

# Dynamics of strings and branes, with application to junctions and vortons

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# 1 Worldsheet curvature analysis

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## 1.1 The first fundamental tensor

The development of geometrical intuition and of computationally efficient methods for use in string and membrane theory has been hampered by a tradition of publishing results in untidy, highly gauge dependent, notation (one of the causes being the undue influence still exercised by Eisenhart's obsolete treatise "Riemannian Geometry" [1]). For the intermediate steps in particular calculations it is of course frequently useful and often indispensable to introduce specifically adapted auxiliary structures, such as curvilinear worldsheet coordinates  $\sigma^i$  ( $i = 0, \dots, d-1$ ) and the associated bitensorial derivatives, namely

$$x^{\mu}_{,i} = \frac{\partial x^{\mu}}{\partial \sigma^i} . \quad (1)$$

It is also frequently useful to introduce (adapted) orthonormal frame vectors, namely an internal subset of vectors  $\iota_A^\mu$  ( $A = 0, \dots, d-1$ ) tangential to the worldsheet and an external subset of vectors  $\lambda_X^\mu$  ( $X = 1, \dots, n-d$ ) orthogonal to the worldsheet, as characterised by

$$\iota_A^\mu \iota_{B\mu} = \eta_{AB}, \quad \iota_A^\mu \lambda_{X\mu} = 0, \quad \lambda_X^\mu \lambda_{Y\mu} = \delta_{XY}, \quad (2)$$

where  $\eta_{AB}$  is a fixed  $d$ -dimensional Minkowski metric and the Kronecker matrix  $\delta_{XY}$  is a fixed  $(n-d)$ -dimensional Cartesian metric, but even in the most recent literature there are still many examples of insufficient effort to sort out the ensuing clutter of indices of different kinds (Greek or Latin, early or late, small or capital) by grouping the various contributions into simple tensorially covariant combinations.

Another inconvenient feature of many publications is that results have been left in a form that depends on some particular gauge choice (such as the conformal gauge for internal string coordinates) which obscures the relationship with other results concerning the same system but in a different gauge.

The strategy adopted here [2] aims at minimising such problems (they can never be entirely eliminated) by working as far as possible with a single kind of tensor index, which must of course be the one that is most fundamental, namely that of the  $n$ -dimensional coordinates,  $x^\mu$ , on the background spacetime with metric  $g_{\mu\nu}$ .

Thus, to avoid dependence on the internal frame index  $A$  (which is lowered and raised by contraction with the fixed d-dimensional Minkowski metric  $\eta_{AB}$  and its inverse  $\eta^{AB}$ ) and on the external frame index  $X$  (which is lowered and raised by contraction with the fixed (n-d)-dimensional Cartesian metric  $\delta_{XY}$  and its inverse  $\delta^{XY}$ ), the separate internal frame vectors  $l_A^\mu$  and external frame vectors  $\lambda_X^\mu$  will as far as possible be eliminated in favour of the frame gauge independent combinations

$$\eta^\mu{}_\nu = l_A^\mu l^A{}_\nu, \quad \perp^\mu{}_\nu = \lambda_X^\mu \lambda^X{}_\nu. \quad (3)$$

The former,  $\eta^\mu{}_\nu$ , is what will be called the (first) *fundamental tensor* of the metric, which acts as the (rank d) operator of tangential projection onto the world sheet, while the latter,  $\perp^\mu{}_\nu$ , is the (rank n-d) operator of projection orthogonal to the world sheet.

The same principle applies to the avoidance of unnecessary involvement of the internal coordinate indices which are lowered and raised by contraction with the induced metric on the worldsheet as given by

$$\eta_{ij} = g_{\mu\nu} x^{\mu}_{,i} x^{\nu}_{,j} , \quad (4)$$

and with its contravariant inverse  $\eta^{ij}$ . After being cast (by index raising if necessary) into its contravariant form, any internal coordinate tensor can be directly projected onto a corresponding background tensor in the manner exemplified by the intrinsic metric itself, which gives

$$\eta^{\mu\nu} = \eta^{ij} x^{\mu}_{,i} x^{\nu}_{,j} . \quad (5)$$

The formula (5) provides an alternative (more direct) prescription for the fundamental tensor that was previously introduced via the use of the internal frame in (3). This approach also provides a direct prescription for the complementary orthogonal projector that was introduced via an external frame in (3) but is also obtainable immediately from (5) as

$$\perp_{\nu}^{\mu} = g_{\nu}^{\mu} - \eta_{\nu}^{\mu}. \quad (6)$$

As well as having the separate operator properties

$$\eta_{\rho}^{\mu} \eta_{\nu}^{\rho} = \eta_{\nu}^{\mu}, \quad \perp_{\rho}^{\mu} \perp_{\nu}^{\rho} = \perp_{\nu}^{\mu} \quad (7)$$

the tensors defined by (5) and (6) will evidently be related by

$$\eta_{\rho}^{\mu} \perp_{\nu}^{\rho} = 0 = \perp_{\rho}^{\mu} \eta_{\nu}^{\rho}. \quad (8)$$

## 1.2 The inner and outer curvature tensors

In so far as we are concerned with tensor fields such as the frame vectors whose support is confined to the  $d$ -dimensional world sheet, the effect of Riemannian covariant differentiation  $\nabla_{\mu}$  along an arbitrary directions on the background spacetime will not be well defined, only the corresponding tangentially projected differentiation operation

$$\overline{\nabla}_{\mu} \stackrel{\text{def}}{=} \eta^{\nu}_{\mu} \nabla_{\nu}, \quad (9)$$

being meaningful for them, as for instance in the case of a scalar field  $\varphi$  for which the tangentially projected gradient is given in terms of internal coordinate differentiation simply by  $\overline{\nabla}^{\mu} \varphi = \eta^{ij} x^{\mu}_{,i} \varphi_{,ji}$ .

An irreducible basis for the various possible covariant derivatives of the frame vectors consists of the *internal rotation* pseudo-tensor  $\rho_{\mu}^{\nu}{}_{\rho}$  and the *external rotation* (or “twist”) pseudo-tensor  $\varpi_{\mu}^{\nu}{}_{\rho}$  as given by

$$\rho_{\mu}^{\nu}{}_{\rho} = \eta^{\nu}{}_{\sigma} \iota^A{}_{\rho} \overline{\nabla}_{\mu} \iota_A{}^{\sigma} = -\rho_{\mu\rho}{}^{\nu},$$

$$\varpi_{\mu}^{\nu}{}_{\rho} = \perp^{\nu}{}_{\sigma} \lambda^X{}_{\rho} \overline{\nabla}_{\mu} \lambda_X{}^{\sigma} = -\varpi_{\mu\rho}{}^{\nu}, \quad (10)$$

together with their *mixed* analogue  $K_{\mu\nu}{}^{\rho}$  which is obtainable in a pair of equivalent alternative forms given by

$$K_{\mu\nu}{}^{\rho} = \perp^{\rho}{}_{\sigma} \iota^A{}_{\nu} \overline{\nabla}_{\mu} \iota_A{}^{\sigma} = -\eta^{\sigma}{}_{\nu} \lambda_X{}^{\rho} \overline{\nabla}_{\mu} \lambda^X{}_{\sigma}. \quad (11)$$

The reason for qualifying the fields (10) as “pseudo” tensors is that although they are tensorial in the ordinary sense with respect to changes of the background coordinates  $x^\mu$  they are not geometrically well defined just by the geometry of the world sheet but are gauge dependent in the sense of being functions of the choice of the internal and external frames  $\iota_A^\mu$  and  $\lambda_X^\mu$ . The gauge dependence of  $\rho_{\mu}^\nu$  and  $\varpi_{\mu}^\nu$  means that both of them can be set to zero at any chosen point on the worldsheet by choice of the relevant frames in its vicinity. However the condition for it to be possible to set these pseudo-tensors to zero throughout an open neighbourhood is the vanishing of the curvatures of the corresponding frame bundles as characterised with respect to the respective invariance subgroups  $SO(1,d-1)$  and  $SO(n-d)$  into which the full Lorentz invariance group  $SO(1,n-1)$  is broken by the specification of the  $d$ -dimensional world sheet orientation.

The existence of a gauge in which  $\rho_{\mu}^{\nu}{}_{\rho}$  vanishes locally depends on the vanishing of the *inner curvature* of Riemannian type, that is obtainable (by a calculation of the type originally developed by Cartan that was made familiar to physicists by Yang Mills theory) as [3]

$$R_{\kappa\lambda}{}^{\mu}{}_{\nu} = 2\eta^{\mu}{}_{\sigma}\eta^{\tau}{}_{\mu}\eta^{\pi}{}_{[\lambda}\overline{\nabla}_{\kappa]}\rho_{\pi}{}^{\sigma}{}_{\tau} + 2\rho_{[\kappa}{}^{\mu\pi}{}_{\lambda]}\pi\nu. \quad (12)$$

The analogue for the “twist” tensor  $\varpi_{\mu}^{\nu}{}_{\rho}$  the *outer curvature* of less familiar type that is given [3] by

$$\Omega_{\kappa\lambda}{}^{\mu}{}_{\nu} = 2\underline{\perp}^{\mu}{}_{\sigma}\underline{\perp}^{\tau}{}_{\mu}\eta^{\pi}{}_{[\lambda}\overline{\nabla}_{\kappa]}\varpi_{\pi}{}^{\sigma}{}_{\tau} + 2\varpi_{[\kappa}{}^{\mu\pi}{}_{\lambda]}\pi\nu. \quad (13)$$

The frame gauge invariance of the expressions (12) and (13) – whereby  $R_{\kappa\lambda}{}^{\mu}{}_{\nu}$  and  $\Omega_{\kappa\lambda}{}^{\mu}{}_{\nu}$  are unambiguously well defined as tensors in the strictest sense of the word – is not obvious from the foregoing formulae, but will be made manifest in the alternative expressions given below.

## 1.3 The second fundamental tensor

Another, even more important, gauge invariance property that is not obvious from the traditional approach – as recapitulated above – is that of the entity  $K_{\mu\nu}{}^\rho$  defined by the mixed analogue (11) of (10), which (unlike  $\rho_{\mu}{}^\nu{}_\rho$  and  $\varpi_{\mu}{}^\nu{}_\rho$ , but like  $R_{\kappa\lambda}{}^\mu{}_\nu$  and  $\Omega_{\kappa\lambda}{}^\mu{}_\nu$ ) is a geometrically well defined tensor in the strict sense. The frame gauge independence of (11) can be seen from its agreement with the alternative – manifestly gauge independent – definition [4]

$$K_{\mu\nu}{}^\rho \stackrel{\text{def}}{=} \eta^\sigma{}_\nu \overline{\nabla}_\mu \eta^\rho{}_\sigma. \quad (14)$$

whereby the entity that we refer to as the *second fundamental tensor* is constructed directly from the the first fundamental tensor  $\eta^{\mu\nu}$  as given by (5).

As this second fundamental tensor,  $K_{\mu\nu}{}^\rho$  will play an important role in the work that follows, it is worth lingering over its essential properties. The expression (14) could of course be meaningfully applied not only to the fundamental projection tensor of a d-surface, but also to any (smooth) field of rank-d projection operators  $\eta^\mu{}_\nu$  as specified by a field of arbitrarily orientated d-surface elements. What distinguishes the integrable case – in which the elements mesh together to form a well defined d-surface through the point under consideration – is the *Weingarten identity*, whereby that the tensor defined by (14) will have the symmetry property

$$K_{[\mu\nu]}{}^\rho = 0, \quad (15)$$

an integrability condition that is derivable [4], [3] as a version of the well known Frobenius theorem.

As well as being symmetric, the tensor  $K_{\mu\nu}{}^\rho$  is obviously tangential on the first two indices and also orthogonal on the last :

$$\perp_\mu^\sigma K_{\sigma\nu}{}^\rho = K_{\mu\nu}{}^\sigma \eta_\sigma{}^\rho = 0. \quad (16)$$

This second fundamental tensor fully determines the tangential derivatives of the first fundamental tensor  $\eta_\nu{}^\mu$  by the formula

$$\bar{\nabla}_\mu \eta_{\nu\rho} = 2K_{\mu(\nu\rho)}, \quad (17)$$

(using round brackets to denote symmetrisation) and it can be seen to be characterisable by the condition that the orthogonal projection of the acceleration of any tangential unit vector field  $\mathbf{u}^\mu$  will be given by

$$\mathbf{u}^\mu \mathbf{u}^\nu K_{\mu\nu}{}^\rho = \perp_\mu^\rho \dot{\mathbf{u}}^\mu, \quad \dot{\mathbf{u}}^\mu = \mathbf{u}^\nu \nabla_\nu \mathbf{u}^\mu. \quad (18)$$

## 1.4 The extrinsic curvature vector and the conformation tensor

It is very practical for a great many purposes to introduce the *extrinsic curvature vector*  $K^\mu$ , defined as the trace of the second fundamental tensor, which is automatically orthogonal to the worldsheet,

$$K^\mu \stackrel{\text{def}}{=} K^\nu{}_\nu{}^\mu = \overline{\nabla}_\mu \eta^{\mu\nu}, \quad \eta^\mu{}_\nu K^\nu = 0. \quad (19)$$

It is useful for many specific purposes to work this out in terms of the intrinsic metric  $\eta_{ij}$  and its determinant  $|\eta|$ . For the tangentially projected gradient of a scalar field  $\varphi$  on the worldsheet, it suffices to use the simple expression

$$\overline{\nabla}^\mu \varphi = \eta^{ij} x^\mu{}_{,i} \varphi_{,j}.$$

However for a tensorial field (unless one is using Minkowski coordinates in a flat spacetime) the gradient will also have contributions involving the background Riemann Christoffel connection

$$\Gamma_{\mu\rho}^{\nu} = g^{\nu\sigma} \left( g_{\sigma(\mu,\rho)} - \frac{1}{2} g_{\mu\rho,\sigma} \right). \quad (20)$$

The curvature vector is thus obtained in explicit detail as

$$K^{\nu} = \frac{1}{\sqrt{\|\eta\|}} \left( \sqrt{\|\eta\|} \eta^{ij} x_{,i}^{\nu} \right)_{,j} + \eta^{ij} x_{,i}^{\mu} x_{,j}^{\rho} \Gamma_{\mu\rho}^{\nu}. \quad (21)$$

This expression is useful for specific computational purposes, but much of the literature on cosmic string dynamics has been made unnecessarily heavy by a tradition of working all the time with long strings of non tensorial terms such as those on the right of (21) rather than exploiting more succinct tensorial expressions, such as  $K^{\nu} = \overline{\nabla}_{\mu} \eta^{\mu\nu}$ .

As an alternative to the universally applicable tensorial approach advocated here, there is of course another more commonly used method of achieving succinctness in particular circumstances, which is to sacrifice gauge covariance by using specialised kinds of coordinate system.

In particular, for the case of a string, i.e. for a 2-dimensional worldsheet, it is standard practise to use conformal coordinates  $\sigma^0$  and  $\sigma^1$  so that the corresponding tangent vectors  $\dot{x}^\mu = x^\mu_{,0}$  and  $x'^\mu = x^\mu_{,1}$  satisfy the restrictions  $\dot{x}^\mu x'_{\mu} = 0$ ,  $\dot{x}^\mu \dot{x}_\mu + x'^\mu x'_{\mu} = 0$ , which implies  $\sqrt{\|\eta\|} = x'^\mu x'_{\mu} = -\dot{x}^\mu \dot{x}_\mu$ , so that (21) simply gives

$$\sqrt{\|\eta\|} K^\nu = x''^\nu - \ddot{x}^\nu + (x'^\mu x'^\rho - \dot{x}^\mu \dot{x}^\rho) \Gamma_{\mu\rho}^\nu.$$

The physical specification of the extrinsic curvature vector (19) for a timelike d-surface in a dynamic theory provides what can be taken as the equations of extrinsic motion of the d-surface [4], [5], the simplest possibility being the “harmonic” condition  $K^\mu = 0$  that is obtained (as will be shown in the following sections) from a surface measure variational principle such as that of the Dirac membrane model [6], or of the Goto-Nambu string model [7] whose dynamic equations in a flat background are therefore expressible with respect to a standard conformal gauge in the familiar form  $x''^\mu - \ddot{x}^\mu = 0$ ,

There is a certain analogy between the Einstein vacuum equations, which impose the vanishing of the trace  $\mathcal{R}_{\mu\nu}$  of the background spacetime curvature  $\mathcal{R}_{\lambda\mu}{}^\rho{}_\nu$ , and the Dirac-Gotu-Nambu equations, which impose the vanishing of the trace  $K^\nu$  of the second fundamental tensor  $K_{\lambda\mu}{}^\nu$ ,

Just as it is useful to separate out the Weyl tensor [8], i.e. the trace free part of the Ricci background curvature which is the only part that remains when the Einstein vacuum equations are satisfied, so also analogously, it is useful to separate out the the trace free part of the second fundamental tensor, namely the extrinsic conformation tensor [3], which is the only part that remains when equations of motion of the Dirac - Goto - Nambu type are satisfied. Explicitly, the trace free *extrinsic conformation* tensor  $\mathcal{C}_{\mu\nu}{}^\rho$  of a d-dimensional imbedding is defined [3] in terms of the corresponding first and second fundamental tensors  $\eta_{\mu\nu}$  and  $K_{\mu\nu}{}^\rho$  as

$$\mathcal{C}_{\mu\nu}{}^\rho \stackrel{\text{def}}{=} K_{\mu\nu}{}^\rho - \frac{1}{d} \eta_{\mu\nu} K^\rho, \quad \mathcal{C}^\nu{}_\nu{}^\mu = 0. \quad (22)$$

Like the Weyl tensor  $\mathcal{W}_{\lambda\mu}{}^{\rho\nu}$  of the background metric (whose definition is given implicitly by (27) below) this conformation tensor has the noteworthy property of being invariant with respect to conformal modifications of the background metric :

$$\begin{aligned}
 g_{\mu\nu} \mapsto e^{2\alpha} g_{\mu\nu} &\quad \Rightarrow \quad K_{\mu\nu}{}^{\rho} \mapsto K_{\mu\nu}{}^{\rho} + \eta_{\mu\nu} \perp^{\rho\sigma} \nabla_{\sigma} \alpha, \\
 \mathcal{C}_{\mu\nu}{}^{\rho} &\mapsto \mathcal{C}_{\mu\nu}{}^{\rho}.
 \end{aligned} \tag{23}$$

This formula is useful [9] for calculations of the kind undertaken by Vilenkin [10] in a standard Robertson-Walker type cosmological background, which can be obtained from a flat auxiliary spacetime metric by a conformal transformation for which  $e^{\alpha}$  is a time dependent Hubble expansion factor.

## 1.5 Codazzi, Gauss, and Schouten identities

As the higher order analogue of (14) we can go on to introduce the *third* fundamental tensor[4] as

$$\Xi_{\lambda\mu\nu}{}^\rho \stackrel{\text{def}}{=} \eta^\sigma{}_\mu \eta^\tau{}_\nu \perp_\alpha{}^\rho \overline{\nabla}_\lambda K_{\sigma\tau}{}^\alpha, \quad (24)$$

which by construction is obviously symmetric between the second and third indices and tangential on all the first three indices. In a spacetime background that is flat (or of constant curvature as is the case for the DeSitter universe model) this third fundamental tensor is fully symmetric over all the first three indices by what is interpretable as the *generalised Codazzi identity*.

In a background with arbitrary Riemann curvature  $\mathcal{R}_{\lambda\mu}{}^{\rho}{}_{\sigma}$  the *generalised Codazzi identity* is expressible [3] as

$$\Xi_{\lambda\mu\nu}{}^{\rho} = \Xi_{(\lambda\mu\nu)}{}^{\rho} + \frac{2}{3}\eta^{\sigma}{}_{\lambda}\eta^{\tau}{}_{(\mu}\eta^{\alpha}{}_{\nu)}\mathcal{R}_{\sigma\tau}{}^{\beta}{}_{\alpha}\perp_{\beta}{}^{\rho} \quad (25)$$

A script symbol  $\mathcal{R}$  is used here in order to distinguish the ( $n$ -dimensional) background Riemann curvature tensor from the intrinsic curvature tensor (12) of the ( $d$ -dimensional) worldsheet to which the ordinary symbol  $R$  has already allocated. For many of the applications that will follow it will be sufficient just to treat the background spacetime as flat, i.e. to take  $\mathcal{R}_{\sigma\tau}{}^{\beta}{}_{\alpha} = 0$ .

For  $n > 2$ , the background curvature tensor will be decomposable (if present) in terms of the background Ricci tensor and its scalar trace,

$$\mathcal{R}_{\mu\nu} = \mathcal{R}_{\rho\mu}{}^{\rho}{}_{\nu}, \quad \mathcal{R} = \mathcal{R}^{\nu}{}_{\nu}, \quad (26)$$

and of its trace free conformally invariant Weyl part  $\mathcal{W}_{\mu\nu}{}^{\rho\sigma}$  – which can be non zero only for  $n \geq 4$  – in the well known [8] form

$$\mathcal{R}_{\mu\nu}{}^{\rho\sigma} = \mathcal{W}_{\mu\nu}{}^{\rho\sigma} + \frac{4}{n-2} g_{[\mu}^{[\rho} \mathcal{R}^{\sigma]}_{\nu]} - \frac{2}{(n-1)(n-2)} \mathcal{R} g_{[\mu}^{[\rho} g^{\sigma]}_{\nu]}. \quad (27)$$

In terms of the tangential projection of this background curvature, the corresponding *internal* curvature tensor (12) takes the form

$$R_{\mu\nu}{}^{\rho\sigma} = 2K^{\rho}{}_{[\mu}{}^{\tau} K_{\nu]\sigma\tau} + \eta^{\kappa}{}_{\mu} \eta^{\lambda}{}_{\nu} \mathcal{R}_{\kappa\lambda}{}^{\alpha}{}_{\tau} \eta^{\rho}{}_{\alpha} \eta^{\tau}{}_{\sigma}, \quad (28)$$

which is the translation into the present scheme of what is well known in other schemes as the *generalised Gauss identity*.

The less well known analogue (attributable [8] to Schouten) for the (trace free conformally invariant) *outer* curvature (13) is expressible [3] in terms of the relevant projection of the background Weyl tensor as

$$\Omega_{\mu\nu}{}^{\rho}{}_{\sigma} = 2 \mathbf{C}_{[\mu}{}^{\tau\rho} \mathbf{C}_{\nu]\tau\sigma} + \eta^{\kappa}{}_{\mu} \eta^{\lambda}{}_{\nu} \mathcal{W}_{\kappa\lambda}{}^{\alpha}{}_{\tau} \perp_{\alpha}{}^{\rho} \perp_{\sigma}{}^{\tau}. \quad (29)$$

In a background that is flat or conformally flat (for which it is necessary, and for  $n \geq 4$  sufficient, that the Weyl tensor should vanish) the vanishing of the extrinsic conformation tensor  $\mathbf{C}_{\mu\nu}{}^{\rho}$  will therefore be sufficient (independently of the behaviour of the extrinsic curvature vector  $K^{\mu}$ ) for vanishing of the outer curvature tensor  $\Omega_{\mu\nu}{}^{\rho}{}_{\sigma}$ , which is the condition for it to be possible to construct fields of vectors  $\lambda^{\mu}$  orthogonal to the surface and such as to satisfy the generalised Fermi-Walker propagation condition to the effect that  $\perp_{\mu}{}^{\rho} \overline{\nabla}_{\nu} \lambda_{\rho}$  should vanish.

