 064024

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right)$$

$$ds^2 = h_{ab} dx^a dx^b + \tilde{r}^2 d\Omega_{n-1}^2,$$

where $\tilde{r} = a(t) r$ and $x^0 = t$, $x^1 = r$

## Apparent Horizon in FRW Universe :

$$h^{ab}\partial_a\tilde{r}\partial_b\tilde{r} = 0.$$

$$\tilde{r}_A = \frac{1}{\sqrt{H^2 + k/a^2}},$$

**Other horizons:**

particle horizon? Hubble horizon? Event horizon?

# Apply the first law to the apparent horizon:

$$-dE = TdS.$$

**Make two ansatzes:**

$$T = \frac{1}{2\pi R_A},$$

$$S = A/4G,$$

**The only problem is to get *dE***

Suppose that the perfect fluid is the source, then

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu},$$

The energy-supply vector is:

$$\Psi_a = T_a{}^b \partial_b \tilde{r} + W \partial_a \tilde{r},$$

The work density is:

$$W = -\frac{1}{2} T^{ab} h_{ab},$$

(S. A. Hayward, S. Mukohyama, et al. 1997-1999)

Then, the amount of energy crossing the apparent horizon within the time interval *dt*

$$-dE \equiv -A\Psi = A(\rho + p) H \tilde{r}_A dt,$$

$$-dE = TdS.$$

$$\dot{H} - \frac{k}{a^2} = -\frac{8\pi G}{n-1}(\rho + p).$$

**By using the continuity equation:**

$$\dot{\rho} + nH(\rho + p) = 0,$$

$$H^2 + \frac{k}{a^2} = \frac{16\pi G}{n(n-1)}\rho.$$

**(Cai and Kim, JHEP 0502 (2005) 050 )**

# Higher derivative theory:

## Gauss-Bonnet Gravity

$$S = \frac{1}{16\pi G}\int d^{n+1}x\sqrt{-g}(R + \alpha R_{GB}) + S_m,$$

## Gauss-Bonnet Term:

$$R_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta}$$

$$\begin{aligned} 8\pi G T_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \alpha\Bigg(\frac{1}{2}g_{\mu\nu}R_{GB} \\ &\quad -2RR_{\mu\nu} + 4R_{\mu\gamma}R^{\gamma}{}_{\nu} + 4R_{\gamma\delta}R^{\gamma}{}_{\mu}{}^{\delta}{}_{\nu} - 2R_{\mu\gamma\delta\lambda}R_{\nu}{}^{\gamma\delta\lambda}\Bigg). \end{aligned}$$

# Black Hole Solution:

$$ds^2 = -e^{\lambda(r)}dt^2 + e^{\nu(r)}dr^2 + r^2 d\Omega_{n-1}^2,$$

$$e^{\lambda(r)} = e^{-\nu(r)} = 1 + \frac{r^2}{2\tilde{\alpha}}\left(1 - \sqrt{1 + \frac{64\pi G\tilde{\alpha}M}{n(n-1)\Omega_n r^n}}\right),$$

Black Hole Entropy:

$$S = \frac{A}{4G}\left(1 + \frac{n-1}{n-3}\frac{2\tilde{\alpha}}{r_+^2}\right),$$

**(R. Myers,1988, R.G. Cai, 2002, 2004)**

# Ansatz:

$$T = 1/(2\pi\tilde{r}_A).$$

$$S = \frac{A}{4G}\left(1 + \frac{n-1}{n-3}\frac{2\tilde{\alpha}}{\tilde{r}_A^2}\right),$$

$$-dE = TdS$$

This time:

$$\left(1+2\tilde{\alpha}(H^2+\frac{k}{a^2})\right)\left(\dot{H}-\frac{k}{a^2}\right) = -\frac{8\pi G}{n-1}(\rho+p).$$

$$H^2+\frac{k}{a^2}+\tilde{\alpha}\left(H^2+\frac{k}{a^2}\right)^2 = \frac{16\pi G}{n(n-1)}\rho.$$

This also holds for more general Lovelock gravity!