## 1.6 Internal Ricci and Conformal Curvatures.

The conclusion of the preceding paragraph is an illustration of the critically significant role of the conformation tensor  $C_{\mu\nu}{}^\rho$  of an imbedding when the background is conformally flat, which suggests the interest of examining its role with respect to the *inner* curvature,  $R_{\kappa\lambda}{}^\mu{}_\nu$  in this conformally flat case, for which the condition that the background Weyl tensor should vanish is necessary – and for  $n \geq 4$  also sufficient [8] – while when the background dimension is  $n = 3$  this condition, namely  $\mathcal{W}_{\kappa\lambda}{}^\mu{}_\nu = 0$ , will hold in any case as an identity. Conformal flatness is of course compatible with the most common applications, in which the background is taken to be flat in the strong sense, as a justifiable approximation in typical circumstances for which the characteristic length scales of the imbedding will be small compared with those of the background curvature.

Leaving aside the trivial (always locally conformally flat) case of a 2-dimensional background, the Gauss relation (28) reduces to

$$R_{\kappa\lambda}{}^{\mu}{}_{\nu} = \eta_{\kappa}{}^{\rho}\eta_{\lambda}{}^{\sigma}\mathcal{W}_{\rho\sigma}{}^{\tau}{}_{\nu}\eta^{\mu}{}_{\tau}\eta^{\nu}{}_{\nu} + 2K_{[\kappa}{}^{\mu\sigma}K_{\lambda]\nu\sigma} + \frac{2}{n-2}\left(\eta_{[\kappa}{}^{\mu}\eta_{\lambda]}{}^{\rho}\eta_{\nu}{}^{\sigma} - \eta_{\nu[\kappa}\eta_{\lambda]}{}^{\rho}\eta^{\mu\sigma}\right)\left(\mathcal{R}_{\rho\sigma} - \frac{\mathcal{R}}{2(n-1)}g_{\rho\sigma}\right).$$

The internal Ricci tensor is thus obtained in terms of the tracefree and trace parts  $\mathcal{C}_{\lambda\mu}{}^{\nu}$  and  $K_{\rho}$  of the second fundamental tensor in the form

$$R_{\mu\nu} = \mathcal{W}_{\mu\nu} - \mathcal{C}_{\mu}{}^{\rho\sigma}\mathcal{C}_{\nu\rho\sigma} + \frac{d-2}{d}\mathcal{C}_{\mu\nu}{}^{\sigma}K_{\sigma} + \frac{d-1}{d^2}K^{\sigma}K_{\sigma}\eta_{\mu\nu} + \frac{d-2}{n-2}\eta_{\mu}{}^{\rho}\eta_{\nu}{}^{\sigma}\mathcal{R}_{\rho\sigma} + \frac{1}{n-2}\left(\eta^{\rho\sigma}\mathcal{R}_{\rho\sigma} - \frac{d-1}{n-1}\mathcal{R}\right)\eta_{\mu\nu},$$

where the background Weyl contribution, if any, is given by the expressions

$$\mathcal{W}_{\mu\nu} = \eta_{\mu}{}^{\sigma}\eta_{\nu}{}^{\kappa}\mathcal{W}_{\rho\sigma}{}^{\tau}{}_{\kappa}\eta^{\rho}{}_{\tau} = -\eta_{\mu}{}^{\sigma}\eta_{\nu}{}^{\kappa}\mathcal{W}_{\rho\sigma}{}^{\tau}{}_{\kappa}\perp^{\rho}{}_{\tau},$$

of which the last version is obtained as a consequence of the tracelessness of the Weyl tensor.

The corresponding Ricci scalar for the internal geometry (whose surface integral in the special (string) case  $d=2$  gives the ordinary Gauss Bonnet type invariant that was mentioned at the end of section 8) is thus finally obtained in the form

$$R = \mathcal{W} - \mathcal{C}_{\lambda\mu}{}^\nu \mathcal{C}^{\lambda\mu}{}_\nu + \frac{d-1}{d} K^\sigma K_\sigma + \frac{d-1}{n-2} \left( 2\eta^{\rho\sigma} \mathcal{R}_{\rho\sigma} - \frac{d}{n-1} \mathcal{R} \right),$$

(which corrects a transcription error whereby a factor of two was omitted in the original version [3]) where the final Weyl contribution is just the trace,

$$\mathcal{W} = \mathcal{W}^\nu{}_\nu = \eta^{\rho\tau} \mathcal{W}_{\rho\sigma}{}^\tau{}_\nu \eta^\nu{}_\tau = \perp^{\rho\tau} \mathcal{W}_{\rho\sigma}{}^\tau{}_\nu \perp^\nu{}_\tau,$$

which can be seen to vanish identically unless both the dimension and the codimension of the worldsheet are greater than one, i.e. unless both  $d \geq 2$  and  $n-d \geq 2$ .

For cases in which the imbedded surface has dimension  $d \leq 3$ , as must always be the case in an ordinary 4-dimensional space-time background, the specification of the Ricci contribution provides all that is needed to specify the complete inner curvature tensor. However to fully specify  $R_{\kappa\lambda}{}^{\mu}{}_{\nu}$  in higher dimensional cases for which the imbedded surface has dimension  $d \geq 4$  it will also be necessary to take account of the generically non zero conformal curvature term  $C_{\kappa\lambda}{}^{\mu}{}_{\nu}$  that will contribute to the total as given by the internal analogue of (27), namely

$$R_{\mu\nu}{}^{\rho\sigma} = C_{\mu\nu}{}^{\rho\sigma} + \frac{4}{d-2} \eta_{[\mu}^{[\rho} R_{\nu]}^{\sigma]} - \frac{2}{(d-1)(d-2)} R \eta_{[\mu}^{[\rho} \eta_{\nu]}^{\sigma]}, \quad (30)$$

The rather greater algebraic effort required to work out this inner conformal curvature contribution is rewarded by the qualitatively tidy form of the result, which turns out to be homogeneously quadratic in the conformation tensor alone.

The contributions of the trace vector  $K^\mu$  and of the background Ricci tensor  $\mathcal{R}_{\mu\nu}$  are again (as in (29)) found to miraculously cancel out altogether, leaving

$$\begin{aligned} \mathcal{C}_{\kappa\lambda}{}^{\mu\nu} = & 2 \mathcal{C}_{[\kappa}{}^{\mu\sigma} \mathcal{C}_{\lambda]}{}^{\nu}{}_{\sigma} - \frac{4}{d-2} \left( \mathcal{C}^{\rho[\mu}{}_{\sigma} \eta^{\nu]}{}_{[\kappa} \mathcal{C}_{\lambda]\rho}{}^{\sigma} + \eta_{[\kappa}{}^{[\mu} \mathcal{W}_{\lambda]}{}^{\nu]} \right) \\ & - \frac{2}{(d-2)(d-1)} \eta_{[\kappa}{}^{\mu} \eta_{\lambda]}{}^{\nu} \left( \mathcal{C}_{\rho\sigma}{}^{\tau} \mathcal{C}^{\rho\sigma}{}_{\tau} - \mathcal{W} \right) + \eta_{\kappa}{}^{\rho} \eta_{\lambda}{}^{\sigma} \mathcal{W}_{\rho\sigma}{}^{\tau}{}_{\nu} \eta^{\mu}{}_{\tau} \eta^{\nu\nu} . \end{aligned}$$

We can thus draw the memorable conclusion that in a conformally flat background the vanishing of the conformation tensor  $\mathcal{C}^{\mu\nu}{}_{\rho}$  is a sufficient condition not only for (local) outer flatness but also for (local) internal conformal flatness, at least for an imbedded surface with dimension  $d \geq 4$ . With a little more work[3] it can be shown that this conclusion also holds for  $d=3$ , while it is trivial for the case of a string worldsheet  $d=2$ , which is always (locally) conformally flat.

## 1.7 Special case of string worldsheet

The following work will deal mainly with the case  $d=2$  of a string, for which an orthonormal tangent frame will consist just of a timelike vector,  $\iota_0^\mu$ , and a spacelike vector,  $\iota_1^\mu$ . Their exterior product vector is the frame independent antisymmetric unit surface element tensor

$$\mathcal{E}^{\mu\nu} = 2\iota_0^{[\mu}\iota_1^{\nu]} = 2(-|\eta|)^{-1/2} x^{[\mu}_{,0} x^{\nu]}_{,1}, \quad (31)$$

whose tangential gradient satisfies

$$\bar{\nabla}_\lambda \mathcal{E}^{\mu\nu} = -2K_{\lambda\rho}^{[\mu} \mathcal{E}^{\nu]\rho}. \quad (32)$$

This is the special  $d=2$  case of formula (B9) in which a sign adjustment factor  $(-1)^{d-1}$  was omitted in the original analysis[3] (of which another misprint was omission in the first term of formula (10.4) of the factor 2).

In this  $d=2$  case the inner rotation pseudo tensor (10) is determined just by a corresponding rotation covector  $\rho_\mu$  according to the specification

$$\rho_{\lambda}^{\mu}{}_{\nu} = \frac{1}{2} \mathcal{E}^{\mu}{}_{\nu} \rho_{\lambda}, \quad \rho_{\lambda} = \rho_{\lambda}^{\mu}{}_{\nu} \mathcal{E}^{\nu}{}_{\mu}. \quad (33)$$

This can be used to see from (12) that the Ricci scalar,

$$R = R^{\nu}{}_{\nu} \quad R_{\mu\nu} = R_{\rho\mu}{}^{\rho}{}_{\nu}, \quad (34)$$

of the 2-dimensional worldsheet will have the well known property of being a pure surface divergence, albeit of a frame gauge dependent quantity :

$$R = \bar{\nabla}_{\mu} (\mathcal{E}^{\mu\nu} \rho_{\nu}). \quad (35)$$

In the specially important case of a string in ordinary 4-dimensional spacetime, i.e. when we have not only  $d=2$  but also  $n=4$ , the antisymmetric background measure tensor  $\varepsilon^{\lambda\mu\nu\rho}$  can be used to determine a scalar (or more strictly, since its sign is orientation dependent, a pseudo scalar) magnitude  $\Omega$  for the outer curvature tensor (13) (despite the fact that its traces are identically zero) according to the specification

$$\Omega = \frac{1}{2} \Omega_{\lambda\mu\nu\rho} \varepsilon^{\lambda\mu\nu\rho} . \quad (36)$$

Under these circumstances one can also define a “twist” covector  $\varpi_\mu$ , that is the outer analogue of  $\rho_\mu$ , according to the specification

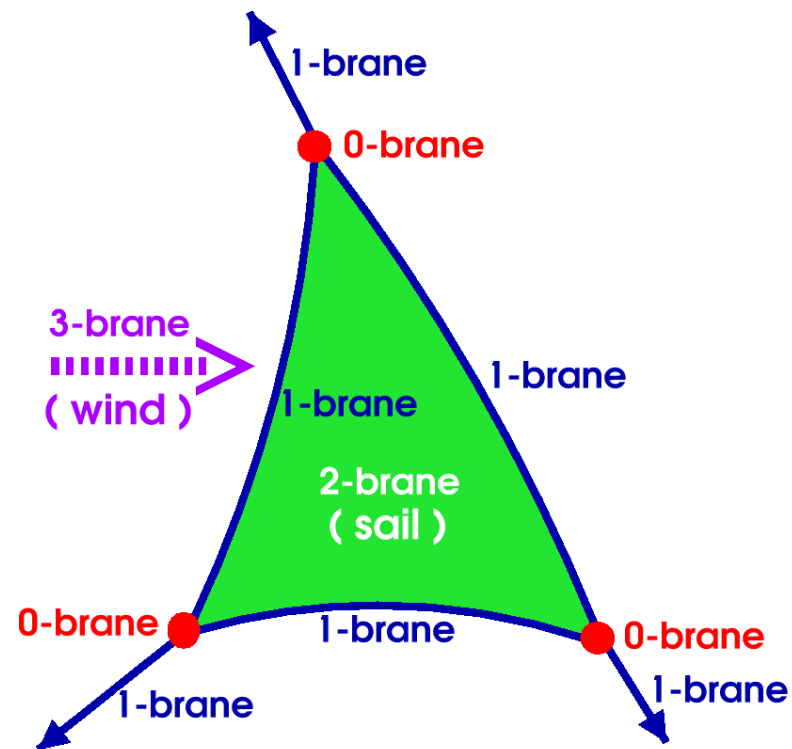
$$\varpi_\nu = \frac{1}{2} \varpi_\nu^{\mu\lambda} \varepsilon_{\lambda\mu\rho\sigma} \mathcal{E}^{\rho\sigma} . \quad (37)$$

Using (37) it can be deduced from (13) that the outer curvature (pseudo) scalar  $\Omega$  of a string worldsheet in 4-dimensions has a divergence property of the same kind as that of its more widely known Ricci analogue (35), the corresponding formula being given by

$$\Omega = \bar{\nabla}_{\mu} (\mathcal{E}^{\mu\nu} \varpi_{\nu}) . \quad (38)$$

It is to be remarked that for a compact spacelike 2-surface the integral of (32) gives the well known Gauss Bonnet invariant, but that the timelike string worldsheets under consideration here will not be characterised by any such global invariant since they will not be compact (being open in the time direction even for a loop that is closed in the spacial sense). The outer analogue of the Gauss Bonnet invariant that arises from (36) for a spacelike 2-surface has been discussed by Penrose and Rindler [11] but again there is no corresponding global invariant in the necessarily non-compact timelike case of a string worldsheet.

## 2 Laws of motion for a regular pure brane complex



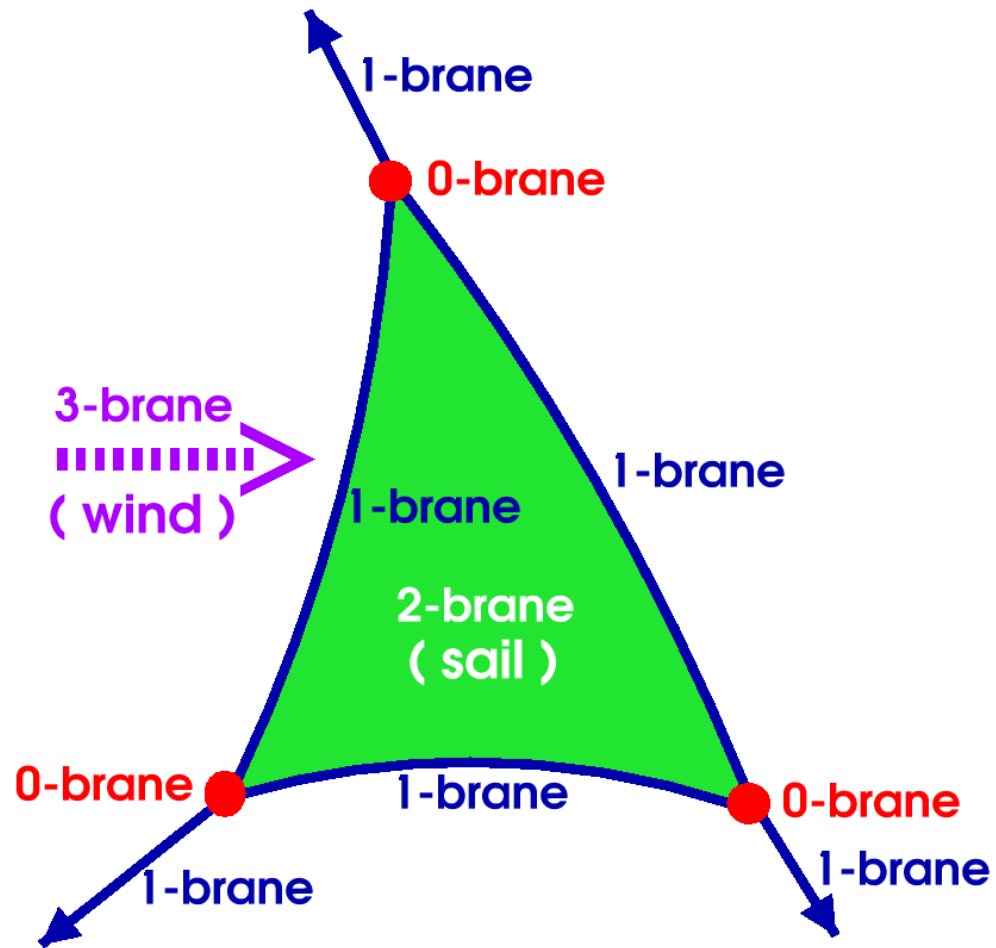
## 2.1 Definition of brane complex

The term  $p$ -brane has come [12], [13] to mean a dynamic system localised on a timelike support surface of dimension  $d=p+1$ , imbedded in a spacetime background of dimension  $n > p$ . Thus a zero - brane means what is commonly referred to as a “point particle”, and a 1-brane means what is commonly referred to as a “string”, while a 2-brane means what is commonly called a “membrane” (whence the generic term “brane”).

At the upper extreme, the “improper” case of an  $(n-1)$ -brane is what is commonly referred to as a “medium” (as exemplified by a simple fluid). The codimension-1 (hypersurface supported) case of an  $(n-2)$ -brane (as exemplified by a cosmological domain wall) is what may be referred to as a “hypermembrane”, while the codimension-2 case of an  $(n-3)$ -brane is what may analogously be referred to as a “hyperstring”.

A set of branes of diverse dimensions will constitute a “geometrically regular” brane complex if the support surface of each  $(d-1)$ -brane member is a smoothly imbedded  $d$ -dimensional timelike submanifold of which the boundary, if any, is a disjoint union of support surfaces of lower dimensional members of the set. For the complex to qualify as regular in the strong dynamic sense, it is also required that a  $p$ -brane member can act directly only on an  $(p-1)$ -brane member on its boundary or on a  $(p+1)$ -brane member on whose boundary it is itself located, though it may be passively influenced by higher dimensional background fields.

Direct mutual interaction between branes with dimension differing by 2 or more will usually lead to divergences, symptomising the breakdown of a strict – meaning thin limit – brane description. To cure that properly, a more elaborate treatment – allowing for finite thickness – would be needed, but it may suffice to use a thin limit approximation whereby the divergence is absorbed in a renormalisation.



Nautical archetype of a regular brane complex in which a 3-brane (the wind) acts (by pressure discontinuity) on a 2-brane (the sail) hemmed by three 1-branes (bolt ropes) terminating on 0-branes (shackles) that are held in place by three more (free) 1-branes (external stay/sheet ropes).

In the case of a brane complex, the total action  $\mathcal{I}$  will be given as a sum of contributions from the various  $(d-1)$ -branes of the complex, of which each has its own Lagrangian  $d$ -surface density scalar  ${}^{(d)}\bar{L}$  say. Each supporting  $d$ -surface will be specified by a mapping  $\sigma \mapsto x\{\sigma\}$  giving the local background coordinates  $x^\mu$  ( $\mu = 0, \dots, n-1$ ) as functions of local internal coordinates  $\sigma^i$  ( $i = 0, \dots, d-1$ ). The corresponding  $d$ -dimensional surface metric tensor  ${}^{(d)}\eta_{ij}$  induced as the pull back of the  $n$ -dimensional background spacetime metric  $g_{\mu\nu}$ , determines the surface measure,  ${}^{(d)}d\bar{\mathcal{S}}$ , in terms of which the total action will be expressible as

$$\mathcal{I} = \sum_d \int {}^{(d)}d\bar{\mathcal{S}} {}^{(d)}\bar{L}, \quad {}^{(d)}d\bar{\mathcal{S}} = \sqrt{\|{}^{(d)}\eta\|} d^d\sigma. \quad (39)$$

For purposes such as the calculation of radiation, it may be useful to replace the *confined* ( $d$ -surface supported) but locally *regular* Lagrangian scalar fields  ${}^{(d)}\overline{L}$  by corresponding unconfined, so no longer regular but *distributional* fields  ${}^{(d)}\hat{L}$ , so as to allow the total action (39) to be represented as a single background spacetime integral,

$$\mathcal{I} = \int d\mathcal{S} \sum_d {}^{(d)}\hat{L}, \quad d\mathcal{S} = \sqrt{\|g\|} d^n x. \quad (40)$$

This requires the distributional action  ${}^{(d)}\hat{L}$  for each  $(d-1)$ -brane of the complex to be given in terms of the regular  $d$ -surface density scalar  ${}^{(d)}\overline{L}$  by the prescription expressible in Dirac notation as

$${}^{(d)}\hat{L} = \|g\|^{-1/2} \int {}^{(d)}d\overline{\mathcal{S}} {}^{(d)}\overline{L} \delta^n[x - x\{\sigma\}]. \quad (41)$$

## 2.2 Current, vorticity, and stress-energy tensor

As well as on its own internal (d-1)-brane surface fields and their derivatives, and those of any attached d-brane, each contribution  ${}^{(d)}\bar{L}$  will also depend (passively) on the spacetime metric  $g_{\mu\nu}$  and perhaps other background fields, such as a Maxwellian gauge potential  $A_\nu$ , or a generalised Kalb-Ramond gauge field  $B_{\nu_1\dots\nu_r}^{[r]} = B_{[\nu_1\dots\nu_r]}^{[r]}$ . In the unpolarised (fine) brane limit considered here the action will not depend on the background field derivatives. These give corresponding fields

$$F_{\mu\nu} = 2\nabla_{[\mu}A_{\nu]}, \quad N_{\mu\nu_1\dots\nu_r}^{[r+1]} = (r+1)\nabla_{[\mu}B_{\nu_1\dots\nu_r]}^{[r]},$$

which are invariant under gauge changes

$$A_\mu \mapsto A_\mu + \nabla_\mu\alpha, \quad B_{\nu_1\nu_2\dots\nu_r}^{[r]} \mapsto B_{\nu_1\nu_2\dots\nu_r}^{[r]} + r!\nabla_{[\nu_1}\chi_{\nu_2\dots\nu_r]},$$

and are automatically closed :

$$\nabla_{[\rho}F_{\mu\nu]} = 0, \quad \nabla_{[\rho}N_{\mu\nu_1\dots\nu_r]}^{[r+1]} = 0.$$

Subject to the internal dynamic equations of motion given by the variational principle stipulating preservation of the action by variations of the independent field variables, the effect of arbitrary infinitesimal “Lagrangian” variations  $\delta_{\underline{L}} A_\nu$ ,  $\delta_{\underline{L}} B_{\nu_1 \dots \nu_r}^{[r]}$ ,  $\delta_{\underline{L}} g_{\mu\nu}$  of the background fields will be to induce a corresponding variation of the simple form

$$\delta \mathcal{I} = \sum_{\text{d}} \int^{(\text{d})} \text{d}\bar{\mathcal{S}} \left( {}^{(\text{d})} \bar{j}^\mu \delta_{\underline{L}} A_\mu + \frac{1}{r!} {}^{(\text{d})} \bar{w}_{[r]}^{\nu_1 \dots \nu_r} \delta_{\underline{L}} B_{\nu_1 \dots \nu_r}^{[r]} + \frac{1}{2} {}^{(\text{d})} \bar{T}^{\mu\nu} \delta_{\underline{L}} g_{\mu\nu} \right),$$

from which, for each  $(\text{d}-1)$ -brane, one can read out the

electromagnetic surface current vector  ${}^{(\text{d})} \bar{j}^\mu$ , the surface flux

(generalised vorticity)  $r$ -vector  ${}^{(\text{d})} \bar{w}_{[r]}^{\nu_1 \dots \nu_r} = {}^{(\text{d})} \bar{w}_{[r]}^{[\nu_1 \dots \nu_r]}$ , and, since

$\delta_{\underline{L}} ({}^{(\text{d})} \text{d}\bar{\mathcal{S}}) = \frac{1}{2} {}^{(\text{d})} \bar{\eta}^{\mu\nu} (\delta_{\underline{L}} g_{\mu\nu}) {}^{(\text{d})} \text{d}\bar{\mathcal{S}}$ , the surface stress momentum energy

tensor

$${}^{(\text{d})} \bar{T}^{\mu\nu} = {}^{(\text{d})} \bar{T}^{\nu\mu} = 2 \frac{\partial {}^{(\text{d})} \bar{L}}{\partial g_{\mu\nu}} + {}^{(\text{d})} \bar{L} {}^{(\text{d})} \bar{\eta}^{\mu\nu}. \quad (42)$$

## 2.3 Conservation of current and vorticity

Arbitrary gauge changes  $\delta_{\underline{L}} A_\nu = \nabla_\nu \alpha$ , and  $B_{\nu_1 \nu_2 \dots}^{[r]} = r! \nabla_{[\nu_1} \chi_{\nu_2 \dots]}$ , with  $\delta_{\underline{L}} g_{\mu\nu} = 0$ , can only leave the action invariant

$$\sum_{\text{d}} \int \text{d}^{(\text{d})} \bar{\mathcal{S}} \left( {}^{(\text{d})} \bar{j}^\nu \nabla_\nu \alpha + {}^{(\text{d})} \bar{w}_{[r]}^{\nu_1 \nu_2 \dots} \nabla_{\nu_1} \chi_{\nu_2 \dots} \right) = 0, \quad \text{if the current}$$

${}^{(\text{d})} \bar{j}^\mu$  and vorticity flux  ${}^{(\text{d})} \bar{w}_{[r]}^{\nu_1 \nu_2 \dots}$  are purely d-surface tangential :

their contractions with the rank (n-d) orthogonal projector

$${}^{(\text{d})} \underline{\perp}_\nu^\mu = g_\nu^\mu - {}^{(\text{d})} \eta^\mu_\nu \quad \text{must vanish, } {}^{(\text{d})} \underline{\perp}_\nu^\mu {}^{(\text{d})} \bar{j}^\nu = 0, \quad {}^{(\text{d})} \underline{\perp}_\nu^\mu {}^{(\text{d})} \bar{w}_{[r]}^{\nu \nu_2 \dots} = 0.$$

Thus decomposing the full gradient operator  $\nabla_\mu$  into tangential and orthogonally projected parts  ${}^{(\text{d})} \bar{\nabla}_\mu = {}^{(\text{d})} \eta^\nu_\mu \nabla_\nu$  and  ${}^{(\text{d})} \underline{\perp}_\mu^\nu \nabla_\nu$ , one sees that the gauge invariance condition for the action takes the form

$$\sum_{\text{d}} \int {}^{(\text{d})} \text{d} \bar{\mathcal{S}} \left( \alpha {}^{(\text{d})} \bar{\nabla}_\nu {}^{(\text{d})} \bar{j}^\nu + \chi_{\nu_2 \dots} {}^{(\text{d})} \bar{\nabla}_\nu {}^{(\text{d})} \bar{w}_{[r]}^{\nu \nu_2 \dots} \right) = \sum_{\text{d}} \int {}^{(\text{d})} \text{d} \bar{\mathcal{S}} {}^{(\text{d})} \bar{\nabla}_\nu \left( {}^{(\text{d})} \bar{j}^\nu \alpha + {}^{(\text{d})} \bar{w}_{[r]}^{\nu \nu_2 \dots} \chi_{\nu_2 \dots} \right),$$

in which integrands on the right are pure d-surface divergences.

For any  $d$ -dimensional support surface  ${}^{(d)}\bar{\mathcal{S}}$ , Green's theorem gives

$$\int {}^{(d)}d\bar{\mathcal{S}} {}^{(d)}\bar{\nabla}_\nu {}^{(d)}\bar{j}^\nu = \oint {}^{(d-1)}d\bar{\mathcal{S}} {}^{(d)}\lambda_\nu {}^{(d)}\bar{j}^\nu, \quad (43)$$

taking the integral on the right over the boundary  $(d-1)$ -surface of  $\partial {}^{(d)}\bar{\mathcal{S}}$  of  ${}^{(d)}\bar{\mathcal{S}}$ , where  ${}^{(d)}\lambda_\nu$  is the (uniquely defined) outward directed unit tangent vector on the  $d$ -surface at its  $(d-1)$ -dimensional boundary.

The gauge invariance condition on the action can thereby be reduced to

$$\sum_p \int {}^{(p)}d\bar{\mathcal{S}} \left\{ \alpha \left( {}^{(p)}\bar{\nabla}_\nu {}^{(p)}\bar{j}^\nu - \sum_{d=p+1} {}^{(d)}\lambda_\nu {}^{(d)}\bar{j}^\nu \right) + \chi_{\nu_2 \dots} \left( {}^{(p)}\bar{\nabla}_\nu {}^{(p)}\bar{w}_{[r]}^{\nu\nu_2 \dots} - \sum_{d=p+1} {}^{(d)}\lambda_\nu {}^{(d)}\bar{w}_{[r]}^{\nu\nu_2 \dots} \right) \right\} = 0$$

where, for any  $p$ -dimensionally supported  $(p-1)$ -brane, the summation “over  $d=p+1$ ” is to be understood as consisting of a contribution from each  $(p+1)$ -dimensionally supported  $p$ -brane attached to it.

The Maxwell gauge invariance condition (independence of  $\alpha$ ) is thus seen to be equivalent to the electric current conservation condition

$${}^{(p)}\overline{\nabla}_{\mu} {}^{(p)}\overline{j}^{\mu} = \sum_{d=p+1} {}^{(d)}\lambda_{\mu} {}^{(d)}\overline{j}^{\mu}, \quad (44)$$

which means that the source of charge injection into any particular  $(p-1)$ -brane is the sum of the currents flowing in from the  $p$ -branes to which it is attached. The generalised Kalb-Ramond gauge invariance condition (independence of  $\chi_{\nu_2 \dots \nu_r}$ ) can similarly be seen to be equivalent to the analogous (generalised vorticity) flux conservation condition

$${}^{(p)}\overline{\nabla}_{\mu} {}^{(p)}\overline{w}_{[r]}^{\mu\nu_2 \dots \nu_r} = \sum_{d=p+1} {}^{(d)}\lambda_{\mu} {}^{(d)}\overline{w}_{[r]}^{\mu\nu_2 \dots \nu_r}. \quad (45)$$

## 2.4 Force and the stress balance equation

The condition of being “Lagrangian” means that  $\delta_{\mathbf{L}}$  is comoving as needed to be meaningful for fields with support confined to a particular brane. However for background fields one can also define an “Eulerian” variation,  $\delta_{\mathbf{E}}$ , with respect to some appropriately fixed reference system, in which the infinitesimal displacement of the brane complex is specified by a vector field  $\xi^\mu$ . The difference will be given by

$$\delta_{\mathbf{L}} - \delta_{\mathbf{E}} = \vec{\xi}\mathcal{L}, \quad (46)$$

where the  $\vec{\xi}\mathcal{L}$  is the Lie differentiation operator, which will be given for the relevant background fields by  $\vec{\xi}\mathcal{L}A_\mu = \xi^\rho \nabla_\rho A_\mu + A_\rho \nabla_\mu \xi^\rho$ ,  $\vec{\xi}\mathcal{L}B_{\mu\nu_2\dots}^{[r]} = \xi^\rho \nabla_\rho B_{\mu\nu_2\dots}^{[r]} + r B_{\rho[\nu_2\dots} \nabla_{\mu]} \xi^\rho$  and  $\vec{\xi}\mathcal{L}g_{\mu\nu} = 2\nabla_{(\mu} \xi_{\nu)}$ .

In a fixed Eulerian background, the background fields will have Lagrangian variations given just by their Lie derivatives with respect to the displacement  $\xi^\mu$ . Subject to the internal field equations, the action variation  $\delta\mathcal{I}$  due to the displacement of the branes will therefore just be  $\sum_{\text{d}} \int^{(\text{d})} \text{d}\bar{\mathcal{S}} \left( {}^{(\text{d})}\bar{j}^\nu \xi^\rho \mathcal{L} A_\nu + \frac{1}{r!} {}^{(\text{d})}\bar{w}_{[r]}^{\mu\nu\dots} \xi^\rho \mathcal{L} B_{\mu\nu\dots}^{[r]} + \frac{1}{2} {}^{(\text{d})}\bar{T}^{\mu\nu} \xi^\rho \mathcal{L} g_{\mu\nu} \right)$ . The postulate that this vanishes for any  $\xi^\mu$  entails the further d-surface tangentiality restriction  ${}^{(\text{d})}\bar{\perp}_\nu^\mu {}^{(\text{d})}\bar{T}^{\nu\rho} = 0$  and the requirement

$$\begin{aligned} & \sum_{\text{d}} \int^{(\text{d})} \text{d}\bar{\mathcal{S}} \left\{ \xi^\rho \left( F_{\rho\nu} {}^{(\text{d})}\bar{j}^\nu + \frac{1}{r!} N_{\rho\mu\nu\dots}^{[r+1]} {}^{(\text{d})}\bar{w}_{[r]}^{\mu\nu\dots} \right. \right. \\ & \quad \left. \left. - {}^{(\text{d})}\bar{\nabla}_\mu {}^{(\text{d})}\bar{T}^\mu{}_\rho - A_\rho {}^{(\text{d})}\bar{\nabla}_\mu {}^{(\text{d})}\bar{j}^\mu - \frac{1}{(r-1)!} B_{\rho\nu\dots}^{[r]} {}^{(\text{d})}\bar{\nabla}_\mu {}^{(\text{d})}\bar{w}_{[r]}^{\mu\nu\dots} \right) \right. \\ & \quad \left. + {}^{(\text{d})}\bar{\nabla}_\mu \left( \xi^\rho \left( A_\rho {}^{(\text{d})}\bar{j}^\mu + \frac{1}{(r-1)!} B_{\rho\nu\dots}^{[r]} {}^{(\text{d})}\bar{w}_{[r]}^{\mu\nu\dots} + {}^{(\text{d})}\bar{T}^\mu{}_\rho \right) \right) \right\} = 0 \quad , \quad (47) \end{aligned}$$

in which the final contribution is a pure surface divergence that can be dealt with using Green's theorem as before.

The conclusion is that invariance of the action under arbitrary displacements  $\xi^\mu$  of the brane complex entails the dynamic equations

$${}^{(p)}\bar{\nabla}_\mu {}^{(p)}\bar{T}^\mu{}_\rho = {}^{(p)}f_\rho, \quad (48)$$

in which total force density,  ${}^{(p)}f_\rho = {}^{(p)}\bar{f}_\rho + {}^{(p)}\check{f}_\rho$ , includes

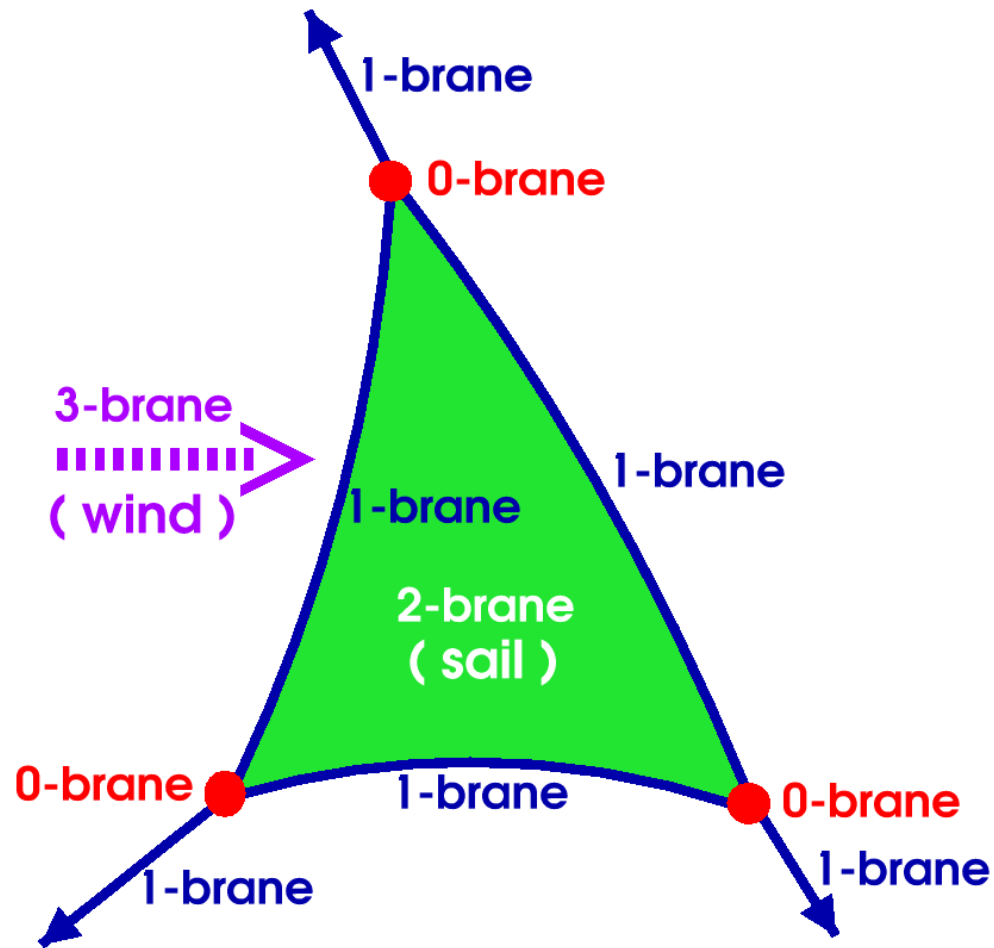
Faraday-Lorenz and Joukowski-Magnus contributions from the

background, as given by  ${}^{(p)}\bar{f}_\rho = F_{\rho\mu} {}^{(p)}\bar{j}^\mu + \frac{1}{r!} N_{\rho\mu\nu\dots}^{[r+1]} {}^{(p)}\bar{w}_{[r]}^{\mu\nu\dots}$ .

On each  $(p-1)$ -brane, the contact force exerted by attached  $p$ -branes is

$${}^{(p)}\check{f}_\rho = \sum_{d=p+1} {}^{(d)}\lambda_\mu {}^{(d)}\bar{T}^\mu{}_\rho, \quad (49)$$

in which it is to be recalled that, on the  $(p+1)$ -dimensional support surface of each attached  $p$ -brane,  ${}^{(d)}\lambda_\mu$  is the unit vector that is directed normally towards the bounding  $(p-1)$ -brane.



Nautical archetype of a regular brane complex in which a 3-brane (the wind) acts (by pressure discontinuity) on a 2-brane (the sail) hemmed by three 1-branes (bolt ropes) terminating on 0-branes (shackles) that are held in place by three more (free) 1-branes (external stay/sheet ropes).

## 2.5 The equation of extrinsic motion

The tangential force balance equations will hold as identities when the internal field equations are satisfied (because a surface tangential displacement has no effect). The non-redundent information governing the extrinsic motion of a  $(p - 1)$ -brane will be given just by the orthogonal part. Integrating by parts, as the surface gradient of the rank- $(n - p)$  orthogonal projector  ${}^{(p)}\underline{\perp}^{\mu}_{\nu}$  will be given in terms of the second fundamental tensor  ${}^{(p)}K_{\mu\nu}{}^{\rho}$  of the  $p$ -surface by  ${}^{(p)}\overline{\nabla}_{\mu} {}^{(p)}\underline{\perp}^{\nu}_{\rho} = - {}^{(p)}K_{\mu\nu}{}^{\rho} - {}^{(p)}K_{\mu}{}^{\rho}{}_{\nu}$ , the extrinsic equations of motion are finally obtained in the form

$${}^{(p)}\overline{T}^{\mu\nu} {}^{(p)}K_{\mu\nu}{}^{\rho} = {}^{(p)}\underline{\perp}^{\rho}_{\mu} {}^{(p)}f^{\mu}. \quad (50)$$

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### 3 Canonical Symplectic Structure

**Abstract** The covariant canonical variational procedure leading to the construction of a conserved bilinear symplectic current was originally developed in the context of field theory by Witten, Zuckerman, and others [1, 2, 3, 4, 5, 6, 7]. The following notes describe the generalisation of this procedure to brane mechanics in the manner initiated by Cartas-Fuentevilla [8, 9] and developed in collaboration with Dani Steer [10]. After a general presentation, including a review of the relationships between the various (Lagrangian, Eulerian and other) relevant kinds of variation, the procedure is illustrated by application to a particular category including the case of branes of purely elastic type.

## 3.1 Canonical formalism for Branes

Consider a generic conservative  $p$ -brane model whose mechanical evolution is governed by an action integral of the form

$$\mathcal{I} = \int \mathcal{L} d^{p+1}\sigma, \quad (51)$$

over a supporting worldsheet with internal co-ordinates  $\sigma^i$  ( $i = 0, 1, \dots, p$ ), and induced metric  $\eta_{ij} = g_{\mu\nu} x_{,i}^{\mu} x_{,j}^{\nu}$  in a background with coordinates  $x^{\mu}$ , ( $\mu = 0, 1, \dots, n - 1$ ), ( $n \geq p + 1$ ) and (flat or curved) space-time metric  $g_{\mu\nu}$ .

The relevant Lagrangian scalar density is given by  $\mathcal{L} = \|\eta\|^{1/2} L$ , where  $L$  is scalar function of a set of field components  $q^A$  – including background coords – and of their surface derivatives,

$$q_{,i}^A = \partial_i q^A = \partial q^A / \partial \sigma^i.$$

The relevant field variables  $q^A$  can be of internal or external kind, the most obvious example of the latter kind being the background coordinates  $x^\mu$  themselves.

The generic action variation,  $\delta\mathcal{L} = \mathcal{L}_A \delta q^A + p_A^i \delta q_{,i}^A$ , specifies partial derivative components  $\mathcal{L}_A$  and corresponding generalised momentum components  $p_A^i$ . The variation principle characterises dynamically admissible “on shell” configurations by the vanishing of the Eulerian derivative

$$\frac{\delta\mathcal{L}}{\delta q^A} = \mathcal{L}_A - p_{A,i}^i. \quad (52)$$

In terms of this Eulerian derivative, the generic Lagrangian variation will have the form

$$\delta \mathcal{L} = \frac{\delta \mathcal{L}}{\delta q^A} \delta q^A + (p_A^i \delta q^A)_{,i}. \quad (53)$$

There will be a corresponding pseudo-Hamiltonian scalar density

$$\mathcal{H} = p_A^i q^A_{,i} - \mathcal{L}, \quad (54)$$

for which

$$\delta \mathcal{H} = q^A_{,i} \delta p_A^i - \mathcal{L}_A \delta q^A. \quad (55)$$

(The covariance of such a pseudo-Hamiltonian distinguishes it from the ordinary kind of Hamiltonian, which depends on the introduction of some preferred time foliation.)

For an on-shell configuration, i.e. when the dynamical equations

$$\frac{\delta \mathcal{L}}{\delta q^A} = 0, \quad (56)$$

are satisfied, the Lagrangian variation will reduce to a pure surface divergence,

$$\delta \mathcal{L} = \left( p_A^i \delta q^A \right)_{,i}, \quad (57)$$

and the corresponding on-shell pseudo-Hamiltonian variation will take the form

$$\delta \mathcal{H} = q_{,i}^A \delta p_A^i - p_{A,i}^i \delta q^A. \quad (58)$$

## 3.2 Symplectic structure

The generic first order variation of the Lagrangian will be given by

$$\delta \mathcal{L} = \frac{\delta \mathcal{L}}{\delta q^A} \delta q^A + \vartheta^i{}_{,i} . \quad (59)$$

in terms of the generalised Liouville 1-form (on the configuration space cotangent bundle) defined by  $\vartheta^i = p_A^i \delta q^A$ .

Now consider a pair of successive independent variations  $\delta'$ ,  $\delta''$ , which will give a second order variation of the form

$$\delta'' \delta' \mathcal{L} = \delta'' \left( \frac{\delta \mathcal{L}}{\delta q^A} \right) \delta' q^A + \frac{\delta \mathcal{L}}{\delta q^A} \delta'' \delta' q^A + \left( \delta'' p_A^i \delta' q^A + p_A^i \delta'' \delta' q^A \right)_{,i} . \quad (60)$$

Thus using the commutation relation  $\delta\delta = \delta\delta$  one gets

$$\delta\left(\frac{\delta\mathcal{L}}{\delta q^A}\right)\delta q^A - \delta\left(\frac{\delta\mathcal{L}}{\delta q^A}\right)\delta q^A = \hat{\omega}^i{}_{,i}, \quad (61)$$

where the symplectic 2-form (on the configuration space cotangent bundle) is defined by  $\hat{\omega}^i = \delta p_A^i \delta q^A - \delta p_A^i \delta q^A$ .

For an on-shell perturbation we thus obtain

$$\frac{\delta\mathcal{L}}{\delta q^A} = 0 \quad \Rightarrow \quad \delta\mathcal{L} = \vartheta^i{}_{,i}, \quad (62)$$

while for a pair of on-shell perturbations we obtain

$$\delta\left(\frac{\delta\mathcal{L}}{\delta q^A}\right) = \delta\left(\frac{\delta\mathcal{L}}{\delta q^A}\right) = 0 \quad \Rightarrow \quad \hat{\omega}^i{}_{,i} = 0. \quad (63)$$

The foregoing surface current conservation law is expressible in shorthand as

$$\varpi^i{}_{,i} = 0, \quad (64)$$

in which the closed (since manifestly exact) symplectic 2-form (59) is specified in concise wedge product notation as

$$\varpi^i = \delta \wedge \vartheta^i = \delta p_A^i \wedge \delta q^A. \quad (65)$$

Some authors prefer to use an even more concise notation system in which it is not just the relevant distinguishing (in our case acute and grave accent) indices that are omitted but even the wedge symbol  $\wedge$  that indicates the antisymmetrised product relation. However such an extreme level of abbreviation is dangerous [8] in contexts in which symmetric products are also involved.

### 3.3 Translation into strictly tensorial form.

To avoid the gauge dependence involved in the use of auxiliary structures such as local frames and internal surface coordinates, by working [11] just with quantities that are strictly tensorial with respect to the background space, one needs to replace the surface current densities whose components  $\mathcal{V}^i$  and  $\mathcal{W}^i$  depend on the choice of the internal coordinates  $\sigma^i$ , by vectorial quantities with strictly tensorial background coordinate components given by

$$\Theta^\nu = \|\eta\|^{-1/2} x^\nu_{,i} \mathcal{V}^i, \quad \Omega^\nu = \|\eta\|^{-1/2} x^\nu_{,i} \mathcal{W}^i,$$

and with strictly scalar divergences given by

$$\overline{\nabla}_\nu \Theta^\nu = \|\eta\|^{-1/2} \mathcal{V}^i_{,i}, \quad \overline{\nabla}_\nu \Omega^\nu = \|\eta\|^{-1/2} \mathcal{W}^i_{,i}.$$

In terms of the surface projected covariant differentiation operator defined in terms of the fundamental tensor

$\eta^{\mu\nu} = \eta^{ij} x_{,i}^{\mu} x_{,j}^{\nu}$  by  $\bar{\nabla}_{\nu} = \eta^{\mu}_{\nu} \nabla_{\mu}$ , one thus obtains a

Liouville current conservation law of the form

$$\bar{\nabla}_{\nu} \Theta^{\nu} = 0 \quad (66)$$

for any symmetry generating perturbation, i.e. for any infinitesimal variation  $\delta q^A$  such that  $\delta \mathcal{L} = 0$ .

Similarly a symplectic current conservation law of the form

$$\bar{\nabla}_{\nu} \Omega^{\nu} = 0 \quad (67)$$

will hold for any pair of perturbations that are on-shell, i.e. such that  $\delta(\delta \mathcal{L} / \delta q^A) = 0$ .

## 3.4 Covariant variation formulae

If the field  $q^A$  is defined over the background – not just confined to brane worldsheet with internal coordinates  $\sigma^i$  – then in terms of the relevant displacement vector,  $\xi^\mu = \delta x^\mu$ , with respect to a given (e.g. Minkowski type) system of background coordinates in terms of which  $\partial_i q^A = x^\mu_{,i} \partial_\mu q^A$ , the simple worldsheet based field component variation  $\delta q^A$  will be given by

$$\delta q^A = \delta_E q^A + \xi^\mu \partial_\mu q^A . \quad (68)$$

where  $\delta_E q^A$  is the relevant Eulerian variation, as defined with respect to the background system.