## ii) Friedmann equation and the first law of thermodynamics

**Consider a FRW universe**

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}\, dx^i dx^j,$$

Apparent horizon

$$\tilde{r}_A = 1/\sqrt{H^2 + k/a^2}.$$

And its surface gravity

$$\kappa = -\frac{1}{\tilde{r}_A}\left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right),$$

**which is defined by**

$$\kappa = \frac{1}{2\sqrt{-h}}\partial_a\left(\sqrt{-h}h^{ab}\partial_b\tilde{r}\right).$$

Consider the Einstein field equations with perfect fluid

$$G_{\mu\nu} = 8\pi G T_{\mu\nu},$$

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu + P g_{\mu\nu},$$

One has the Friedmann equation and the continuity equation

$$H^2 + \frac{k}{a^2} = \frac{16\pi G}{n(n-1)}\rho,$$

$$\dot{\rho} + nH(\rho + P) = 0.$$

$$\frac{1}{\tilde{r}_A^2} = \frac{16\pi G}{n(n-1)}\rho.$$

$$\frac{1}{\tilde{r}_A^3} d\tilde{r}_A = \frac{8\pi G}{n-1}(\rho + P)H dt.$$

Multiplying both side hands by a factor $n\Omega_n \tilde{r}_A^n (1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A})$,

$$\frac{\kappa}{2\pi} d\left(\frac{n\Omega_n \tilde{r}_A^{n-1}}{4G}\right) = -n\Omega_n \tilde{r}_A^n (\rho + P) H \left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right) dt$$

Using the definition

$$T = \kappa/2\pi \qquad S = A/4G \qquad A = n\Omega_n \tilde{r}_A^{n-1}$$

One has

$$TdS = -n\Omega_n \tilde{r}_A^n (\rho + P) H \left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right) dt.$$

Now consider the energy inside the apparent horizon

$$E = \Omega_n \tilde{r}_A^n \rho.$$

$$dE = n\Omega_n \tilde{r}_A^{n-1} \rho d\tilde{r}_A - n\Omega_n \tilde{r}_A^n (\rho + P) H dt.$$

$$\Rightarrow \quad dE = TdS + WdV. \qquad W \equiv (\rho - P)/2.$$

(Unified first law of thermodynamics, Hayward, 1998,1999)

The case with a Gauss-Bonnet term?

$$S = \frac{1}{16\pi G}\int d^{n+1}x\sqrt{-g}(R+\alpha R_{GB}) + S_m.$$

Black hole has an entropy of form

$$S = \frac{A}{4G}\left(1 + \frac{n-1}{n-3}\frac{2\tilde{\alpha}}{r_+^2}\right),$$

Consider the Friedmann equation in GB gravity

$$(H^2 + \frac{k}{a^2}) + \tilde{\alpha}(H^2 + \frac{k}{a^2})^2 = \frac{16\pi G}{n(n-1)}\rho.$$

$$\frac{1}{\tilde{r}_A^2} + \tilde{\alpha}\frac{1}{\tilde{r}_A^4} = \frac{16\pi G}{n(n-1)}\rho$$

$$\frac{1}{\tilde{r}_A^3}d\tilde{r}_A + 2\tilde{\alpha}\frac{1}{\tilde{r}_A^5}d\tilde{r}_A = \frac{8\pi G}{(n-1)}(\rho+P)Hdt,$$

Once again, multiplying a factor with $n\Omega_n \tilde{r}_A^n (1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A})$

$$\frac{\kappa}{2\pi} d\left( (\frac{n\Omega_n \tilde{r}_A^{n-1}}{4G})(1 + \frac{n-1}{n-3}\frac{2\tilde{\alpha}}{\tilde{r}_A^2}) \right) = -n\Omega_n \tilde{r}_A^n (\tilde{\rho} + \tilde{P}) H (1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}) dt.$$

Defining

$$\begin{aligned} S &= \frac{n\Omega_n \tilde{r}_A^{n-1}}{4G}(1 + \frac{n-1}{n-3}\frac{2\bar{\alpha}}{\tilde{r}_A^2}), \\ T &= \frac{\kappa}{2\pi}, \end{aligned}$$

$$TdS = -n\Omega_n \tilde{r}_A^n (\rho + P) H (1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}) dt.$$

$$dE = TdS + WdV.$$

It also holds for Lovelock case !