When one is dealing with a background field that is not simply a scalar but of a more general tensorial nature, it will commonly be desirable to go on to convert the Eulerian variation formula  $\delta_{\text{E}} = \delta - \vec{\xi} \cdot \partial$  into terms of covariant derivation as given by  $\vec{\xi} \cdot \nabla = \vec{\xi} \cdot \partial + \{\vec{\xi} \cdot \Gamma\}$  where  $\{\vec{\xi} \cdot \Gamma\}$  is a purely algebraic operator involving contractions with the 2-index quantity  $(\vec{\xi} \cdot \Gamma)^\mu{}_\nu = \xi^\rho \Gamma_{\rho\nu}{}^\mu$ , as exemplified, for a vectorial (e.g. Killing) field  $k^\mu$ , or a covectorial (e.g. Maxwellian) form  $A_\mu$ , by  $\{\vec{\xi} \cdot \Gamma\} k^\mu = (\vec{\xi} \cdot \Gamma)^\mu{}_\nu k^\nu$ , and  $\{\vec{\xi} \cdot \Gamma\} A_\mu = -(\vec{\xi} \cdot \Gamma)^\nu{}_\mu A_\nu$ .

Instead of using the connection dependent covariant derivative, one can work with the corresponding Lie derivative, as given by a prescription of the form  $\vec{\xi}\mathcal{L} = \vec{\xi} \cdot \nabla - \{\nabla\xi\}$ , in which the operator  $\{\nabla\xi\}$  acts by contractions with the displacement gradient tensor  $\nabla_\nu\xi^\mu$ , in the way exemplified respectively for a vector  $k^\mu$ , or a 1-form (i.e. covector)  $A_\mu$ , by the formulae  $\{\nabla\xi\}k^\mu = k^\nu\nabla_\nu\xi^\mu$ , and  $\{\nabla\xi\}A_\mu = -A_\nu\nabla_\mu\xi^\nu$ . It can be seen that connection cancels out, so that the Lie derivative will be equivalently expressible in terms just of partial derivative components  $\partial_\nu\xi^\mu$  as

$$\vec{\xi}\mathcal{L} = \vec{\xi} \cdot \partial - \{\partial\xi\}. \quad (69)$$

Another kind of variation that is particularly important in the context of brane mechanics – because (unlike the Eulerian, covariant, and Lie derivatives) it is always well defined even for fields whose support is confined to the brane worldsheet – is what is known as the Lagrangian variation, meaning change with respect to background coordinates that are dragged by displacement. In the case of a field that is not confined to the brane worldsheet, so that its Eulerian variation is well defined, this latter kind will be related to the corresponding Lagrangian variation by the well known Lie derivation formula

$$\delta_{\mathbf{L}} = \delta_{\mathbf{E}} + \vec{\xi} \mathcal{L} . \quad (70)$$

Yet another possibility that may be useful is to express the Eulerian (fixed background point) variation in the form

$$\delta_{\mathbf{E}} = \delta_{\Gamma} - \vec{\xi} \cdot \nabla, \quad (71)$$

where parallelly transported variation is defined – not just for background field, but also for tensor confined to brane – by

$$\delta_{\Gamma} = \delta + \{\vec{\xi} \cdot \Gamma\}, \quad (72)$$

using the operator notation introduced above. This parallel variation  $\delta_{\Gamma}$  shares with the Lagrangian variation  $\delta_{\mathbf{L}}$  the important property of being well defined not just for background fields but also for fields whose support is confined to the brane worldsheet.

The Lagrangian variation  $\delta_{\mathbf{L}}$  will always be expressible directly in terms of the corresponding parallel variation  $\delta_{\mathbf{\Gamma}}$  by a relation of the form

$$\delta_{\mathbf{L}} = \delta_{\mathbf{\Gamma}} - \{\nabla \vec{\xi}\}, \quad (73)$$

in which it can be seen that connection dependence cancels out, leaving an expression of the simple form

$$\delta_{\mathbf{L}} = \delta - \{\partial \vec{\xi}\}, \quad (74)$$

where the action of the algebraic operator  $\{\partial \vec{\xi}\}$  is exemplified for a vector  $k^\mu$ , or a covector  $A_\mu$ , by the formulae  $\{\partial \vec{\xi}\}k^\mu = k^\nu \partial_\nu \xi^\mu$ , and  $\{\partial \vec{\xi}\}A_\mu = -A_\nu \partial_\mu \xi^\nu$ .

In conclusion of this overview of the relationships between the various kinds of infinitesimal variations that are commonly useful, it is to be mentioned that in literature dealing with purely non relativistic contexts in which it is possible (though not necessarily wise) to work exclusively with space coordinates of strictly Cartesian (orthonormal) type, the variations of the kind referred to here as “parallel” are generally described as “Lagrangian” by many authors. That usage does not necessarily lead to confusion, because for scalars the distinction does not arise, and because such authors systematically eschew the use (and the technical advantages) of Lagrangian variations of the fully comoving kind (that is considered here) by working exclusively with tensor components that are evaluated in terms only of orthonormal frames.

### 3.5 Evaluation via Lagrangian variations.

In typical applications, the relevant set of configuration components  $q^A$  will include a set of brane field components  $\varphi^\alpha$  as well as the background coords  $x^\mu$ , so that in terms of displacement vector

$\xi^\mu = \delta x^\mu$  the Liouville current will take the form

$$\Theta^\nu = \|\eta\|^{-1/2} x_{,i}^\nu (p_\alpha^i \delta\varphi^\alpha + p_\mu^i \xi^\mu) = \pi_\alpha^\nu \delta\varphi^\alpha + \pi_\mu^\nu \xi^\mu, \text{ in}$$

which the latter version replaces the original momentum components by the corresponding background tensorial momentum variables, which are

$$\text{given by } \pi_\alpha^\nu = \|\eta\|^{-1/2} x_{,i}^\nu p_\alpha^i \text{ and } \pi_\mu^\nu = \|\eta\|^{-1/2} x_{,i}^\nu p_\mu^i.$$

To obtain an analogously tensorial formula for the symplectic current 2-form, it is convenient, as a first step, to take advantage of the symmetry property  $\Gamma_{\mu \rho}^{\nu} = \Gamma_{\rho \mu}^{\nu}$ , of the Riemannian background connection, which allows substitution of parallel variation

$\delta_{\Gamma} p_{\mu}^i = \delta p_{\mu}^i - \Gamma_{\mu \rho}^{\nu} p_{\nu}^i \xi^{\rho}$  for  $\delta p_{\mu}^i$  so as to provide an expression of the form

$$\Omega^{\nu} = \|\eta\|^{-1/2} x_{,i}^{\nu} (\delta p_{\alpha}^i \wedge \delta \varphi^{\alpha} + \delta_{\Gamma} p_{\mu}^i \wedge \xi^{\mu}). \quad (75)$$

The next step is to evaluate the relevant momentum variations in terms of the corresponding Lagrangian variations, using the formulae

$$\|\eta\|^{-1/2} x_{,i}^{\nu} \delta p_{\alpha}^i = \delta_{\underline{L}} \pi_{\alpha}^{\nu} + \pi_{\alpha}^{\nu} \overline{\nabla}_{\rho} \xi^{\rho}, \quad (76)$$

and

$$\|\eta\|^{-1/2} x_{,i}^{\nu} \delta p_{\mu}^i = \delta_{\underline{L}} \pi_{\mu}^{\nu} - \pi_{\rho}^{\nu} \nabla_{\mu} \xi^{\rho} + \pi_{\mu}^{\nu} \overline{\nabla}_{\rho} \xi^{\rho}. \quad (77)$$

The advantage of Lagrangian variations is their convenience for relating the relevant intrinsic physical quantities via the appropriate equations of state.

## 3.6 Application to hyperelastic case

The hyperelastic category [12] (generalising the case of an ordinary elastic solid which includes the special case of an ordinary barotropic perfect fluid) consists of brane models in which – with respect to a suitably comoving internal reference system  $\sigma^i$  – there are no independent surface fields at all – meaning that the  $\varphi^\alpha$  and the  $p_\alpha{}^i$  are absent – and in which the only relevant background field is the metric  $g_{\mu\nu}$  that is specified as a function of the external coordinates  $x^\mu$ . In any such case, the generic variation of the Lagrangian is determined just by the surface stress momentum energy density tensor  $\bar{T}^{\mu\nu}$  according to the standard prescription

$\delta\mathcal{L} = \frac{1}{2} \|\eta\|^{1/2} \bar{T}^{\mu\nu} \delta g_{\mu\nu}$ , whereby  $\bar{T}^{\mu\nu}$  is specified in terms of partial derivation of the action density with respect to the metric.

In a fixed background (i.e. in the absence of any Eulerian variation of the metric) the Lagrangian variation of the metric will be given, according to the formula (70), by  $\delta_{\underline{L}} g_{\mu\nu} = \vec{\xi} \mathcal{L} g_{\mu\nu} = 2 \nabla_{(\mu} \xi_{\nu)}$ . Comparing this to canonical prescription  $\delta \mathcal{L} = \mathcal{L}_{\mu} \xi^{\mu} + p_{\mu}^i \xi^{\mu}_{,i}$  with  $\xi^{\mu} = \delta x^{\mu}$  shows that the relevant partial derivatives will be given by the (non-tensorial) formulae  $\mathcal{L}_{\mu} = \|\eta\|^{1/2} \Gamma_{\mu}^{\nu} \bar{T}_{\nu}^{\rho}$  and  $p_{\mu}^i = \|\eta\|^{1/2} \bar{T}_{\mu\nu} \eta^{ij} x^{\nu}_{,j}$ .

It can thus be seen that in the hyperelastic case, the canonical momentum tensor  $\pi_{\mu}^{\nu}$  and the Liouville current  $\Theta^{\nu}$  will be given just in terms of surface stress tensor  $\bar{T}^{\mu\nu}$  by the very simple formulae

$$\pi_{\mu}^{\nu} = \bar{T}_{\mu}^{\nu}, \quad \Theta^{\nu} = \bar{T}_{\mu}^{\nu} \xi^{\mu}. \quad (78)$$

In order to proceed, we must consider the second order metric variation, whereby (following Friedman and Schutz [13]) the hyper Cauchy tensor (generalised elasticity tensor)  $\bar{\mathfrak{E}}^{\mu\nu\rho\sigma} = \bar{\mathfrak{E}}^{\rho\sigma\mu\nu}$  is specified [14] in terms of Lagrangian variations by a partial derivative relation of the form

$$\delta_{\underline{L}} \left( \|\eta\|^{1/2} \bar{T}^{\mu\nu} \right) = \|\eta\|^{1/2} \bar{\mathfrak{E}}^{\mu\nu\rho\sigma} \delta_{\underline{L}} g_{\rho\sigma} . \quad (79)$$

The symplectic current is thereby obtained in the form

$$\Omega^\nu = \mathfrak{D}_\mu^\nu \wedge \xi^\mu , \quad (80)$$

where

$$\mathfrak{D}_\mu^\nu = 2 \bar{\mathfrak{E}}_{\mu\rho}^{\nu\sigma} \bar{\nabla}_\sigma \xi^\rho + \bar{T}^{\nu\rho} \bar{\nabla}_\rho \xi_\mu . \quad (81)$$

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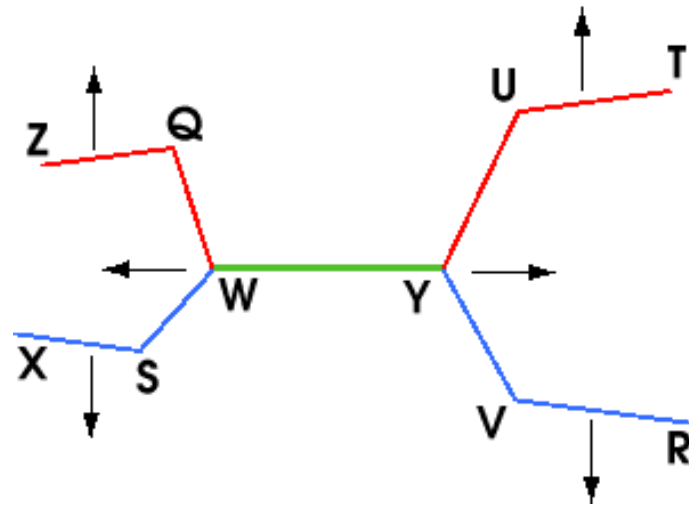
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## 4 Application to string junctions and intercommutations



The prototype application is to a point particle, labelled by 0, at the junction between strings labelled by an index  $j$  that runs from 1 to 2 for a V junction, or from 1 to 3 for a Y junction (or even 1 to 4 for an X junction). The particle position  $x^\mu$  will have proper time derivative  $\dot{x}^\mu$  and acceleration  $\ddot{x}^\mu = \dot{x}^\nu \nabla_\nu \dot{x}^\mu$  given, with  $\dot{x}^\nu \ddot{x}_\nu = 0$ , by

$$\eta_{0\nu}^\sigma \nabla_\sigma \bar{T}_{0\mu}^\nu = F_{\mu\nu} \bar{j}_0^\nu + \sum_j \lambda_{j\nu} \bar{T}_{j\mu}^\nu, \quad (82)$$

in conjunction with the charge conservation law

$$\eta_{0\nu}^\sigma \nabla_\sigma \bar{j}_0^\nu = \sum_j \lambda_{j\nu} \bar{j}_j^\nu. \quad (83)$$

where the  $\lambda_{j\nu}$  are outward directed string tangent vectors, subject to orthonormality conditions  $\lambda_j^\nu \lambda_{j\nu} = 1$ ,  $\dot{x}^\nu \lambda_{j\nu} = 0$ ,  $\dot{x}^\nu \dot{x}_\nu = -1$ .

The zero-brane fundamental tensor, energy tensor, and charge vector are given in terms of the particle mass  $m_0$  and charge number  $z_0$  by

$$\eta_0^{\mu\nu} = -\dot{x}^\mu \dot{x}^\nu, \quad \bar{T}_0^{\mu\nu} = m_0 \dot{x}^\mu \dot{x}^\nu, \quad \bar{j}_0^\nu = e z_0 \dot{x}^\nu,$$

where  $e$  is a charge coupling constant, while  $K_{0\mu\nu}{}^\rho = -\eta_{0\mu\nu} \ddot{x}^\rho$ .

We thus get  $\dot{m}_0 \dot{x}_\mu + m_0 \ddot{x}_\mu - e z_0 F_{\mu\nu} \dot{x}^\nu = \sum_j \lambda_{j\nu} \bar{T}_{j\mu}^\nu$ ,

and  $e \dot{z}_0 = \sum_j \lambda_{j\nu} \bar{j}_j^\nu$ , while as a consequence we shall have

$\dot{m}_0 = -\dot{x}^\mu \sum_j \lambda_{j\nu} \bar{T}_{j\mu}^\nu$ . If no strings attached, right hand sides will drop out, so the proper time derivatives  $\dot{m}_0$  and  $\dot{z}_0$  must vanish, leaving a dynamic equation of the familiar form  $m_0 \ddot{x}_\mu = e z_0 F_{\mu\nu} \dot{x}^\nu$ .

Our concern here is with the opposite extreme, for which the left hand sides drop out in the absence of a substantial particle at the junction,

i.e. when  $m_0 = 0$  and  $z_0 = 0$ , so that we shall be left with the

junction conditions  $\sum_j \lambda_{j\nu} \bar{T}_{j\mu}^\nu = 0$  and  $\sum_j \lambda_{j\nu} \bar{j}_j^\nu = 0$ .

For any string model the 2 dimensional worldsheet will have unit surface bivector and (first) fundamental tensor  $\eta^\mu{}_\nu = \mathcal{E}^\mu{}_\rho \mathcal{E}^\rho{}_\nu$  expressible in terms of any orthonormal tangent diad  $u^\nu, \tilde{u}^\nu$  as  $\mathcal{E}^{\mu\nu} = 2 u^{[\mu} \tilde{u}^{\nu]}$  and  $\eta^\mu{}_\nu = -u^\mu u^\nu + \tilde{u}^\mu \tilde{u}^\nu$ . The symmetric surface stress energy tensor will be expressible in the form

$$\bar{T}^{\mu\nu} = \beta_+^{(\mu} \beta_-^{\nu)}. \quad (84)$$

in terms of a pair of bicharacteristic vectors having the generically timelike form  $\beta_\pm^\mu = \sqrt{\mathcal{U}} u^\mu \pm \sqrt{\mathcal{T}} \tilde{u}^\mu$ , in terms of a preferred diad such that

$$\bar{T}^{\mu\nu} = \mathcal{U} u^\mu u^\nu - \mathcal{T} \tilde{u}^\mu \tilde{u}^\nu \quad (85)$$

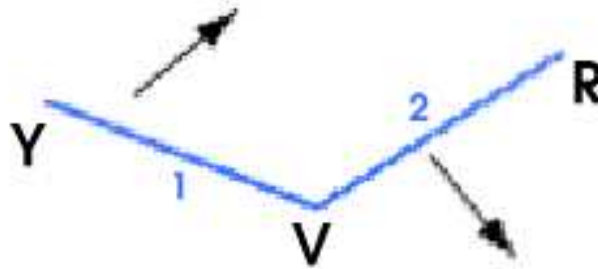
where  $\mathcal{U}$  is the surface energy density and  $\mathcal{T}$  the string tension.

The magnitudes of the bicharacteristic vectors will then be given by  $\beta_{+\mu}\beta_+{}^\mu = \beta_{-\nu}\beta_-{}^\nu = -(\mathcal{U} - \mathcal{T})$ , while their scalar product would in any case be given by  $\beta_{+\mu}\beta_-{}^\mu = \bar{T}^\nu{}_\nu = -(\mathcal{U} + \mathcal{T})$ .

At the junction, for the  $j$ th string, we shall have  $u_j^\nu = \gamma_j(\dot{x}^\nu - v_j\lambda_j^\nu)$  and  $\tilde{u}_j^\nu = \gamma_j(\lambda_j^\nu - v_j\dot{x}^\nu)$  where  $\gamma_j = (1 - v_j^2)^{-1/2}$ , so with  $\beta_{j\pm}^\nu = \beta_{j\pm}^0\dot{x}^\nu + \beta_{j\pm}^1\lambda_j^\nu$  its force contribution will be

$$\begin{aligned} \bar{T}_j^{\mu\nu}\lambda_{j\nu} &= \gamma_j^2(\mathcal{U}_j v_j^2 - \mathcal{T}_j)\lambda_j^\nu - \gamma_j^2 v_j(\mathcal{U}_j - \mathcal{T}_j)\dot{x}^\nu, \\ &= \beta_{j+}^1\beta_{j-}^1\lambda_j^\mu + \beta_{j+}^{(1)}\beta_{j-}^{(0)}\dot{x}^\mu. \end{aligned} \quad (86)$$

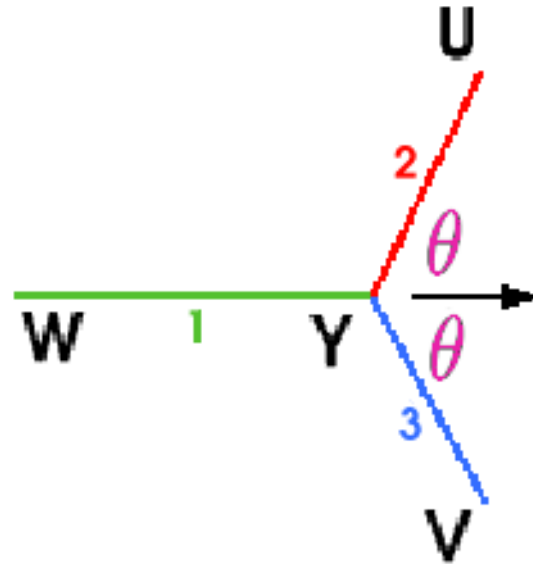
At a V junction between just 2 string segments in different directions,  $\lambda_1^\mu \neq -\lambda_2^\mu$ , the coefficient of the first term in (86) and thus one of its factors, must vanish:  $\beta_{j+}^1 = 0$  say, so the junction worldline is itself bicharacteristic:  $\dot{x}^\nu \propto \beta_{j+}^\nu$ .



On either side of a V junction (kink) between string segments YV and VR the relative flow speed  $v$  must satisfy the condition  $v^2 = c_E^2$  where  $c_E = \sqrt{T/U}$  is the extrinsic (wiggle) propagation speed. The force balance condition is then just that there be no jump discontinuity in the energy transmission rate, which means

$$\left[ \sqrt{UT} \right]_1^2 = 0. \quad (87)$$

If the string state depends on electric and/or other surface currents, their conservation conditions will just be equivalent to continuity of the relevant variables.



Now consider a Y junction (bifurcation) where a first string segment WY splits into two branches YV and YU , which (for simplicity) we suppose to be symmetric, both deviating by the same angle  $\theta$  from the direction of WY in the junction rest frame, so that  $\lambda_2^\mu + \lambda_3^\nu = -2 \cos \theta \lambda_1^\nu$ . Dropping the suffices 2,3 for the symmetrically related branches, the force balance conditions will be expressible as

$$\gamma_1^2 v_1 (\mathcal{U}_1 - \mathcal{T}_1) = -2 \gamma^2 v (\mathcal{U} - \mathcal{T}), \quad (*)$$

and

$$\gamma_1^2 (\mathcal{U}_1 v_1^2 - \mathcal{T}_1) = 2 \cos \theta \gamma^2 (\mathcal{U} v^2 - \mathcal{T}). \quad (a)$$

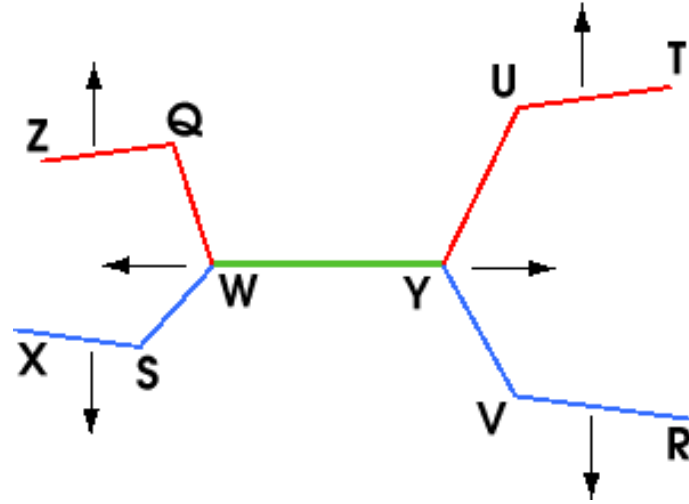
If there is a current  $\nu^\mu = \nu \mathbf{u}^\mu$  of particles that are conserved not just internally but also at junctions, then in terms of the specific enthalpy,  $h = (\mathcal{U} - \mathcal{T})/\nu$ , the energy conservation condition (\*) will reduce to

$$\gamma_1 h_1 = \gamma h, \quad (b)$$

by the condition  $\sum_j \lambda_{j\nu} \nu_j^\nu = 0$ , which, for symmetric Y-junction, is

$$\gamma_1 \nu_1 \nu_1 = -2 \gamma \nu \nu. \quad (c)$$

In the ordinary elastic case, for which  $\mathcal{U}$  depends just on  $\mathcal{T}$ , such a number density  $\nu$  and the corresponding chemical potential or effective mass  $\mu = d\mathcal{U}/d\nu$ , are specifiable by  $\ln\{\nu\} = \int d\mathcal{U}/(\mathcal{U} - \mathcal{T})$ , with the identification  $\mu = h$ . This gives two conserved currents,  $\nu^\mu = \nu \mathbf{u}^\mu$  and  $\tilde{\mu}^\nu = \mu \tilde{\mathbf{u}}^\nu$  (of which one or other will be identifiable, in the electromagnetic case envisaged by Witten, with the current  $\mathbf{z}^\nu$  above). The two equations (b) and (c) suffice to determine the change in velocity and state in the elastic case, and the preceding equation (a) then fixes the angle  $\theta$ .