# iii) Corrected entropy-area relation and modified Friedmann equation

[RGC, L.M. Cao and Y.P. Hu JHEP **0808, 090 (2008)**]

Corrected entropy-area relation:

$$S = \frac{A}{4G} + \alpha \ln \frac{A}{4G},$$

→ Friedmann equations

Loop quantum cosmology:

$$H^2 = \frac{8\pi G}{3}\rho\left(1 - \frac{\rho}{\rho_{\rm crit}}\right),$$

→ Entropy formula

$\rho_{\rm crit} = \sqrt{3}/(32\pi G^2\gamma^3)$, $\gamma$ is the so-called Barbero-Immirzi parameter.

# From corrected entropy-area relation to modified Friedmann equation

$$S = \frac{A}{4G} + \alpha \ln \frac{A}{4G},$$

→ Friedmann equations

we define the work density $W$ and energy-supply vector $\Psi$ as

$$W = -\frac{1}{2}T^{ab}h_{ab}, \quad \Psi_a = T_a^b \partial_b \tilde{r} + W \partial_a \tilde{r},$$

For a FRW universe with a perfect fluid:

$$W = \frac{1}{2}(\rho - p), \quad \Psi_a = -\frac{1}{2}(\rho + p)H\tilde{r}dt + \frac{1}{2}(\rho + p)adr.$$

The amount of energy crossing the apparent horizon within *dt*

$$\delta Q = -A\Psi = A(\rho + p)H\tilde{r}_A dt,$$

where A is the area of the apparent horizon.

Assume the temperature $T = \frac{1}{2\pi\tilde{r}_A}.$

and the Clausius relation $\delta Q = TdS,$

$$A(\rho + p)H\tilde{r}_A dt = \frac{1}{2\pi\tilde{r}_A}(\frac{1}{4G} + \frac{\alpha}{A})dA$$

$$\left(1 + \frac{4G\alpha}{A}\right)(\dot{H} - \frac{k}{a^2}) = -4\pi G(\rho + p).$$

$$H^2 + \frac{k}{a^2} + \frac{\alpha G}{2\pi}(H^2 + \frac{k}{a^2})^2 = \frac{8\pi G}{3}\rho,$$

# Bouncing universe?

$$H^2 + \frac{k}{a^2} + \frac{\alpha G}{2\pi}(H^2 + \frac{k}{a^2})^2 = \frac{8\pi G}{3}\rho,$$

$$H^2 + \frac{k}{a^2} = \frac{\pi}{\alpha G}\left(-1 + \sqrt{1 + \frac{16\alpha G^2}{3}\rho}\right).$$

$$H^2 + \frac{k}{a^2} \approx \frac{8\pi G}{3}\rho\left(1 - \frac{4\alpha G^2}{3}\rho\right).$$

$$H^2 = \frac{8\pi G}{3}\rho\left(1 - \frac{\rho}{\rho_{\rm crit}}\right),$$

Loop quantum cosmology

## More general case:

$$S = \frac{A}{4G} + \alpha \ln \frac{A}{4G} + \beta \frac{4G}{A},$$

$$\left(1 + \alpha \frac{4G}{A} - \beta \frac{16G^2}{A^2}\right)(\dot{H} - \frac{k}{a^2}) = -4\pi G(\rho + p),$$

$$H^2 + \frac{k}{a^2} + \frac{\alpha G}{2\pi}\left(H^2 + \frac{k}{a^2}\right)^2 - \frac{\beta G^2}{3\pi^2}\left(H^2 + \frac{k}{a^2}\right)^3 = \frac{8\pi G}{3}\rho,$$