As in hep-th/0601153, using frame with unit vectors

$e_t^\mu$ ,  $e_x^\mu$ ,  $e_y^\mu$ ,  $e_z^\mu$ , consider symmetric collision of strings ZT and XR with directions deviating from  $x$  direction by angles  $\pm\alpha$  in  $y$  direction, and velocities  $\pm v_z$  in  $z$  direction, after formation of connecting segment WY at rest in the  $x$  direction. In terms of time  $t$  and internal space coordinate  $\sigma$ , the unperturbed segments ZQ and UT of the first string will thus have position given by

$$x^\mu = t \left( e_t^\mu + v_z e_z^\mu \right) + \sigma \left( \cos \alpha e_x^\mu + \sin \alpha e_y^\mu \right),$$

with preferred internal frame given by  $u^\mu = \gamma_z \left( e_t^\mu + v_z e_z^\mu \right)$ .

Relative to the preferred frame, the kink U has characteristic speed  $c_E$ , and so is given by  $\sigma = \gamma_z^{-1} c_E t$ . This gives the unit bicharacteristic :  $\beta_+^\nu \propto u_+^\nu = \gamma_E \gamma_z (e_t^\mu + v_z e_z^\nu) + c_E \gamma_E (\cos \alpha e_x^\nu + \sin \alpha e_y^\nu)$ .

The connecting string segment WY will have tangent vector  $\lambda_1^\nu = \gamma_1 (e_x^\nu + v_1 e_t^\nu)$  normal to the worldline of the junction Y, of which the tangent  $\dot{x}^\nu = \gamma_1 (e_t^\nu + v_1 e_x^\nu)$  is also, like  $u_+^\nu$ , tangent to the segment YU, in which, with

$\gamma_+ = -\dot{x}_\nu u_+^\nu = \gamma_E \gamma_1 (\gamma_z - c_E v_1 \cos \alpha)$ , the normal frame vector will be  $\lambda^\nu = (\gamma_+^2 - 1)^{-1/2} (\gamma_+ \dot{x}^\nu - u_+^\nu)$ . The preferred frame vector with speed  $c_E$  relative to  $u_+^\nu$  in YU is thus

$u^\nu = \gamma_E (\gamma_+ - c_E \sqrt{\gamma_+^2 - 1}) (\dot{x}^\nu - v \lambda^\nu)$  with velocity

$$v = (v_+ - c_E) / (1 - c_E v_+) \quad \text{where} \quad v_+ = \sqrt{1 - \gamma_+^{-2}},$$

which is needed for (b) and (c). Finally the angle needed for (a) is given

by  $\cos \theta = -\lambda_1^\nu \lambda_\nu = \gamma_E \gamma_1 (\gamma_+^2 - 1)^{-1/2} (c_E \cos \alpha - v_1 \gamma_z)$ .

Eliminating  $\theta$ ,  $v$ , and its Lorentz factor  $\gamma = \gamma_E \gamma_+ (1 - c_E v_+)$ , from (\*) and (a) thus gives the two conditions

$$\begin{aligned} \gamma_1^2 v_1 (\mathcal{U}_1 - \mathcal{T}_1) &= 2\gamma_+^2 (c_E - v_+) (1 - c_E v_+) \mathcal{U}, \text{ and} \\ \gamma_1 (\mathcal{U}_1 v_1^2 - \mathcal{T}_1) &= 2\gamma_E \gamma_+ ((1 + c_E^2) v_+ - 2c_E) (c_E \cos \alpha - v_1 \gamma_z) \mathcal{U}, \end{aligned}$$

which suffices for determination on WY of  $v_1$  and  $\mathcal{T}_1$  provided the latter is given as a function of  $\mathcal{U}_1$  by some equation of state.

A prototypical example is provided by warm string model for thermal distribution of wiggles with temperature  $\Theta$  and entropy density  $s$  on an underlying Nambu-Goto model with Kibble mass  $m$  ( $= \sqrt{\hbar} \text{“}\mu\text{”}$ ) meaning that  $\mathcal{T} = \mathcal{U} = m^2/\hbar$ , for which macroscopic averaging gives

$$\frac{\hbar \mathcal{T}}{m^2} = \frac{m^2}{\hbar \mathcal{U}} = c_E = \sqrt{1 - \frac{2\pi \Theta^2}{3 m^2}} = \left(1 + \frac{3\hbar^2 s^2}{2\pi m^2}\right)^{-1/2}. \quad (88)$$

In the ultrarelativistic Nambu-Goto limit,  $\mathcal{U} - \mathcal{T} \rightarrow 0$ ,  $c_E \rightarrow 1$ ,  $\gamma_E = (1 - c_E^2)^{-1/2} \rightarrow \infty$ , there is no current and remaining Y junction condition is (a), which reduces to  $\mathcal{T}_1 = 2 \cos \theta \mathcal{T}$  (as for ordinary static equilibrium) while preceding formula simply gives  $\cos \theta = (\cos \alpha - v_1 \gamma_z) / (\gamma_z - v_1 \cos \alpha)$ . We thus recover the Copeland-Kibble-Steer formula for speed of Y junction along x axis :

$$v_1 = \frac{2\mathcal{T} \cos \alpha - \mathcal{T}_1 \gamma_z}{2\mathcal{T} \gamma_z - \mathcal{T}_1 \cos \alpha}. \quad (89)$$

In generic elastic case,  $\mathcal{T}_1$  is not fixed, but depends on internal state of connecting string, WY. This will be determined by 2 more equations, (b) and (c). That would be OK for static equilibrium with an adjustable angle, but it over-determines the case of a dynamic collision with  $\alpha$  given in advance. So (as in an ordinary shock) treatment of such a collision will generically require use of a non-conservative model!

As 2-dim analogue of ordinary perfect fluid, non-conservative string models have energy density  $\mathcal{U}$  depending, not just on conserved number density  $\nu$ , but also on another number density  $s$  representing entropy, subject to  $\gamma_1 \nu_1 s_1 + 2 \gamma \nu s \geq 0$ . Thus (generalising warm string limit with  $\nu = 0$ ) the generic variation,  $d\mathcal{U} = \mu d\nu + \Theta ds$ , specifies chemical potential  $\mu$  (i.e. effective mass per particle) and a thermal potential  $\Theta$  (i.e temperature) on the string, whose tension will be  $\mathcal{T} = \mathcal{U} - \mu\nu - \Theta s$ , while enthalpy per particle in (b) will then be  $h = \mu + \Theta s/\nu$ . The Y junction condition is given, for  $\iota = 2$ , by

$$(\mathcal{T}_1 - \iota \cos \theta \mathcal{T}) \left( \frac{h_1}{\nu_1} + \frac{h}{\iota \nu} \right) = (h^2 - h_1^2) \frac{\iota \nu h_1 - \cos \theta \nu_1 h}{\iota \nu h_1 - \nu_1 h}. \quad (90)$$

(Taub shock condition in unbent string given by  $\iota = 1$  with  $\theta = 0$ .)

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# 5 Brane perturbation by external radiation fields

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## 5-1. Motivation

This section will show how the force density exerted on a brane by linearised interaction with radiation fields of commonly considered (gravitational, electromagnetic, axionic and dilatonic type) is generally describable by a relatively simple expression that is well behaved for fields of purely external origin. This work is needed for the subsequent treatment, in Section 6, of the problem of divergent behaviour that arises from self-interaction via corresponding fields of internal origin, for branes of codimension 2 (the hyperstring case) or more. For such cases (including ordinary strings in 4 dimensional spacetime) it will be shown how the necessary regularisation can be conveniently performed using a method based on the use of an appropriately regularised gradient operator given by  $\widehat{\nabla}_\nu = \overline{\nabla}_\nu + \frac{1}{2}K_\nu$  where  $\overline{\nabla}_\nu$  is the usual tangential gradient operator and  $K_\nu$  is the extrinsic curvature vector.

## 5-2. Linearised long range radiation fields.

As well as being restricted to the degenerate Goto-Nambu case, for which  $\mathcal{T} = \mathcal{U}$ , most early work on string self interaction considered only the physically exotic but technically simple effects of axion type coupling, which in 4 dimensions is of pseudo scalar type, mediated [13] by an antisymmetric Kalb-Ramond gauge field, having components  $B_{\mu\nu}$  (with respect to local background spacetime coordinates  $x^\mu$  say) like that of ordinary relativistic fluid mechanics [14], with coupling as introduced into superstring theory by Wess and Zumino, but previously known in aerofoil theory from the work of Joukowski. The fluid model for the axion case is that of Zel'dovich's "stiff" limit, with perturbation propagation at the speed of light. Such axion coupling is treated here as a special case, with  $q = 2$ , of a multi index antisymmetric field  $B_{\mu_1 \dots \mu_q}$ . We also consider dilaton coupling, as mediated by a scalar field  $\phi$ , for which, as observed by Buonano and Damour [15], the early literature was less reliable.

The other fields considered here are the physically familiar cases of ordinary electromagnetic interactions, as governed by Maxwell's equations for a gauge field  $A_\mu$ , and weak gravitational interactions as governed by the linearised Einstein equations for a small amplitude but perhaps rapidly varying perturbation  $h_{\mu\nu} = \delta g_{\mu\nu}$  of a slowly varying background spacetime metric  $g_{\mu\nu}$  satisfying the vacuum equation of vanishing Ricci tensor,

$$\mathcal{R}_{\mu\nu} = 0. \quad (91)$$

The electromagnetic and gravitational interactions share with the simpler axionic and dilatonic prototypes the essential property of being governed – with a suitable choice of gauge – by propagation equations that will be of simple D'Alembertian kind in the weak field limit for which the background metric  $g_{\mu\nu}$  is not just empty but actually flat.

In the formulation of the theory it is essential to distinguish the total,

$$g_{\text{E}\mu\nu} = g_{\mu\nu} + h_{\mu\nu}, \quad (92)$$

which is known as the Einstein metric, from the conformally related Dicke metric given by

$$g_{\text{D}\mu\nu} = e^{2\phi} g_{\text{E}\mu\nu}, \quad (93)$$

where  $\phi$  is the dilatonic field scalar. Our present treatment is based on linearisation not just with respect to  $A_\mu$  and to  $B_{\mu_1 \dots \mu_q}$  but also with respect to  $h_{\mu\nu}$  and to  $\phi$ , which is assumed here to be small, so that it will suffice to take the Dicke metric to be

$$g_{\text{D}\mu\nu} = g_{\mu\nu} + h_{\mu\nu} + 2\phi g_{\mu\nu}. \quad (94)$$

The Dalembertian propagation property on which our work will depend is attributable to the form of the  $n$ -dimensional action contribution

$$\mathcal{I}_r = \int \hat{L}_r \|g\|^{1/2} d^n x, \quad (95)$$

that governs the behaviour of the long range radiation, where  $\hat{L}_r$  is the relevant kinetic contribution the Lagrangian density of the relevant fields, which it will be convenient to express in terms of a unit system that has been rationalised in the usual way, so as to simplify the explicit form of the solution for a static point source.

For a given (empty)  $n$ -dimensional background metric  $g_{\mu\nu}$ , the radiation action density  $\hat{L}_r$  will consist of a sum of contributions that are homogeneously quadratic functions of the relevant gradient fields.

The relevant gravitational field tensor and dilatonic field vector are

$$h_{\mu\nu;\rho} = \nabla_{\rho} h_{\mu\nu}, \quad \phi_{;\rho} = \nabla_{\rho} \phi, \quad (96)$$

while the electromagnetic flux 2-form and the axionic type flux  $(q + 1)$ -form are defined by the usual exterior derivation formulae

$$F_{\mu\nu} = 2\nabla_{[\mu} A_{\nu]}, \quad N_{\mu_0\mu_1\dots\mu_q} = (q + 1)\nabla_{[\mu_0} B_{\mu_1\dots\mu_q]}, \quad (97)$$

using square brackets to denote index antisymmetrisation, where  $\nabla_{\rho}$  is the operator of covariant differentiation as specified with respect to the connection specified by the empty background metric  $g_{\mu\nu}$ . The choice of connection does not matter for the definition of  $\phi_{;\mu}$ , nor (due to the antisymmetrisation) to that of  $F_{\mu\nu}$  and  $N_{\mu_1\dots\mu_q}$ , but it does matter for that of the gravitational field tensor  $h_{\mu\nu;\rho}$ . It is to be understood that the same empty background metric  $g_{\mu\nu}$  is used for this, and for index raising and lowering.

The rationalisation is achieved by the introduction of the relevant numerical denominator  $\Omega_d$  defined as the surface area of a unit  $d$ -sphere where  $d=n-2$  in a spacetime of dimension  $n$ , so that for ordinary spacetime with  $n=4$  we shall have  $d=2$ . The values needed for higher dimensional applications are obtainable using the recurrence relation

$$\Omega_{d+1} = \frac{2\pi}{d} \Omega_{d-1} \quad (98)$$

from the prototype case of a circle, as given by  $\Omega_1 = 2\pi$ , and from the case relevant to ordinary spacetime, namely a 2-sphere, as given by  $\Omega_2 = 4\pi$ . The general result is thus given for odd and even dimensions respectively by

$$\Omega_{2j+1} = 2\pi^{j+1}/j!, \quad \Omega_{2j} = 2^{2j}\pi^j(j-1)!/(2j-1)!. \quad (99)$$

In units such that, as well as the speed of light  $c$ , the Dirac Planck constant  $\hbar$  is also set to unity, the long range radiation field contribution to the action density will thus be expressible in terms of constant mass weighting parameters,  $M_G$ ,  $M_D$ ,  $M_F$ ,  $M_{[q]}$ , , in the form

$$\hat{L}_r = \frac{1}{2\Omega_{n-2}} \left\{ \frac{M_G^{n-2}}{2} \mathcal{R}_E^{(2)} - M_D^{n-2} \phi_{;\mu} \phi_{;\mu} - \frac{M_F^{n-4}}{2} F^{\mu\nu} F_{\mu\nu} - \frac{M_{[q]}^{n-2-2q}}{(q+1)!} N^{\mu_0 \dots \mu_q} N_{\mu_0 \dots \mu_q} \right\}$$

where  $\mathcal{R}_E^{(2)}$  is the residual Ricci scalar contribution, as obtained from the measure weighted Ricci scalar of the Hilbert action of the Einstein metric (92) by expanding to quadratic order in  $h_{\mu\nu}$  and subtracting off a divergence to remove second order derivatives, which leaves

$$\mathcal{R}_E^{(2)} = \frac{1}{2} h^{\mu\nu;\rho} \left( h_{\rho\mu;\nu} - \frac{1}{2} h_{\mu\nu;\rho} - g_{\mu\nu} h^{\rho\sigma}{}_{;\sigma} + \frac{1}{2} g_{\mu\nu} h^\sigma{}_{\sigma;\rho} \right). \quad (100)$$

(This formula is not as well known as it ought to be : it is absent even from the comprehensive treatise of Misner, Thorne, and Wheeler [16].)

The constant mass scale parameters,  $M_G$ ,  $M_D$ ,  $M_F$ ,  $M_{[q]}$ , should not be confused with the masses of the corresponding bosonic particles namely gravitons, dilatons, photons, and generalised axions. These are all usually assumed to be very small, and are taken here to vanish in order for the associated fields to be qualifiable as long range. In ordinary spacetime with  $n=4$ , the first of these parameters, namely the gravitational coupling mass  $M_G$ , will be identifiable with the ordinary Planck mass  $M_P = G^{-1/2}$  where  $G$  is Newton's constant. The analogous quantity  $M_D$  is what may be called the Dicke mass, which is usually supposed to be very large, at least comparable with the gravitational mass. In ordinary spacetime with  $n=4$ , the application to the solar system provides severe observational limits that can be interpreted [17] as implying that the dimensionless Brans-Dicke parameter  $\omega = 2GM_D^2 + 3/2$  should be very large compared with unity, and hence that the dilatation mass scale  $M_D$  itself should be large compared with the Planck mass.

The quantity  $M_F$  is what (in memory of the first great pioneer of electromagnetic unification) might appropriately be termed the Faraday mass scale. In higher dimensional applications derived from superstring theory this quantity is commonly found to be comparable with the gravitational mass scale, but in applications to ordinary spacetime with  $n=4$  it plays no role at all, since in this case the dimensions are such that it entirely cancels out. The other mass scale,  $M_{[q]}$ , is what, in the usual case  $q=3$ , might appropriately be referred to as the Joukowski coupling mass (in memory of the pioneering aerofoil theorist who first brought such coupling to light). In the relevant 4-dimensional applications, for which the corresponding unrationalised pseudo-scalar axion coupling constant, as used by Battye and Shellard [18, 19], will be given by  $f_a^2 = 2\pi M_{[3]}^2$ , the value of this axion coupling mass scale is usually supposed to be considerably below the Planck mass scale.

To obtain the ensuing field equations it is necessary to work out the Eulerian derivatives with respect to the respective fields on which the action contributions depend. These Eulerian derivatives (whose vanishing is the condition for the field equations to be satisfied in the source free case) are evidently given for the dilaton contribution by

$$\frac{\delta}{\delta\phi} \left( -\phi^{;\rho}\phi_{;\rho} \right) = 2\nabla_{\rho}\nabla^{\rho}\phi, \quad (101)$$

and for the electromagnetic contribution by the familiar expression

$$\frac{\delta}{\delta A_{\mu}} \left( -F^{\rho\sigma}F_{\rho\sigma} \right) = 4\nabla_{\rho}F^{\rho\mu}. \quad (102)$$

For the generalised axionic contribution one obtains

$$\frac{\delta}{\delta B_{\mu_1 \dots \mu_q}} \left( - N^{\rho_0 \dots \rho_q} N_{\rho_0 \dots \rho_q} \right) = 2(q + 1) \nabla_{\rho} N^{\rho \mu_1 \dots \mu_q}, \quad (103)$$

while for the gravitational contribution, with a little more work, using the vacuum property (91) of the background metric, one gets

$$\frac{\delta}{\delta h_{\mu\nu}} \mathcal{R}_{\text{E}}^{(2)} = \frac{1}{2} \nabla_{\rho} \nabla^{\rho} \check{h}^{\mu\nu} - \nabla_{\rho} \nabla^{(\mu} \check{h}^{\nu)\rho} + \frac{1}{2} g^{\mu\nu} \nabla_{\rho} \nabla_{\sigma} \check{h}^{\rho\sigma}, \quad (104)$$

using the notation

$$\check{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h_{\rho}^{\rho}. \quad (105)$$

It is evident that both the field action integral and the Eulerian derivatives are unaffected by the gauge transformations

$$A_\mu \mapsto A_\mu + \nabla_\mu \chi, \quad B_{\mu_1 \mu_2 \dots \mu_q} \mapsto B_{\mu_1 \mu_2 \dots \mu_q} + q \nabla_{[\mu_1} \alpha_{\mu_2 \dots \mu_q]}, \quad (106)$$

for an arbitrary scalar field  $\chi$  and an arbitrary antisymmetric covector field  $\alpha_{\mu_2 \dots \mu_q}$ , while subject to the background emptiness condition (91) it can be checked that such invariance also holds for gravitational gauge transformations of the form

$$h_{\mu\nu} \mapsto h_{\mu\nu} + 2\nabla_{(\mu} \xi_{\nu)}, \quad (107)$$

using round brackets to denote index symmetrisation, for an arbitrary displacement vector field  $\xi^\mu$ .

### 5-3. Linearised interactions

We consider cases in which linearised long range fields with kinetics governed by the Lagrangian (101) interact linearly with a material system via a Lagrangian contribution provided by a master function  $\hat{\Lambda}$  say, that depends, in a not necessarily linear manner, on a set of internal fields representing quantities such as currents, as well as on the background metric. We consider only the unpolarised case, for which the dependence on the metric is purely algebraic. The linearity of the interaction means that the interaction Lagrangian,  $\hat{L}_c$  say, has the form

$$\hat{L}_c = \hat{j}^\mu A_\mu + \frac{1}{q!} \hat{w}^{\mu_1 \dots \mu_q} B_{\mu_1 \dots \mu_q} + \frac{1}{2} \hat{T}^{\mu\nu} h_{\mu\nu} + \hat{T} \phi, \quad (108)$$

where the coefficients  $\hat{j}^\mu$ ,  $\hat{w}^{\mu_1 \dots \mu_q}$ ,  $\hat{T}^{\mu\nu}$ , and  $\hat{T}$  depend only on the background metric and internal fields involved in  $\hat{\Lambda}$ .

The total action of this underlying model thus takes the form

$$\mathcal{I}_t = \mathcal{I}_r + \mathcal{I} \quad (109)$$

with

$$\mathcal{I} = \int \hat{L} \|g\|^{1/2} d^n x, \quad (110)$$

using the notation

$$\hat{L} = \hat{\Lambda} + \hat{L}_c. \quad (111)$$

Although it was originally motivated by applications to the (most practically interesting) case of ordinary spacetime, for which  $n=4$ , the results developed here are particularly suitable for applications of a more speculative kind to cases for which the background dimension  $n$  has higher values (including the values  $n=10$  and  $n=11$  that are of particular academic interest in the context of superstring theory and M-theory).

The field equations are given by invariance with respect to local variation of the linear external fields  $A_\mu$ ,  $B_{\mu_1 \dots \mu_q}$ ,  $h_{\mu\nu}$ ,  $\phi$ , and of any internal fields that may be (in general non-linearly) involved in the specification of  $\hat{\Lambda}$  and also of the sources  $\hat{j}^\mu$ ,  $\hat{w}^{\mu_1 \dots \mu_q}$ ,  $\hat{T}^{\mu\nu}$ , and  $\hat{T}$ .

Invariance under local transformations  $A_\mu \mapsto A_\mu + \nabla_\mu \chi$ , for arbitrary  $\chi$ , requires the electromagnetic current  $\hat{j}^\mu$  to be conserved,

$$\nabla_\mu \hat{j}^\mu = 0. \quad (112)$$

Analogous invariance when  $B_{\mu_1 \mu_2 \dots \mu_q} \mapsto B_{\mu_1 \mu_2 \dots \mu_q} + q \nabla_{[\mu_1} \alpha_{\mu_2 \dots \mu_q]}$ , imposes on  $\hat{w}^{\mu_1 \dots \mu_q}$  (describable, in view of the ordinary fluid case [14], as a generalised vorticity flux) to satisfy the conservation law

$$\nabla_\mu \hat{w}^{\mu \mu_2 \dots \mu_q} = 0. \quad (113)$$

Unlike the electromagnetic and axionic source terms,  $\hat{j}^\mu$  and  $\hat{w}^{\mu_1 \dots \mu_q}$  the gravitational source term  $\hat{T}^{\mu\nu}$  is not something whose choice admits any latitude. In order for the theory under consideration to be considered as the linearisation of a generally covariant model in which the gravitational field equations are obtained by requiring invariance with respect to variations of the total (Einstein) metric, it can be seen from the form of (92) – bearing in mind the metric dependence of the measure  $\|g\|^{1/2}$  in (110) – that this gravitational source term must be given by the geometric stress momentum energy tensor as defined by

$$\hat{T}^{\mu\nu} = 2\|g\|^{-1/2} \frac{\partial(\hat{\Lambda}\|g\|^{1/2})}{\partial g_{\mu\nu}} = 2 \frac{\partial \hat{\Lambda}}{\partial g_{\mu\nu}} + \hat{\Lambda} g^{\mu\nu}. \quad (114)$$

The scalar source coefficient  $\hat{T}$  might have diverse forms for various conceivable scalar coupling theories, but to qualify as dilatonic in the sense associated particularly with the work of Dicke, it must be derived from a model with matter Lagrangian specified in terms not of the Einstein metric, (92) but of the Dicke metric (94), which means that the coupling coefficient in (108) will be obtainable unambiguously as the trace

$$\hat{T} = \hat{T}_{\mu}^{\mu}. \quad (115)$$

In the modern context of models derived in the low energy limit from superstring theory or M-theory, the term dilatonic often refers to theories whose scalar couplings are rather more general than envisaged by Dicke, but in most such cases (although they may be relevant for the numerical evaluation of the dilaton coupling coefficient) such underlying technicalities do not modify the form of the linearised limit dealt with here.

The equations of motion for the internal fields of the material system will depend on  $\hat{\Lambda}$ , but whatever that may be, in an empty background of dimension  $n$ , using the usual Lorenz gauge condition provides an electromagnetic field equation of familiar D'Alembertian form :

$$\nabla^\mu A_\mu = 0 \Rightarrow \nabla_\sigma \nabla^\sigma A_\mu = -\Omega_{n-2} M_F^{4-n} \hat{j}_\mu, \quad (116)$$

where for  $n=4$  the coefficient is simply  $\Omega_2 = 4\pi$ . For the axionic field, the analogous gauge condition again gives the well known [13] D'Alembertian form :

$$\nabla^\mu B_{\mu\mu_2\dots\mu_q} = 0 \Rightarrow \nabla_\sigma \nabla^\sigma B_{\mu_1\dots\mu_q} = -\Omega_{n-2} M_{[q]}^{2q+2-n} \hat{w}_{\mu_1\dots\mu_q}. \quad (117)$$

The dilaton field equation is immediately obtainable in the simple form

$$\nabla_{\sigma} \nabla^{\sigma} \phi = -\Omega_{n-2} M_D^{2-n} \hat{T}, \quad (118)$$

but for the gravitational perturbation field, there is again a question of gauge. Assuming the empty  $n$ -dimensional background metric  $g_{\mu\nu}$  is flat (but without any restriction as to whether or not the coordinates are of Minkowski type) the de Donder gauge condition,

$$\nabla^{\mu} h_{\mu\nu} = \frac{1}{2} \nabla_{\nu} h^{\mu}_{\mu} \quad (119)$$

provides a gravitational field equation of the well known form

$$\nabla_{\sigma} \nabla^{\sigma} h_{\mu\nu} = -4\Omega_{n-2} M_G^{2-n} \left( \hat{T}_{\mu\nu} - \frac{1}{n-2} \hat{T} g_{\mu\nu} \right). \quad (120)$$

It is to be remarked that, whereas the pure gravitational radiation contribution is exactly gauge invariant, the gravitational coupling contribution would be strictly invariant with respect to the gauge transformations (107) only when  $\nabla_{\mu} \hat{T}^{\mu\nu}$  is exactly zero, which holds just in the limit of infinitely weak coupling.

This means that there will be a second order (and therefore, in the context of the linearised approximation, unimportant) discrepancy between the traditionally used (since computationally more convenient) gravitational source equation (120) and the equation that would be obtained by rigorous application of the formula (100).

## 5-4. Distributional Sources

A generic p-brane is characterised by (p+1)-dimensional worldsheet with imbedding given locally by  $x^\mu = \bar{x}^\mu \{ \sigma \}$  in terms of intrinsic coordinates  $\sigma^i$  ( $i = 0, 1, \dots, p$ ), so that the induced surface metric will have the form

$$\eta_{ij} = g_{\mu\nu} \bar{x}^\mu_{,i} \bar{x}^\nu_{,j} . \quad (121)$$

As in the more familiar case of point particle models, or “zero-branes”, the problem of ultraviolet divergences arises for higher dimensional “p-branes”, starting with string models, for which p=1, because in all these cases the relevant source densities  $\hat{j}^\mu$ ,  $\hat{w}^{\mu_2 \dots \mu_q}$ , and  $\hat{T}^{\mu\nu}$  are not regular functions but Dirac type distributions that vanish outside the relevant world sheets, except of course in the case of an ordinary medium, with the maximal space dimension p=n-1, where n is the background space-time dimension.

For any “p-branes”, the relevant source distributions will be expressible using the terminology of Dirac delta “functions” in the form exemplified in the case of the electromagnetic current by

$$\hat{j}^{\mu} = \|g\|^{-1/2} \int \bar{j}^{\mu} \delta^n[x - \bar{x}\{\sigma\}] \|\eta\|^{1/2} d^{p+1}\sigma, \quad (122)$$

in which  $\|\eta\|$  is the determinant of the induced metric (121), while the analogous expression for the generalised vorticity flux source is

$$\hat{w}^{\mu_1 \dots \mu_q} = \|g\|^{-1/2} \int \bar{w}^{\mu_1 \dots \mu_q} \delta^n[x - \bar{x}\{\sigma\}] \|\eta\|^{1/2} d^{p+1}\sigma, \quad (123)$$

where the surface current density  $\bar{j}^{\mu}$ , and the antisymmetric surface vorticity flux multivector  $\bar{w}^{\mu_1 \dots \mu_q}$  are regular tensorial fields on the worldsheet (but undefined off it).

For the stress momentum energy source one will similarly have

$$\hat{T}^{\mu\nu} = \|g\|^{-1/2} \int \bar{T}^{\mu\nu} \delta^n[x - \bar{x}\{\sigma\}] \|\eta\|^{1/2} d^{p+1}\sigma, \quad (124)$$

where the symmetric surface stress momentum energy density  $\bar{T}^{\mu\nu}$ , is a *regular* tensorial function on the worldsheet (but undefined off it).

The distributional nature of these source terms in cases for which  $p < n-1$  is a corollary of the similarly distributional nature of the action density  $\hat{L}$ , which is defined, according to (111), to include both the purely internal contribution  $\hat{\Lambda}$  and the cross coupling contribution  $\hat{L}_c$ .

The distributional action density will thus be expressible in the form

$$\hat{L} = \|g\|^{-1/2} \int \bar{L} \delta^n[x - \bar{x}\{\sigma\}] \|\eta\|^{1/2} d^{p+1}\sigma, \quad (125)$$

with

$$\bar{L} = \bar{\Lambda} + \bar{L}_c \quad (126)$$

where  $\bar{\Lambda}$  and  $\bar{L}_c$  are well behaved scalars on the worldsheet but undefined off it. The intrinsic Lagrangian  $\bar{\Lambda}$  will depend just on the relevant internal fields (such as currents) confined to the worldsheet, and on its induced metric. The cross coupling contribution will be given in terms of the confined fields  $\bar{j}^\mu$ ,  $\bar{w}^{\mu_2 \dots \mu_q}$ ,  $\bar{T}^{\mu\nu}$  and  $\bar{T} = \bar{T}_\mu{}^\mu$  by

$$\bar{L}_c = \bar{j}^\mu A_\mu + \frac{1}{q!} \bar{w}^{\mu_1 \dots \mu_q} B_{\mu_1 \dots \mu_q} + \frac{1}{2} \bar{T}^{\mu\nu} h_{\mu\nu} + \bar{T} \phi. \quad (127)$$

In terms of these well behaved worldsheet functions, the corresponding source contribution (110) to the action can be expressed directly, without recourse to heavy distributional machinery, as a simple  $(p+1)$ -surface integral in the form

$$\mathcal{I} = \int \bar{L} \|\eta\|^{1/2} d^{p+1}\sigma. \quad (128)$$

In particular, the regular surface stress energy tensor  $\bar{T}^{\mu\nu}$  needed for the purpose of applying (127) is obtainable directly from the world sheet master function, without the use of distributions, using the formula

$$\bar{T}^{\mu\nu} = 2\|\eta\|^{-1/2} \frac{\partial \bar{\Lambda} \|\eta\|^{1/2}}{\partial g_{\mu\nu}}. \quad (129)$$

Provided the current  $\bar{j}^\mu$  and the multivector  $\bar{w}^{\mu_2 \dots \mu_q}$  are appropriately constructed as automatically conserved fluxes, the corresponding terms in (108) will not introduce any extra metric dependence. The last two terms in (127) will however introduce an extra metric dependence whose treatment needs the use of the hyper-Cauchy tensor (a relativistic generalisation of the ordinary Cauchy elasticity tensor) defined by

$$\bar{\mathfrak{E}}^{\mu\nu\rho\sigma} = \|\eta\|^{-1/2} \frac{\partial}{\partial g_{\mu\nu}} \left( \bar{T}^{\rho\sigma} \|\eta\|^{1/2} \right), \quad (130)$$

or equivalently, in manifestly symmetric form, by

$$\bar{\mathfrak{E}}^{\mu\nu\rho\sigma} = \bar{\mathfrak{E}}^{\rho\sigma\mu\nu} = 2 \|\eta\|^{-1/2} \frac{\partial^2 (\bar{\Lambda} \|\eta\|^{1/2})}{\partial g_{\mu\nu} \partial g_{\rho\sigma}}. \quad (131)$$

If the relevant radiation fields  $A_\mu$ ,  $B_{\mu_1 \dots \mu_q}$ ,  $h_{\mu\nu}$  and  $\phi$  are just regular background fields attributable to external sources, the treatment of such a system will be straightforward, but this will not be the case for the radiation fields produced by the brane itself, since they will be singular just where their evaluation is needed. Their form can easily be obtained explicitly in the case of a flat  $p$ -brane in a flat  $n$ -dimensional spacetime, using Minkowski coordinates such that the string worldsheet embedding is given with respect to corresponding worldsheet Minkowski coordinates  $\sigma^i$  by  $x^i = \sigma^i$  for  $i = 0, 1, \dots, p$  and  $x^a = 0$  for  $a = p+1, \dots, n$ . In such a system the metric will have components

$$g_{ij} = \eta_{ij}, \quad g_{ia} = 0, \quad g_{ab} = \delta_{ab}, \quad (132)$$

where  $\eta_{ij}$  are the internal Minkowski metric components and  $\delta_{ab}$  are the components of the Kronecker unit matrix.

To obtain a Lorentz covariant formulation, it is convenient to work with the fundamental rank  $p+1$  tangential projection tensor  $\eta^\mu{}_\nu$  of the worldsheet, whose components in this system will be given by

$$\eta_{ij} = \eta_{ij}, \quad \eta_{ia} = 0, \quad \eta_{ab} = 0, \quad \text{and with the}$$

complementary orthogonal projection tensor  $\perp^\mu{}_\nu$  of rank  $n-p-1$ , whose components in this system will be given by

$$\perp_{ij} = 0, \quad \perp_{ia} = 0, \quad \perp_{ab} = \delta_{ab}. \quad \text{This will have the}$$

separate projection tensor properties  $\eta^\mu{}_\nu \eta^\nu{}_\rho = \eta^\mu{}_\rho$ ,  $\perp^\mu{}_\nu \perp^\nu{}_\rho = \perp^\mu{}_\rho$  as

well as being related by the complementarity conditions  $\perp^\mu{}_\nu \eta^\nu{}_\rho = 0$ ,

$\perp^\mu{}_\nu + \eta^\mu{}_\nu = g^\mu{}_\nu$ . It provides a Lorentz covariant expression whereby

the radial distance  $r$  from the worldsheet is given by  $r^2 = \perp_{\mu\nu} x^\mu x^\nu$ .

The corresponding formula for the outward radially directed unit vector

$r^{;\rho} = \nabla^\rho r$  will be

$$r^{;\rho} = r^{-1} \perp_\sigma^\rho x^\sigma . \quad (133)$$

The introduction of these fields enables the maximally symmetric (static, asymptotically vanishing) solution of the simple Dalemberertian equation (118) for a uniform  $p$ -brane supported stress energy distribution of the form (124) to be expressed in the form

$$\phi^{;\rho} = - \frac{\Omega_{n-2} M_D^{2-n}}{\Omega_{n-2-p} r^{n-1-p}} \overline{T} \perp_\sigma^\rho x^\sigma . \quad (134)$$

in which the angle factor  $\Omega_{n-2}/\Omega_{n-2-p}$  evidently reduces to unity in the case of a zero brane (i.e. a point source).

The analogous expression for the electromagnetic field is

$$A^{\mu;\rho} = -\frac{\Omega_{n-2} M_F^{4-n}}{\Omega_{n-2-p} r^{n-1-p}} \bar{j}^{\mu} \perp_{\sigma}^{\rho} x^{\sigma}, \quad (135)$$

while axionic field is given by

$$B^{\mu_1 \dots \mu_q; \rho} = -\frac{\Omega_{n-2} M_{[q]}^{2q+2-n}}{\Omega_{n-2-p} r^{n-1-p}} \bar{w}^{\mu_1 \dots \mu_q} \perp_{\sigma}^{\rho} x^{\sigma}. \quad (136)$$

Finally, the expression for the gravitational field is

$$h^{\mu\nu;\rho} = -4 \frac{\Omega_{n-2} M_D^{2-n}}{\Omega_{n-2-p} r^{n-1-p}} \left( \bar{T}_{\mu\nu} - \frac{1}{n-2} \bar{T} g_{\mu\nu} \right) \perp_{\sigma}^{\rho} x^{\sigma}. \quad (137)$$

## 5-5. Brane worldsheet geometry

For treating the dynamical equations ensuing from the action (128), it is useful to recapitulate the relevant the geometric machinery[20, 21] (whose unavailability at the time of the early work[22, 23] on the Goto-Nambu case accounts for some of the confusion that was not cleared until more recently [10, 12] ).

In the simple case of a point particle in an electromagnetic field  $F_{\mu\nu}$ , the dynamical equation takes the familiar form  $M \dot{u}_\mu = Q F_{\mu\nu} u^\nu$ , where  $M$  and  $Q$  are its constant as mass and charge,  $u^\mu = \dot{x}^\mu$  is the proper time derivative of position along its timelike worldline, i.e. the future directed tangent vector with unit normalisation,  $u^\mu u_\nu = -1$ , while the acceleration vector is  $\dot{u}^\mu = u^\nu \nabla_\nu u^\mu$ .

However for strings and higher branes, a less specialised kinematic description [6, 21] is needed.

Instead of the unique vectors  $u^\mu$  and  $\dot{u}_\mu$ , a generic brane with timelike worldsheet metric  $\eta_{ij}$  is described by its first fundamental tensor

$$\eta^{\mu\nu} = \eta^{ij} \bar{x}^\mu_{,i} \bar{x}^\nu_{,j}, \quad (138)$$

and the corresponding second fundamental tensor

$$K_{\mu\nu}{}^\rho = \eta^\sigma{}_\nu \bar{\nabla}_\mu \eta^\rho{}_\sigma, \quad (139)$$

as defined, using the tangential differentiation operator

$$\bar{\nabla}_\mu = \eta_\mu{}^\nu \nabla_\nu, \quad (140)$$

in such a way as to have the Weingarten symmetry property

$$K_{[\mu\nu]}{}^\rho = 0. \quad (141)$$

As well the non-trivial Weingarten symmetry property, which is the condition for worldsheet integrability, the first two indices have the property of tangentiality, while the last index is worldsheet orthogonal,

$$\perp_{\mu}^{\lambda} K_{\lambda\nu}{}^{\rho} = 0 = K_{\mu\nu}{}^{\sigma} \eta_{\sigma}{}^{\rho}, \quad (142)$$

where the orthogonal projection tensor is given by

$$\perp_{\nu}^{\mu} = g^{\mu}{}_{\nu} - \eta^{\mu}{}_{\nu}. \quad (143)$$

The trace of the second fundamental tensor is the curvature vector, which must evidently be worldsheet orthogonal,

$$K^{\rho} = K_{\mu}{}^{\mu\rho} = \overline{\nabla}_{\nu} \eta^{\nu\rho}, \quad \eta^{\rho}{}_{\sigma} K^{\sigma} = 0, \quad (144)$$

and which will vanish for free motion in the simple Dirac-Nambu-Goto case of a string or brane with no internal structure.

The first fundamental tensor can be used to rewrite the expressions (129) and (130) for the surface stress and hyper-Cauchy tensors more explicitly as

$$\bar{T}^{\mu\nu} = 2 \frac{\partial \bar{\Lambda}}{\partial g_{\mu\nu}} + \bar{\Lambda} \eta^{\mu\nu}. \quad (145)$$

and

$$\bar{\mathfrak{e}}^{\mu\nu\rho\sigma} = \frac{\delta \bar{T}^{\rho\sigma}}{\delta g_{\mu\nu}} + \frac{1}{2} \bar{T}^{\rho\sigma} \eta^{\mu\nu}. \quad (146)$$

Their tangentiality properties,

$$\perp_{\mu}^{\lambda} \bar{T}^{\mu\nu} = 0, \quad \perp_{\mu}^{\lambda} \bar{\mathfrak{e}}^{\mu\nu\rho\sigma} = 0. \quad (147)$$

contrast with the orthogonality property (144) of the curvature vector.

Assuming that the worldsheet supported fields  $\bar{j}^\mu$  and  $\bar{w}^{\mu_1 \dots \mu_q}$  are constructed from internal worldsheet fields in such a way as to acquire the corresponding tangentiality properties

$$\perp_{\mu}^{\lambda} \bar{j}^{\mu} = 0, \quad \perp_{\mu}^{\lambda} \bar{w}^{\mu_1 \dots \mu_q} = 0, \quad (148)$$

it can be seen that the corresponding distributional conservation laws (112) and (113) can be equivalently expressed in terms of tangentially projected differentiation as the regular worldsheet flux conservation laws

$$\bar{\nabla}_{\mu} \bar{j}^{\mu} = 0, \quad (149)$$

and

$$\bar{\nabla}_{\mu} \bar{w}^{\mu_1 \dots \mu_q} = 0. \quad (150)$$

## 5-6. The effect of local displacements.

The motion of the worldsheet is determined by action invariance with respect to local displacements  $\xi^\mu$ , whose effect can be conveniently analysed using a Lagrangian treatment in which not just the internal coordinates  $\sigma^i$  but also the background coordinates  $x^\mu$  are considered to be dragged along by the displacement. Thus the relevant field variations are given just by Lie derivatives with respect to the vector field  $\xi^\mu$ . For the electromagnetic and axionic fields one obtains

$$\delta A_\mu = \xi^\sigma \nabla_\sigma A_\mu + A_\sigma \nabla_\mu \xi^\sigma, \quad (151)$$

$$\delta B_{\mu_1 \mu_2 \dots \mu_q} = \xi^\sigma \nabla_\sigma B_{\mu_1 \mu_2 \dots \mu_q} + q B_{\sigma [\mu_2 \dots \mu_q} \nabla_{\mu_1]} \xi^\sigma. \quad (152)$$

For the gravitational and dilatonic fields one obtains

$$\delta h_{\mu\nu} = \xi^\sigma \nabla_\sigma h_{\mu\nu} + 2h_{\sigma(\mu} \nabla_{\nu)} \xi^\sigma, \quad (153)$$

$$\delta\phi = \xi^\sigma \nabla_\sigma \phi, \quad (154)$$

while for the background metric itself one has the well known formula

$$\delta g_{\mu\nu} = 2\nabla_{(\mu} \xi_{\nu)}. \quad (155)$$

The postulate that the internal field equations are satisfied means that perturbations of the relevant internal fields have no effect on the action integral  $\mathcal{I}$ , with the implication that for the purpose of evaluating the variation  $\delta\mathcal{I}$  there will be no loss of generality taking the variations of these so far unspecified internal fields simply to be zero.

The worldsheet flux conservation laws (149) and (150) imply [20] that  $\bar{j}^\mu$  and  $\bar{w}^{\mu_1 \dots \mu_q}$  are related by Hodge type duality to the exterior derivatives of corresponding worldsheet differential forms (that will generically be of order  $p-1$  and  $p-q+1$  respectively). In the usual cases these differential forms will be among (or depend only on) the relevant independently variable internal fields whose variation is postulated to have no effect on the action  $\mathcal{I}$  when the internal field equations are satisfied, so when evaluating  $\delta\mathcal{I}$  these variations can without loss of generality be postulated to vanish. This means that the variations of the corresponding vector and bivector surface densities will also vanish, i.e. we shall have

$$\delta\left(\|\eta\|^{1/2}\bar{j}^\mu\right) = 0, \quad \delta\left(\|\eta\|^{1/2}\bar{w}^{\mu_1 \dots \mu_q}\right) = 0. \quad (156)$$

It follows that the electromagnetic and axionic contributions from (127) to the variation of the integrand in (128) will be given by

$$\delta \left( \|\eta\|^{1/2} \bar{j}^{\mu} A_{\mu} \right) = \|\eta\|^{1/2} \bar{j}^{\mu} \delta A_{\mu}, \quad (157)$$

and

$$\delta \left( \frac{\|\eta\|^{1/2}}{q!} \bar{w}^{\mu_1 \dots \mu_q} B_{\mu_1 \dots \mu_q} \right) = \frac{\|\eta\|^{1/2}}{q!} \bar{w}^{\mu_1 \dots \mu_q} \delta B_{\mu_1 \dots \mu_q}. \quad (158)$$

It is noteworthy that the variation of the background metric does not contribute to (157) or (158). The metric variation will however be of paramount importance for the evaluation of the corresponding contribution from the gravitational and dilatonic coupling terms in (126).

It can be seen from (146) that the gravitational contribution to the variation of the integrand in (128) will be given by the – until recently [3] not so well known – expression

$$\delta \left( \frac{1}{2} \|\eta\|^{1/2} \bar{T}^{\mu\nu} h_{\mu\nu} \right) = \frac{1}{2} \|\eta\|^{1/2} \left( \bar{T}^{\mu\nu} \delta h_{\mu\nu} + \bar{\mathcal{E}}^{\mu\nu\rho\sigma} h_{\rho\sigma} \delta g_{\mu\nu} \right), \quad (159)$$

while, despite its deceptively simple scalar nature, the dilatonic coupling gives rise to a corresponding contribution that works out to be given by the – even less well known – expression

$$\delta \left( \|\eta\|^{1/2} \bar{T} \phi \right) = \|\eta\|^{1/2} \left( \bar{T} \delta \phi + (\bar{T}^{\mu\nu} + \bar{\mathcal{E}}^{\mu\nu}) \phi \delta g_{\mu\nu} \right), \quad (160)$$

using the notation

$$\bar{\mathcal{E}}^{\mu\nu} = \bar{\mathcal{E}}^{\mu\nu\rho}{}_{\rho}. \quad (161)$$

Under the same conditions, i.e. assuming that the internal field equations are satisfied, the variation of the background metric will provide the only contribution from the intrinsic Lagrangian term in (126) to the variation of the integrand in (128). As can be seen from (129), this last contribution will simply be given by an expression of the familiar form

$$\delta \left( \|\eta\|^{1/2} \Lambda \right) = \frac{1}{2} \|\eta\|^{1/2} \overline{T}^{\mu\nu} \delta g_{\mu\nu} . \quad (162)$$

To evaluate the integrated effect of the contributions (157), (158), (160) and (162) the next step is the routine procedure of substituting the relevant Lie derivative formulae (151), (152), (153), (154) and (155) followed by absorption of the terms involving derivatives of the displacement fields into pure worldsheet current divergences.