further

$$S = f(x), \quad x = \frac{A}{4G},$$

$$\left(\dot{H} - \frac{k}{a^2}\right) f'(x) = -4\pi G(\rho + p),$$

$$\frac{8\pi G}{3}\rho = -\frac{\pi}{G}\int \frac{f'}{x^2} dx.$$

# From modified Friedmann equation to corrected entropy-area relation

$$H^2 = \frac{8\pi G}{3}\rho\left(1 - \frac{\rho}{\rho_{\rm crit}}\right),$$

→ Entropy formula

The unified first law

$$dE = A\Psi + W dV,$$

$$E = \frac{\tilde{r}}{2G}(1 - h^{ab}\partial_a\tilde{r}\partial_b\tilde{r}),$$

The first law of apparent horizon (R.G. Cai and L.M. Cao, hep-th/0612144)

$$\langle dE, \xi\rangle = \frac{\kappa}{8\pi G}\langle dA, \xi\rangle + \langle W dV, \xi\rangle,$$

$$\xi = \partial_t - (1 - 2\epsilon)Hr\partial_r \text{ with } \epsilon = \dot{\tilde{r}}_A/2H\tilde{r}_A,$$

# Rewriting the modified Friedmann equation

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}(\rho + \rho_e),$$

$$\rho_e = -\rho^2/\rho_{\rm crit}.$$

$$\Psi_e = \frac{\rho}{\rho_{\rm crit}}(\rho + p)H\tilde{r}dt - \frac{\rho}{\rho_{\rm crit}}(\rho + p)adr,$$

$$W_e = \frac{\rho}{\rho_{\rm crit}}p.$$

$$\begin{aligned}
\delta Q \equiv \langle A\Psi_m, \xi\rangle &= \frac{\kappa}{8\pi G}\langle dA, \xi\rangle - \langle A\Psi_e, \xi\rangle \\
&= -\frac{HA\epsilon(1-\epsilon)}{2\pi G}\frac{1}{\sqrt{\tilde{r}_A^2 - \frac{3}{2\pi G\rho_{\rm crit}}}} \\
&= T\langle\frac{2\pi\tilde{r}_A^2}{G}\frac{d\tilde{r}_A}{\sqrt{\tilde{r}_A^2 - \frac{3}{2\pi G\rho_{\rm crit}}}}, \xi\rangle \\
&= T\langle dS, \xi\rangle
\end{aligned}$$

$$S=\frac{\pi\tilde{r}_A}{G}\sqrt{\tilde{r}_A^2-\frac{3}{2\pi G\rho_{\rm crit}}}+\frac{3}{2\pi G^2\rho_{\rm crit}}\ln(\tilde{r}_A+\sqrt{\tilde{r}_A^2-\frac{3}{2\pi G\rho_{\rm crit}}})+C,$$

It is easy to show

$$dE_m=TdS+W_mdV,$$

$$S=\frac{A}{4G}+\frac{3}{4\pi G^2\rho_{\rm crit}}\ln\frac{A}{4G}+o(\frac{1}{A})+C_0.$$

Compare with

$$S=\frac{A}{4G}+\alpha\ln\frac{A}{4G},$$

$$\alpha'=3/4\pi G^2\rho_{\rm crit}$$

## iv ) Hawking radiation of apparent horizon in FRW universe

We know Hawking radiation is always associated with event horizon of spacetime:

(1) Black hole, (2) de Sitter space, (3) Rindler horizon

Question: how about apparent horizon in FRW?

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right)$$

$$T = \frac{1}{2\pi \tilde{r}_A}, \quad S = \frac{A}{4G},$$

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right)$$

Define $\tilde{r} = ar$,

$$ds^2 = -\frac{1-\tilde{r}^2/\tilde{r}_A^2}{1-k\tilde{r}^2/a^2}dt^2 - \frac{2H\tilde{r}}{1-k\tilde{r}^2/a^2}dtd\tilde{r} + \frac{1}{1-k\tilde{r}^2/a^2}d\tilde{r}^2 + \tilde{r}^2 d\Omega_2^2.$$

when k=0, it is quite similar to the Painleve-de Sitter metric (M. Parikh, PLB 546, 189 (2002)