Using the conservation properties (149) and (150) one gets

$$\bar{j}^\mu \delta A_\mu = \xi^\mu F_{\mu\nu} \bar{j}^\nu + \bar{\nabla}_\mu (\xi^\nu A_\nu \bar{j}^\mu), \quad (163)$$

for the contribution in (157) while that in (158) is given by

$$\begin{aligned} \frac{1}{q!} \bar{w}^{\mu_1 \dots \mu_q} \delta B_{\mu_1 \dots \mu_q} &= \frac{1}{q!} \xi^\mu N_{\mu \mu_1 \dots \mu_q} \bar{w}^{\mu_1 \dots \mu_q} \\ &+ \frac{1}{(q-1)!} \bar{\nabla}_\mu (\xi^\nu B_{\nu \mu_2 \dots \mu_q} \bar{w}^{\mu \mu_2 \dots \mu_q}). \end{aligned} \quad (164)$$

For the first term in (159) and in (160) one gets

$$\frac{1}{2} \bar{T}^{\mu\nu} \delta h_{\mu\nu} = \xi^\mu \left( \frac{1}{2} \bar{T}^{\nu\rho} \nabla_\mu h_{\nu\rho} - \bar{\nabla}_\nu (\bar{T}^{\nu\rho} h_{\mu\rho}) \right) + \bar{\nabla}_\mu (\xi^\nu \bar{T}^{\mu\rho} h_{\nu\rho}). \quad (165)$$

and

$$\bar{T} \delta \phi = \xi^\mu \bar{T} \nabla_\mu \phi. \quad (166)$$

For the other term — from the metric variation — in (159) one gets

$$\frac{1}{2} \bar{\mathfrak{e}}^{\mu\nu\rho\sigma} h_{\rho\sigma} \delta g_{\mu\nu} = -\xi^\mu \bar{\nabla}_\nu (\bar{\mathfrak{e}}^\nu{}_\mu{}^{\rho\sigma} h_{\rho\sigma}) + \bar{\nabla}_\mu (\xi^\nu \bar{\mathfrak{e}}^\mu{}_\nu{}^{\rho\sigma} h_{\rho\sigma}), \quad (167)$$

while the corresponding expression for the last term in (160) is

$$(\bar{T}^{\mu\nu} + \bar{\mathfrak{e}}^{\mu\nu}) \phi \delta g_{\mu\nu} = -\xi^\mu \bar{\nabla}_\nu \left( 2(\bar{T}^\nu{}_\mu + \bar{\mathfrak{e}}^\nu{}_\mu) \phi \right) + \bar{\nabla}_\mu \left( 2\xi^\nu (\bar{T}^\mu{}_\nu + \bar{\mathfrak{e}}^\mu{}_\nu) \phi \right). \quad (168)$$

Finally, as well as these less familiar contributions from from the metric variation, there is of course the ordinary intrinsic part, for which one obtains from (162) an expression of the usual form

$$\frac{1}{2} \bar{T}^{\mu\nu} \delta g_{\mu\nu} = -\xi^\mu \bar{\nabla}_\nu \bar{T}^\nu{}_\mu + \bar{\nabla}_\mu (\xi^\nu \bar{T}^\mu{}_\nu). \quad (169)$$

## 5-7. The force density formula.

We can now evaluate the variation of the action integral (128) due to a displacement confined to a bounded region of the worldsheet. The relevant  $(p+1)$ -dimensional version of Green's theorem removes the divergence contributions, leaving an expression of the standard form

$$\delta\mathcal{I} = \int \xi^\mu (\bar{f}_\mu - \bar{\nabla}_\nu \bar{T}^\nu{}_\mu) \|\eta\|^{1/2} d^{(p+1)}\sigma. \quad (170)$$

The variation principle thereby provides the corresponding dynamical equation, which takes the standard form

$$\bar{\nabla}_\nu \bar{T}^{\mu\nu} = \bar{f}^\mu, \quad (171)$$

in which the vector  $\bar{f}^\mu$  represents the total force density exerted by the various radiation fields involved.

Using the foregoing expressions, this force density can immediately be read out in the form

$$\overline{f}^{\mu} = \overline{f}_{\text{F}}^{\mu} + \overline{f}_{[q]}^{\mu} + \overline{f}_{\text{G}}^{\mu} + \overline{f}_{\text{D}}^{\mu}, \quad (172)$$

in which the contributions from the various fields involved are as follows. To start with the electromagnetic contribution can be seen from (163) to be given by an expression of the familiar Lorentz type form

$$\overline{f}_{\text{F}}^{\mu} = F^{\mu}_{\nu} \overline{j}^{\nu}, \quad (173)$$

while the axionic contribution can be seen from (163) to be given by an expression of the analogously simple form

$$\overline{f}_{[q]}^{\mu} = \frac{1}{q!} N^{\mu}_{\mu_1 \dots \mu_q} \overline{w}^{\mu_1 \dots \mu_q}, \quad (174)$$

which, in the case  $q = 3$ , is also well known to specialists in axion theory.

The gravitational contribution can be seen from (165) and (167) to be

$$\bar{f}_G^\mu = \frac{1}{2} \bar{T}^{\nu\sigma} \nabla^\mu h_{\nu\sigma} - \bar{\nabla}_\nu \left( \bar{T}^{\nu\sigma} h_\sigma^\mu + \bar{\mathfrak{e}}^{\mu\nu\rho\sigma} h_{\rho\sigma} \right). \quad (175)$$

a formula that was not so well known until relatively recently [3]. The same applies to the formula [24] for the dilatonic contribution, which can be seen from (168) and (166) to be given by

$$\bar{f}_D^\mu = \bar{T} \nabla^\mu \phi - \bar{\nabla}_\nu \left( 2(\bar{T}^{\mu\nu} + \bar{\mathfrak{e}}^{\mu\nu}) \phi \right). \quad (176)$$

If the system of equations governing the dynamics of the internal fields in the brane worldsheet involves, let us say,  $q_i$  independent degrees of freedom, then the complete set of dynamical equations governing the evolution of the brane will involve a total of  $q_i + p + 1$  degrees of freedom so as to include what is needed to determine the geometrical evolution of the  $p + 1$  dimensional supporting worldsheet.

Since it involves  $n$  components (in an  $n$  dimensional background) the force law (171) will, by itself provide a complete system of equations of motion for the system whenever  $q_i < n - p$ . In particular, it will provide just the number of equations of motion that is needed in cases for which  $q_i = n - p - 1$ . There will even be some redundancy, in that the equations of the system (171) will not be mutually independent, for cases in which  $q_i \leq n - p - 2$ , and in particular for a brane of the simple Dirac-Nambu-Goto type, in which there are no internal fields, i.e.  $q_i = 0$ .

The historical delay in formulating the general purpose gravitational and dilatonic force formulae (175) and (176) arose because the workers involved had long been accustomed to handling relevant fields such as  $F_{\mu\nu}$  and  $\bar{T}^{\mu\nu}$  and also more recently the axionic 3-form  $N_{\mu\nu\rho}$ , but not to the utilisation of the generalised Cauchy tensor  $\bar{\mathcal{G}}^{\mu\nu\rho\sigma}$  (which originally emerged [25] in the relativistic theory of elastic solidity).

## 5-8. The case of a simple Dirac-Nambu-Goto type brane

Before concentrating on strings (with  $p=1$ ) it is worth considering the simplest dimensionally unrestricted application, which is to a  $p$ -brane of the Dirac-Nambu-Goto type, for which the relevant master function is simply constant, so given in terms of a corresponding Kibble mass  $M_K$  by

$$\bar{\Lambda} = -M_K^{p+1}. \quad (177)$$

(In the context of superstring theory  $M_K$  is usually supposed to be of the order of magnitude of the Planck mass  $M_P$ , whereas in the context of cosmic string theory the Kibble mass is expected to have the same order of magnitude as the Higgs mass,  $M_X$  say, associated with the underlying “spontaneously broken” symmetry of the vacuum.)

In this special ( $q = 0$ ) Dirac-Nambu-Goto case, the surface stress momentum energy tensor is of course simply proportional to the fundamental tensor, with coefficient given by the surface tension  $\mathcal{T}$ ,

$$\bar{T}^{\mu\nu} = -\mathcal{T} \eta^{\mu\nu}, \quad \mathcal{T} = M_K^{p+1} \quad (178)$$

so its trace will be given by  $\bar{T} = - (p+1) \mathcal{T}$ .

The corresponding the hyper-Cauchy tensor is found[3] to be

$$\bar{\mathcal{E}}^{\mu\nu\rho\sigma} = \mathcal{T} \left( \eta^{\mu(\rho} \eta^{\sigma)\nu} - \frac{1}{2} \eta^{\mu\nu} \eta^{\rho\sigma} \right). \quad (179)$$

Its trace will be given simply by  $\bar{\mathcal{E}}^{\mu\nu} = \frac{1-p}{2} \mathcal{T} \eta^{\mu\nu}$  which evidently vanishes in the string case,  $p=1$ . It is to be noted however that the combination involved in the expression (176) for the dilatonic force density will be given by  $\bar{T}^{\mu\nu} + \bar{\mathcal{E}}^{\mu\nu} = -\frac{p+1}{2} \mathcal{T} \eta^{\mu\nu}$ , which never vanishes.

For the Dirac-Nambu-Goto case, the dynamical equation of motion (171) can be seen from (147) to reduce to the form

$$M_K^{(p+1)} K^\rho = -f^\rho, \quad (180)$$

in which, by (175), the gravitational contribution to the surface force on the right hand side will be expressible[3] in the form

$$f_G^\mu = f_I^\mu + f_{II}^\mu, \quad (181)$$

with

$$f_I^\mu = M_K^{p+1} \perp^{\mu\nu} \eta^{\rho\sigma} \left( \nabla_\rho h_{\nu\sigma} - \frac{1}{2} \nabla_\nu h_{\rho\sigma} \right), \quad (182)$$

$$f_{II}^\mu = M_K^{p+1} \left( \perp^{\mu\nu} K^\rho + \frac{1}{2} \eta^{\rho\nu} K^\mu - K^{\nu\rho\mu} \right) h_{\nu\rho}. \quad (183)$$

If a dilatonic field  $\phi$  is present, its effect will be given in the Dirac-Nambu-Goto case by

$$f_{\text{D}}^{\mu} = f_{\text{III}}^{\mu} + f_{\text{IV}}^{\mu}, \quad (184)$$

with

$$f_{\text{III}}^{\mu} = -(\mathbf{p} + 1) M_{\text{K}}^{\mathbf{p}+1} \perp^{\mu\nu} \nabla_{\nu} \phi, \quad (185)$$

$$f_{\text{IV}}^{\mu} = (\mathbf{p} + 1) M_{\text{K}}^{\mathbf{p}+1} \phi K^{\mu}, \quad (186)$$

It was observed by Battye[26] that the early work on gravitational perturbations of strings cited by Vilenkin and Shellard in their 1994 treatise [11] was seriously flawed by the use for estimating  $f_{\text{G}}^{\mu}$  of a formula (7.7.3) not only lacking the contribution  $f_{\text{II}}^{\mu}$  but also without the orthogonal projection operator  $\perp^{\mu\nu}$  in the expression (182) for  $f_{\text{I}}^{\mu}$ .

When due just to high frequency radiation[26] of external origin, the contributions  $f_{\text{II}}^{\mu}$  and  $f_{\text{IV}}^{\rho}$  might indeed be negligible compared with  $f_{\text{I}}^{\rho}$  and  $f_{\text{III}}^{\rho}$ . However, even in this limit, the naive tradition [11] of assuming the gravitational wave force to be simply proportional to  $\eta^{\rho\sigma} \left( \nabla_{\rho} h_{\nu\sigma} - \frac{1}{2} \nabla_{\nu} h_{\rho\sigma} \right)$ , without the orthogonal projection, is as unsatisfactory as postulating that a point particle with unit velocity vector  $u^{\mu}$  has acceleration  $\dot{u}^{\mu} = u^{\nu} \nabla_{\nu} u^{\mu}$  simply proportional to the gradient  $\nabla^{\mu} \varphi$  of some scalar field  $\varphi$ , the trouble being that, unlike  $\dot{u}^{\mu}$ , this gradient would not automatically be orthogonal to  $u^{\mu}$  (whereas acceleration proportional to  $\perp^{\mu\nu} \nabla_{\nu} \varphi$  would be admissible).

Anyway the high frequency approximation is quite unjustifiable in the context of self interaction, with which the unsound early work [22, 23] was concerned. A more valid treatment is provided below.

# 6 Allowance for radiative self-interaction

6-1 Regularisation of self interaction

6-2 The regularised gradient operator

6-3 Stress renormalisation

6-4 Action renormalisation

## 6-1. Regularisation of self interaction

By separating out the locally source free contributions due to passing radiation, as indicated by a widetilde, the relevant fields take the form

$$\begin{aligned}
 A_\mu &= \widetilde{A}_\mu + \overbrace{A_\mu} , & B_{\mu\nu} &= \widetilde{B}_{\mu_1 \dots \mu_q} + \overbrace{B_{\mu_1 \dots \mu_q}} , \\
 h_{\mu\nu} &= \widetilde{h}_{\mu\nu} + \overbrace{h_{\mu\nu}} , & \phi &= \widehat{\phi} + \overbrace{\phi} ,
 \end{aligned} \tag{187}$$

using an overbrace for the contributions given by the retarded Green function solutions of the relevant source equations (116), (117), (120) and (118). If the coupling is very weak, the latter can be neglected, but if significant they cause trouble, as the relevant source fields in the field equations (116), (117), (120) and (118) are not the regular worldsheet supported tensor fields  $\overline{j}^\mu$ ,  $\overline{w}^{\mu\nu}$ , and  $\overline{T}^{\mu\nu}$ , but the corresponding distributions,  $\hat{j}^\mu$ ,  $\hat{w}^{\mu_1 \dots \mu_q}$  and  $\hat{T}^{\mu\nu}$  given by (122) (123) and (124), so the resulting field contributions diverge in the thin worldsheet limit.

However, as in the familiar point particle case (with  $p = 0$ ) so also for higher dimensional branes, one can obtain a realistically regularised result by taking account of the fact that the physical system in question (which will be a vacuum vortex defect in the cosmic string case) will actually not be infinitely thin, but will be of finite thickness characterisable by an appropriate microscopic “ultraviolet” cut off length scale,  $a$  say.

This authorises replacements of the form exemplified by

$\overbrace{A_\mu} \mapsto \widehat{A}_\mu = \langle \overbrace{A_\mu} \rangle$ , where the angle brackets indicate an appropriately weighted [27] averaging over the finite thickness to give a regularised value, indicated by a widehat, that is a well behaved but worldsheet confined field. As far as effects on the brane are concerned, the decomposition (187) is thus replacable by its regularised analogue :

$$A_\mu = \widetilde{A}_\mu + \widehat{A}_\mu, \quad B_{\mu\nu} = \widetilde{B}_{\mu_1 \dots \mu_q} + \widehat{B}_{\mu_1 \dots \mu_q},$$

$$h_{\mu\nu} = \widetilde{h}_{\mu\nu} + \widehat{h}_{\mu\nu}, \quad \phi = \widetilde{\phi} + \widehat{\phi}.$$

Allowance for the finite thickness  $a$  will be sufficient for regularisation in the point particle case, and more generally so long as the codimension  $n - p - 1$  is 3 or more, i.e. when  $n - p - 1 > 2$  so that the divergence will be of power law type. However in the logarithmically divergent case of a *hyperstring* meaning a brane of codimension 2, i.e when  $n - p - 1 = 2$  (as in the case of an ordinary string, with  $p = 1$  in a spacetime of dimension  $n = 4$ ) it will also be necessary to introduce a long range “infrared” cut off length scale  $\Delta$  say that might represent the macroscopic mean distance between neighbouring hyperstrings.

In the hypersurface case of codimension 1, i.e.  $n - p - 1 = 1$  (including the case of an ordinary membrane with  $p = 2$  in a spacetime of dimension  $n = 4$ ) there is no divergence, but only a jump discontinuity, which can be dealt with [28] using Darmois-Israel type matching conditions. Leaving the latter aside, all the other cases can be dealt with [27] as follows.

For  $n - p \geq 3$ , the relevant Green function integrals will be proportional to a regularisation factor of the form  $\hat{l} a^{p+3-n}$  where  $\hat{l}$  is a dimensionless number that depends on the particular technical details of the regularisation procedure. In the generic case this quantity  $\hat{l}$  will be of order unity, and indeed it can be set exactly to unity

$$\hat{l} = 1, \quad (188)$$

by adjustment of the precise definition of the effective thickness lengthscale  $a$ , except in the logarithmically divergent hyperstring case  $p + 3 = n$ , for which its magnitude will be given by an expression of the form

$$\hat{l} = \ln \{ \Delta^2 / a^2 \}. \quad (189)$$

The prototype example of the application of this procedure is provided by the D'Alembertian equation (116) for Maxwell type potential  $A_\mu$  for which (as the higher dimensional generalisation of what was pointed out by Witten in his original discussion [8] of “superconducting” cosmic strings) the effective self field contribution  $\hat{A}_\mu$  say on the worldsheet will be obtained in the simple form

$$\hat{A}_\mu = \frac{\hat{l} M_F^{4-n}}{a^{n-p-3}} \overline{j}_\mu. \quad (190)$$

The corresponding axionic type contribution will similarly be obtainable from (117) as

$$\hat{B}_{\mu_1 \dots \mu_q} = \frac{\hat{l} M_{[q]}^{2q+2-n}}{a^{n-p-3}} \overline{w}_{\mu_1 \dots \mu_q}. \quad (191)$$

In the same way, it can be seen from (120) that the corresponding expression for the regularised gravitational self field, will have the form

$$\hat{h}_{\mu\nu} = \frac{4\hat{l} M_G^{2-n}}{a^{n-p-3}} \left( \overline{T}_{\mu\nu} - \frac{1}{n-2} \overline{T}_{\sigma}{}^{\sigma} g_{\mu\nu} \right), \quad (192)$$

while finally by (118) for the regularised dilatonic self field one gets

$$\hat{\phi} = \frac{\hat{l} M_D^{2-n}}{a^{n-p-3}} \overline{T}. \quad (193)$$

(If the microscopic electromagnetic source distribution were very different from its gravitational counterpart, the corresponding values of  $\hat{l}$  that would be most appropriate might be different. However for most purposes, and particularly in the hyperstring case for which the dependence on the cut off is only logarithmic, it will be adequate to use the same regularisation factor  $\hat{l}$  for all relevant kinds of field.)

## 6-2. The regularised gradient operator

Splitting the relevant fields into a source free external part, denoted by a widetilde, and a regularised world sheet supported self interaction contribution, denoted by a widehat, provides a corresponding split

$$\begin{aligned}
 f_{\text{F}}^{\mu} &= \widetilde{f}_{\text{F}}^{\mu} + \widehat{f}_{\text{F}}^{\mu}, & f_{[q]}^{\mu} &= \widetilde{f}_{[q]}^{\mu} + \widehat{f}_{[q]}^{\mu}, \\
 f_{\text{G}}^{\mu} &= \widetilde{f}_{\text{G}}^{\mu} + \widehat{f}_{\text{G}}^{\mu}, & f_{\text{D}}^{\mu} &= \widetilde{f}_{\text{D}}^{\mu} + \widehat{f}_{\text{D}}^{\mu},
 \end{aligned} \tag{194}$$

of the resulting force densities, as specified by the general formulae (173), (174), (175), (176). As these formulae involve gradients, to get the regularised self force contributions  $\widehat{f}_{\text{F}}^{\mu}$ ,  $\widehat{f}_{[q]}^{\mu}$ ,  $\widehat{f}_{\text{G}}^{\mu}$  and  $\widehat{f}_{\text{D}}^{\mu}$ , knowledge just of the self fields  $\widehat{A}_{\mu}$ ,  $\widehat{B}_{\mu_1 \dots \mu_2}$ ,  $\widehat{h}_{\mu\nu}$  and  $\widehat{\phi}$  is not quite sufficient. These regularised values are defined only on the worldsheet, and so do not have gradients other than in tangential directions, as given by the  $\overline{\nabla}_{\mu}$ .

The solution to the problem of averaging the self field gradient was first found, for a Maxwellian field, in the case  $n = 4$  (for which the retarded Green function is particularly simple) where it is given [1] by

$$\widehat{\nabla}_{\mu} A_{\nu} \equiv \langle \nabla_{\mu} \overbrace{A_{\nu}} \rangle = \overline{\nabla}_{\mu} \hat{A}_{\nu} + \frac{1}{2} K_{\mu} \hat{A}_{\nu}.$$

It was subsequently shown [27] that the same formula remains valid, without change, for higher values of the background dimension  $n$ .

This means that, when dealing with a regularised mean field due to self-interaction, one can make the replacement  $\nabla_{\mu} \mapsto \widehat{\nabla}_{\mu}$  using the generic formula

$$\widehat{\nabla}_{\mu} = \overline{\nabla}_{\mu} + \frac{1}{2} K_{\mu}, \quad (195)$$

whose application in earlier applications [29, 30, 22, 23, 18, 19, 15, 31] was implicit, but masked by use of specialised gauges for what was not so easily recognisable as the curvature vector,  $K_{\mu}$ .

In the actual application of the formula (195) , it is sufficient, in the case of the scalar field  $\hat{\phi}$ , to use the simple expression  $\overline{\nabla}^\mu \hat{\phi} = \eta^{ij} \overline{x}^\mu_{,i} \hat{\phi}_{,j}$  for the tangentially projected gradient.

However for a tensorial field there will also be contributions involving the background Riemann Christoffel connection

$\Gamma_{\mu \rho}^\nu = g^{\nu\sigma} \left( g_{\sigma(\mu,\rho)} - \frac{1}{2} g_{\mu\rho,\sigma} \right)$  unless one is using Minkowski coordinates in a flat spacetime background.

The connection will also be involved in the explicit expression – in terms of the intrinsic metric  $\eta_{ij}$  and its determinant  $|\eta|$  – for the curvature vector, which is obtainable from (147) in the form

$$K^\nu = \|\eta\|^{-1/2} \left( \|\eta\|^{1/2} \gamma^{ij} \overline{x}^\nu_{,i} \right)_{,j} + \eta^{ij} \overline{x}^\mu_{,j} \overline{x}^\rho_{,i} \Gamma_{\mu \rho}^\nu.$$

The prototype application [1] of the formula (195) gave the regularised electromagnetic field on a string worldsheet in the form

$$\hat{F}_{\mu\nu} = 2\hat{\nabla}_{[\mu}\hat{A}_{\nu]} = \hat{l} \left( 2\overline{\nabla}_{[\mu}\bar{j}_{\nu]} + K_{[\mu}\bar{j}_{\nu]} \right). \quad (196)$$

Using the surface current version,  $\overline{\nabla}_{\nu}\bar{j}^{\nu} = 0$ , of the conservation law (112), the electromagnetic self force contribution in (173) is thereby expressible as a worldsheet divergence in the form

$$\hat{f}_{\text{F}}^{\mu} = -\overline{\nabla}_{\nu}\hat{T}_{\text{F}}^{\mu\nu}, \quad (197)$$

where the relevant stress momentum energy density contribution from the electromagnetic self interaction is given, via (190), by

$$\hat{T}_{\text{F}}^{\mu\nu} = \hat{A}^{\mu}\bar{j}^{\nu} - \frac{1}{2}\hat{A}_{\rho}\bar{j}^{\rho}\eta^{\mu\nu}. \quad (198)$$

By analogous use of the surface flux version,  $\overline{\nabla}_\mu \overline{w}^{\mu\mu_2\dots\mu_q} = 0$ , of the generalised vorticity conservation law (113), it can be seen that the axionic self force contribution in (174) will also be expressible as a world sheet divergence in the form

$$\widehat{f}_{[q]}^\mu = -\overline{\nabla}_\nu \widehat{T}_{[q]}^{\mu\nu}, \quad (199)$$

in which, via (191), as the analogue of (198), one will have

$$\widehat{T}_{[q]}^{\mu\nu} = \frac{1}{(q-1)!} \widehat{B}_{\rho_2\dots\rho_q}^\mu \overline{w}^{\nu\rho_2\dots\rho_q} - \frac{1}{2q!} \widehat{B}_{\rho_1\dots\rho_q} \overline{w}^{\rho_1\dots\rho_q} \eta^{\mu\nu}. \quad (200)$$

Applying same procedure to the gravitational self force contribution in (175) one finds[4] that it too is expressible as a world sheet divergence :

$$\hat{f}_G{}^\mu = -\overline{\nabla}_\nu \hat{T}_G{}^{\mu\nu}, \quad (201)$$

with stress momentum energy density contribution given, via (192), by

$$\hat{T}_G{}^{\mu\nu} = \hat{h}_\sigma{}^\mu \overline{T}^{\nu\sigma} - \frac{1}{4} \hat{h}_{\rho\sigma} \overline{T}^{\rho\sigma} \eta^{\mu\nu} + \hat{h}_{\rho\sigma} \overline{\mathfrak{E}}^{\rho\sigma\mu\nu}. \quad (202)$$

Finally for the dilatonic contribution in (176) one similarly obtains

$$\hat{f}_D{}^\mu = -\overline{\nabla}_\nu \hat{T}_D{}^{\mu\nu}, \quad (203)$$

with stress momentum energy density tensor given, via (193), by

$$\hat{T}_D{}^{\mu\nu} = 2\hat{\phi} \left( \overline{T}^{\mu\nu} - \frac{1}{4} \overline{T} \eta^{\mu\nu} + \overline{\mathfrak{E}}^{\mu\nu} \right). \quad (204)$$

### 6-3. Stress renormalisation

The forgoing formulation of the self force contributions as divergences allows the force balance equation (171) to be rewritten in the form

$$\overline{\nabla}_\nu \tilde{T}^{\mu\nu} = \tilde{f}^\mu, \quad (205)$$

with nothing but well behaved source free contribution on the right,  $\tilde{f}^\mu = \tilde{f}_F^\mu + \tilde{f}_{[q]}^\mu + \tilde{f}_G^\mu + \tilde{f}_D^\mu$ , and with the “bare” stress momentum energy density tensor  $\overline{T}^{\mu\nu}$  replaced by an effectively renormalised “dressed” stress momentum energy density tensor

$$\tilde{T}^{\mu\nu} = \overline{T}^{\mu\nu} + \hat{T}^{\mu\nu}, \quad (206)$$

in which self interaction is allowed for by the extra “dressing” term,  $\hat{T}^{\mu\nu} = \hat{T}_F^{\mu\nu} + \hat{T}_{[q]}^{\mu\nu} + \hat{T}_G^{\mu\nu} + \hat{T}_D^{\mu\nu}$ .