There is a Kodama vector: $K^a \equiv -\epsilon^{ab}\nabla_b \tilde{r} = \sqrt{1-k\tilde{r}^2/a^2}(\partial/\partial t)^a$

$$K_a K^a = -(1-\tilde{r}^2/\tilde{r}_A^2).$$

Now let us consider a particle with mass $m$ in FRW universe.
The Hamilton-Jacobi equation:

$$g^{\mu\nu}\partial_\mu \mathbf{S}\partial_\nu \mathbf{S} + m^2 = 0.$$

By use of the Kodama vector, one could define

$$\omega = -K^a\partial_a \mathbf{S} = -\sqrt{1-k\tilde{r}^2/a^2}\partial_t \mathbf{S}, \quad k_{\tilde{r}} = (\partial/\partial\tilde{r})^a\partial_a \mathbf{S} = \partial_{\tilde{r}}\mathbf{S}.$$

Then the action:

$$\mathbf{S} = -\int \frac{\omega}{\sqrt{1-k\tilde{r}^2/a^2}}dt + \int k_{\tilde{r}}d\tilde{r}.$$

$$-\frac{\omega^2}{1-k\tilde{r}^2/a^2}+\frac{2H\tilde{r}\omega}{\sqrt{1-k\tilde{r}^2/a^2}}k_{\tilde{r}}+(1-\frac{\tilde{r}^2}{\tilde{r}_A^2})k_{\tilde{r}}^2+m^2=0,$$

$$k_{\tilde{r}}=\frac{-H\tilde{r}\pm\sqrt{H^2\tilde{r}^2+(1-\tilde{r}^2/\tilde{r}_A^2)[1-m^2(1-k\tilde{r}^2/a^2)/\omega^2]}}{(1-\tilde{r}^2/\tilde{r}_A^2)\sqrt{1-k\tilde{r}^2/a^2}}\omega,$$

Consider the incoming mode, the action has a pole at the apparent horizo

$$\begin{aligned}\mathrm{Im}\mathbf{S} &= -\mathrm{Im}\int\frac{H\tilde{r}+\sqrt{H^2\tilde{r}^2+(1-\tilde{r}^2/\tilde{r}_A^2)[1-m^2(1-k\tilde{r}^2/a^2)/\omega^2]}}{(1-\tilde{r}^2/\tilde{r}_A^2)\sqrt{1-k\tilde{r}^2/a^2}}\omega d\tilde{r}\\ &= \pi\tilde{r}_A\omega.\end{aligned}$$

(Parikh and Wilczek,2000)

In WKB approximation, the emission rate Gamma is the square of the tunneling amplitude:

$$\Gamma \propto \exp(-2\mathrm{Im}\mathbf{S}).$$

The emission rate can be cast in a form of thermal spectrum

$$\Gamma \sim \exp(-\omega/T),$$

$$T = \frac{1}{2\pi\tilde{r}_A}.$$

→ The end

The null geodesic method leads to the same result!

(RGC, L.M. Cao and Y.P. Hu, CQG **26,155018 (2009**) )

# Further remarks:

1) From dQ=TdS to Friedmann equations, here S=A/4G and

$$T = \frac{1}{2\pi R_A},$$

2) The Friedmann equation can be recast to a universal form

$$dE = TdS + WdV.$$

3) There is a Hawking radiation for the apparent horizon in FRW univers

4) In Einstein gravity and Lovelock gravity, the expression of S has a same form as the black hole entropy

5) In brane world scenario, that form still holds, and we can obtain an expression of horizon entropy associated with apparent horizon, and expect it also holds for brane world black hole horizon.

# My papers on this subject:

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4) A. Sheykhi, B. Wang and RGC, NPB 779, 1 (2007),
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5) R.G. Cai, L. M. Cao and Y.P. Hu, JHEP0808, 090 (2008)
6) R.G. Cai, Prog. Theor. Phys. Suppl. 172, 100 (2008)
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11) RGC, L. M. Cao and N. Ohta, **PRD81,061501 (2010**)
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# Thank You !