## 6-4. Action renormalisation

Although there may still be other finite contributions from self interaction, the foregoing work shows how the dominant part that diverges in the thin brane limit can be allowed for, after regularisation, by an effective renormalisation of the surface stress energy momentum density tensor. This “dressed” tensor will now be shown to be derivable, in the usual way (129), from a correspondingly “dressed” action,

$$\tilde{\mathcal{I}} = \int \tilde{L} \|\eta\|^{1/2} d^{p+1}\sigma, \quad (207)$$

such that only source free external fields appear in the expression

$$\tilde{L} = \tilde{\Lambda} + \tilde{A}_\mu \bar{j}^\mu + \frac{1}{q!} \tilde{B}_{\mu_1 \dots \mu_q} \bar{w}^{\mu_1 \dots \mu_q} + \frac{1}{2} \tilde{h}_{\mu\nu} \bar{T}^{\mu\nu} + \tilde{\phi} \bar{T}. \quad (208)$$

but with a “dressed” intrinsic Lagrangian  $\tilde{\Lambda}$  instead of the “bare”  $\Lambda$ .

The renormalised intrinsic action density will have the form

$$\tilde{\Lambda} = \Lambda + \hat{\Lambda} \quad (209)$$

with

$$\hat{\Lambda}_{\text{F}} + \hat{\Lambda}_{[q]} + \hat{\Lambda}_{\text{G}} + \hat{\Lambda}_{\text{D}} . \quad (210)$$

For the regularised field  $\hat{A}_{\mu}$  to act like its external analogue  $\tilde{A}_{\mu}$  as the coefficient of the current variation in the formula  $\delta\Lambda_{\text{F}} = \hat{A}_{\mu}\delta j^{\mu}$ , the corresponding action term must be given the quadratic form

$$\hat{\Lambda}_{\text{F}} = \frac{1}{2}\hat{A}_{\mu}\bar{j}^{\mu} , \quad (211)$$

with  $\hat{A}_{\mu}$  proportional to  $\bar{j}_{\mu}$  according to (190), where the factor  $1/2$  is interpretable as allowance for the double counting that would otherwise arise from the summing of pairwise particle interactions.

By analogous reasoning, again involving insertion of the factor  $\frac{1}{2}$ , the axionic self interaction contribution will be

$$\widehat{\Lambda}_{[q]} = \frac{1}{2q!} \widehat{B}_{\mu-1\dots\mu_q} \overline{w}^{\mu_1\dots\mu_q}, \quad (212)$$

with  $\widehat{B}_{\mu-1\dots\mu_q}$  proportional to  $\overline{w}^{\mu_1\dots\mu_q}$  according to (191).

The corresponding gravitational contribution [5] will similarly be

$$\widehat{\Lambda}_{\text{G}} = \frac{1}{4} \widehat{h}_{\mu\nu} \overline{T}^{\mu\nu}, \quad (213)$$

with  $\widehat{h}_{\mu\nu}$  given by (192), and finally the dilatonic contribution will be

$$\widehat{\Lambda}_{\text{D}} = \frac{1}{2} \widehat{\phi} \overline{T}, \quad (214)$$

with  $\widehat{\phi}$  given by (193).

Explicitly, in terms of the small lengthscale  $a$  characterising the thickness of the brane, these self interaction terms take the forms

$$\widehat{\Lambda}_{\text{F}} = \frac{\widehat{l} M_{\text{F}}^{4-n}}{2 a^{n-p-3}} \overline{j}_{\mu} \overline{j}^{\mu}, \quad (215)$$

$$\widehat{\Lambda}_{[q]} = \frac{\widehat{l} M_{[q]}^{2q+2-n}}{2q! a^{n-p-3}} \overline{w}_{\mu_1 \dots \mu_q} \overline{w}^{\mu_1 \dots \mu_q}, \quad (216)$$

$$\widehat{\Lambda}_{\text{G}} = \frac{\widehat{l} M_{\text{G}}^{2-n}}{a^{n-p-3}} \left( \overline{T}_{\mu\nu} \overline{T}^{\mu\nu} - \frac{1}{n-2} \overline{T}^2 \right), \quad (217)$$

$$\widehat{\Lambda}_{\text{D}} = \frac{\widehat{l} M_{\text{D}}^{2-n}}{2 a^{n-p-3}} \overline{T}^2. \quad (218)$$

in which  $\widehat{l}$  is some dimensionless number of order unity.

It can be checked directly, using the definitions (145), (146) and the conditions (156), that the surface stress momentum energy density contributions (198) (200) (202, (204) are indeed given exactly by the preceding action prescriptions (211), (212), (213), (214). This confirmation of the coherence of our renormalisation scheme means that the “dressed” brane model characterised by the action density  $\hat{\Lambda}$  given by (210) will indeed have a corresponding “dressed” surface stress momentum energy density tensor  $\tilde{T}^{\mu\nu}$  of the required form (206).

A special category, generalising ordinary axionic coupling of a string in 4-dimensions [24], has worldsheet dimension  $p + 1$  equal to the number  $q$  of indices of the field  $B_{\nu_1 \dots \nu_q}$ , whose source must then be proportional to the surface measure tensor,  $\overline{w}^{\nu_0 \dots \nu_p} = \bar{\kappa} \mathcal{E}^{\nu_0 \dots \nu_p}$ , where  $\bar{\kappa}$  is constant by the conservation law (150). Its self-action contribution is thus also a constant :

$$\hat{\Lambda}_{[p+1]} = -\frac{1}{2} \hat{l} M_{[p+1]}^{2p+4-n} \bar{\kappa}^2 a^{3+p-n} .$$

The manifestly negative definite nature of the constant self-action contribution when  $q = p + 1$  makes it possible in particular cases [24] for it to be cancelled by the corresponding, manifestly positive definite, dilatonic contribution (218) in the Dirac-Nambu-Goto case (178), for which the latter has the constant value

$$\hat{\Lambda}_D = \hat{l} \frac{(p+1)^2}{2} M_K^{2(p+1)} M_D^{2-n} a^{3+p-n} .$$

In this Dirac-Nambu-Goto case, the gravitational self action contribution will be

$$\hat{\Lambda}_G = \hat{l} \frac{(n-p-3)(p+1)}{(n-2)} M_K^{2(p+1)} M_G^{2-n} a^{3+p-n} ,$$

which evidently vanishes by itself in the hyperstring case, for which  $p = n - 3$ , and hence in particular (contrary to what was asserted in the early work quoted by Vilenkin and Shellard [11]) in the particular case of a Nambu-Goto string in ordinary spacetime with dimension  $n = 4$ .

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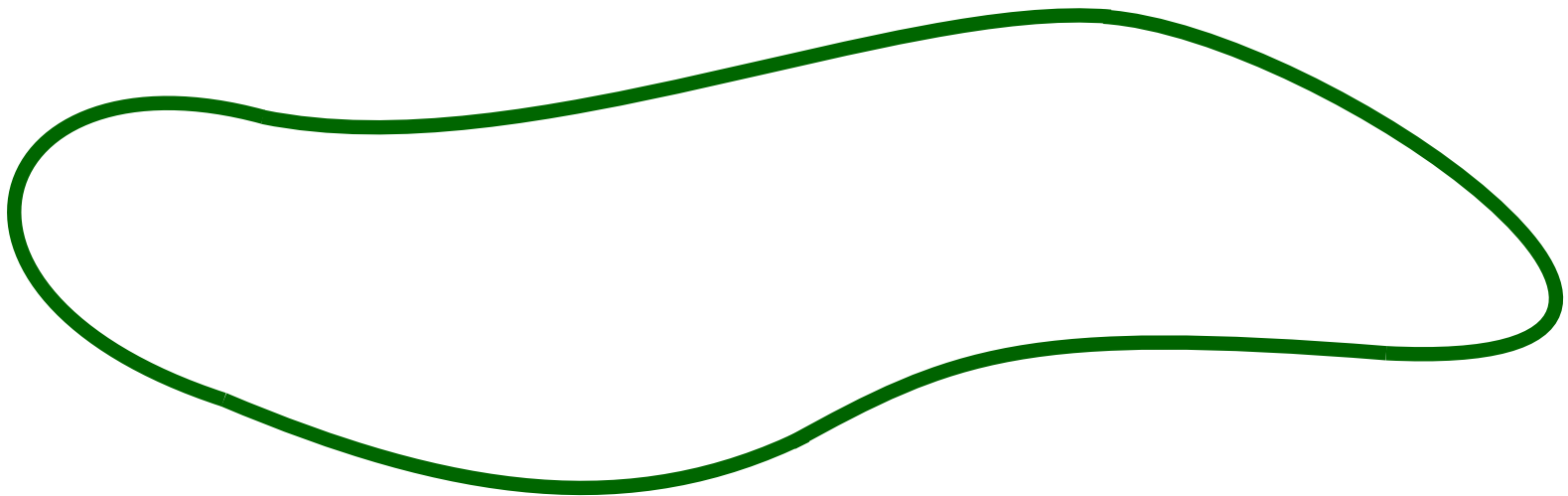
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# 7 Dynamics and vorton equilibrium states of elastic string loops



## 7-1 Kinematics of thin string or brane

Classical p-brane model qualifies as 'thin' if support confined near timelike worldsheet of dim  $p+1$ , coords  $\sigma^i$ ,  $i=0,1, \dots, p$ , with  $p=1$  in case of string.

In  $n$  dim background, with coords  $x^\mu$ ,  $\mu = 0, 1, \dots, n-1$ , metric  $g_{\mu\nu}$ , the brane embedding induces worldsheet metric  $\eta_{ij} = g_{\mu\nu} x_{,i}^\mu x_{,j}^\nu$ , whose contravariant inverse induces (first) fundamental tensor  $\eta^{\mu\nu} = \eta^{ij} x_{,i}^\mu x_{,j}^\nu$  giving tangential projection tensor  $\eta_\mu^\nu = g_\mu^\nu - \perp_\mu^\nu$  and tangential deriv operator  $\bar{\nabla}_\mu = \eta_\mu^\nu \nabla_\nu$ .

Hence construct second fundamental tensor  $K_{\mu\nu}^\rho = \eta_\nu^\sigma \bar{\nabla}_\mu \eta_\sigma^\rho$ , with symmetry  $K_{[\mu\nu]}^\rho = 0$  as condition for integrability, and projection properties  $\perp_\mu^\sigma K_{\sigma\nu}^\rho = 0 = K_{\mu\nu}^\sigma \eta_\sigma^\rho$ .

## 7-2. Dynamics of thin string or brane

Brane governed by worldsheet action  $\mathcal{I} = \int L \sqrt{-\eta} d^{p+1}\sigma$  will have surface stress energy tensor given by  $\bar{T}^{\mu\nu} = 2 \partial L / \partial g_{\mu\nu} + L \eta^{\mu\nu}$ .

Evolution of worldsheet subject to external surface force density  $\bar{f}^\mu$  given by universally applicable dynamical equation  $\bar{T}^{\mu\nu} K_{\mu\nu}{}^\rho = \perp_\sigma^\rho \bar{f}^\sigma$ .

In generic string case,  $\exists$  orthonormal timelike and spacelike eigenvectors  $u^\mu = x^\mu_{,i} u^i$ ,  $\tilde{u}^\mu = x^\mu_{,i} \tilde{u}^i$ ,  $\tilde{u}^i = \varepsilon^{ij} u_j$  associated with energy density  $\mathcal{U}$  and string tension  $\mathcal{T}$  such that  $\eta^{\mu\nu} = -u^\mu u^\nu + \tilde{u}^\mu \tilde{u}^\nu$  and  $\bar{T}^{\mu\nu} = \mathcal{U} u^\mu u^\nu - \mathcal{T} \tilde{u}^\mu \tilde{u}^\nu$ .

In particular, a Nambu Goto (internally isotropic) string has

$\mathcal{U} = \mathcal{T} = m^2$  for some fixed mass  $m$ , so in terms of worldsheet curvature vector  $K^\rho = K^\nu{}_\nu{}^\rho$  its dynamical equation will just be

$$K^\rho = m^{-2} \perp_\sigma^\rho \bar{f}^\sigma.$$

### 7-3. Free motion of elastic string models

Witten's conduction mechanism provides string Lagrangian  $L$  depending only on scalar  $w = \gamma^{ij} \varphi_{,i} \varphi_{,j}$ , providing adjoint  $\Lambda = L + w\kappa$  with  $\kappa = -2 dL/dw$ , in terms of which transverse (wiggle) and longit (sound) propagation speeds are respectively  $v = \sqrt{T/U}$  and  $v_L = \sqrt{-dT/dU}$ , where  $T = -L$  and  $U = -\Lambda$  when  $w < 0$ , while  $T = -\Lambda$  and  $U = -L$  when  $w > 0$ .

In all cases phase gradient proportional to surface current,  $\bar{c}^\mu = x^\mu_{,i} \bar{c}^i$ ,  $\bar{c}^i = \kappa \gamma^{ij} \varphi_{,j} = -\partial L \partial \varphi_{,i}$ , that is conserved,  $(\sqrt{-\gamma} \bar{c}^i)_{,i} = 0$ , when no ext force, so that  $\bar{T}^{\mu\nu} K_{\mu\nu}{}^\rho = 0$  with  $\bar{T}^{\mu\nu} = 2\kappa^{-1} \bar{c}^\mu \bar{c}^\nu + L \eta^{\mu\nu}$ .

## 7-4. Free motion of circular elastic string loops

Circular loop invariant under action of axisymmetry generating Killing vector  $\varrho^\mu \partial / \partial x^\mu = \partial / \partial \phi$  has conserved phase winding number  $N = \varrho^\nu \bar{\nabla}_\nu \varphi$  and charge number  $C = 2\pi \varrho^\mu \mathcal{E}_{\mu\nu} \bar{c}^\nu$ , with  $\mathcal{E}^{\mu\nu} = 2u^{[\mu} \tilde{u}^{\nu]}$ , giving relation  $w \varrho^2 = N^2 - (C/2\pi\kappa)^2$  between radius  $\varrho = \sqrt{\varrho^\nu \varrho_\nu}$  and  $w$ , whose sign is determined by ratio  $b = 2\pi\kappa_0 N/C$ . Product is angular momentum  $J = NC = \varrho^\nu \Pi_\nu$ , where  $\Pi^\mu = 2\pi \varrho^\nu \mathcal{E}_{\nu\rho} \bar{T}^{\rho\mu}$ . Will also have conserved mass  $M = -k^\nu \Pi_\nu$  in stationary background with Killing vector  $k^\mu \partial / \partial x^\mu = \partial / \partial t$ . Proper time variation of  $\dot{\varrho}$  then given (in flat background) by  $M^2 \dot{\varrho}^2 = M^2 - \Upsilon^2$  with field  $\Upsilon$  given implicitly via  $w$  as function of  $\varrho$  by  $\Upsilon = C^2/2\pi\kappa\varrho - 2\pi L\varrho$ .

## 7-5. Logarithmic equation of state for cosmic string

Linear action formula,  $L = -m^2(1 + \delta_*^2 w)$  proposed for weak current by E. Witten, but no good – since would give subsonic  $v^2 < v_L^2$ , contrary to effect of Witten's conduction mechanism as calculated by P. Peter. More realistic supersonic wiggle propagation  $v^2 > v_L^2$  from logarithmic formula

$L = -m^2 - \frac{1}{2}m_*^2 \ln \{1 + \delta_*^2 w\} \Rightarrow \kappa = m_*^2 \delta_*^2 / (1 + \delta_*^2 w)$ ,  
valid over finite range,  $\exp\{-2m^2/m_*^2\} < 1 + \delta_*^2 w < 2$ . Radial dependence in circular case given explicitly by

$$1/\kappa = 2(\pi m_*/rC)^2 (-\varrho^2 + \sqrt{\varrho^4 + (C/\pi m_*^2)^2 (\varrho^2/\delta_*^2 + N^2)}).$$

For ratio  $|b|$  in “safe” range,  $\exp\{-2m^2/m_*^2\} < |b| < 2$ ,  
(including “chiral” value  $|b| = 1$ ) the ring can oscillate with unbounded energy  $M$ .

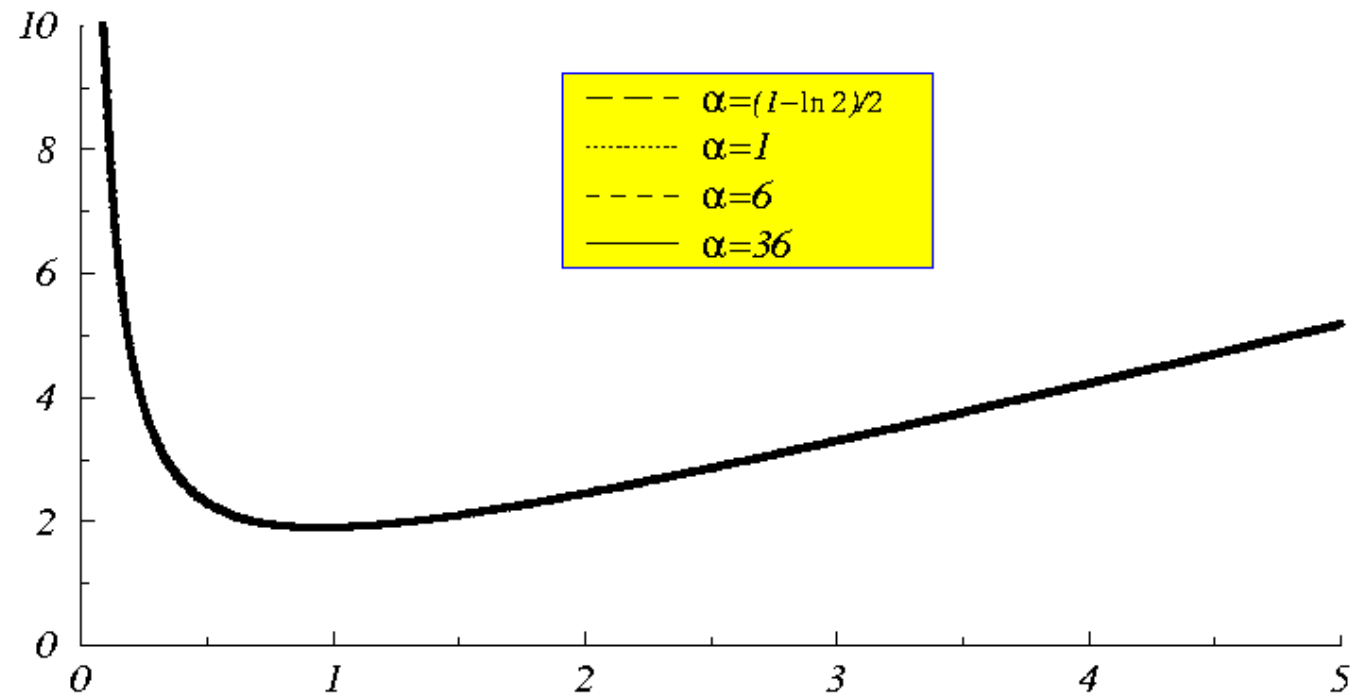


FIG. 1 – Plot of effective potential  $\mathcal{V}$  against  $\varrho$  with various values of ratio  $\alpha = m^2/m_\star^2$  for  $|b| = 2\pi\kappa_0|N/C|$  in “safe” range not too far from chiral value  $|b| = 1$ . [hep-ph/9609401]

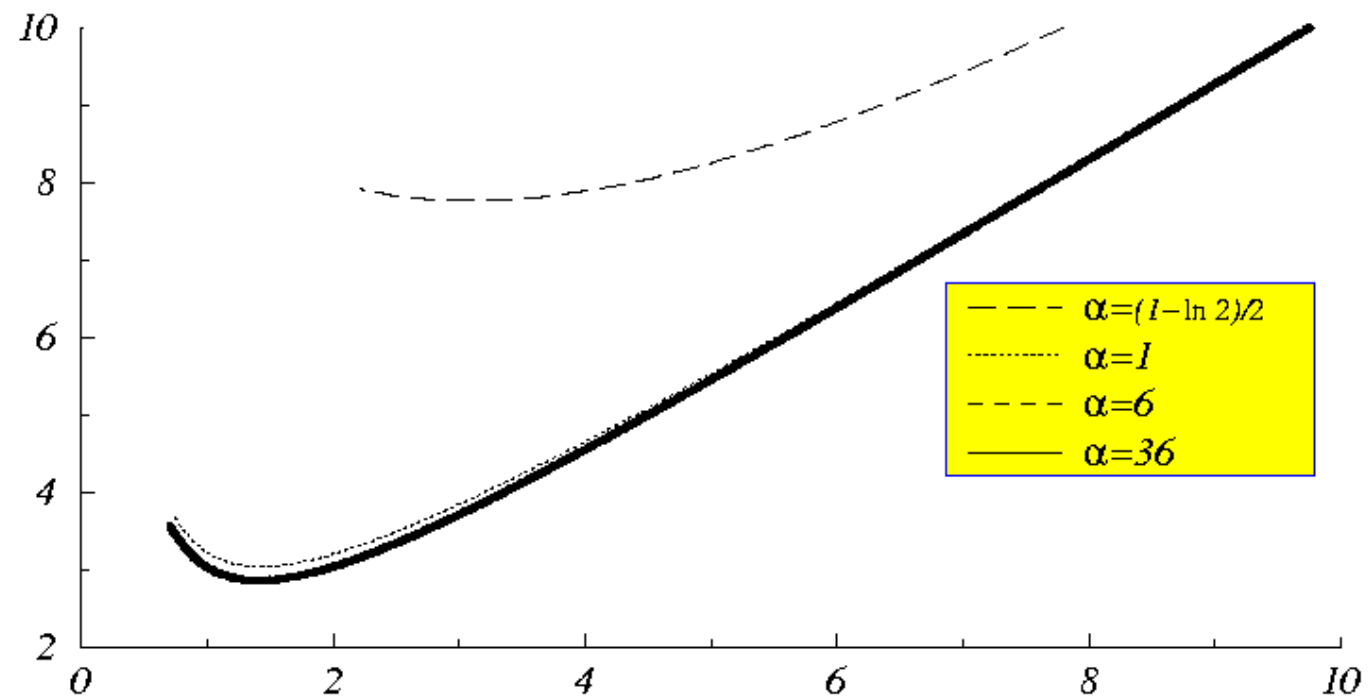


FIG. 2 – Plot of effective potential  $V$  against  $\rho$  with various values of ratio  $\alpha = m^2/m_*^2$  for  $|b| = 2\pi\kappa_0|N/C|$  outside “safe” range, where oscillation possible only for low values of energy  $M$ .

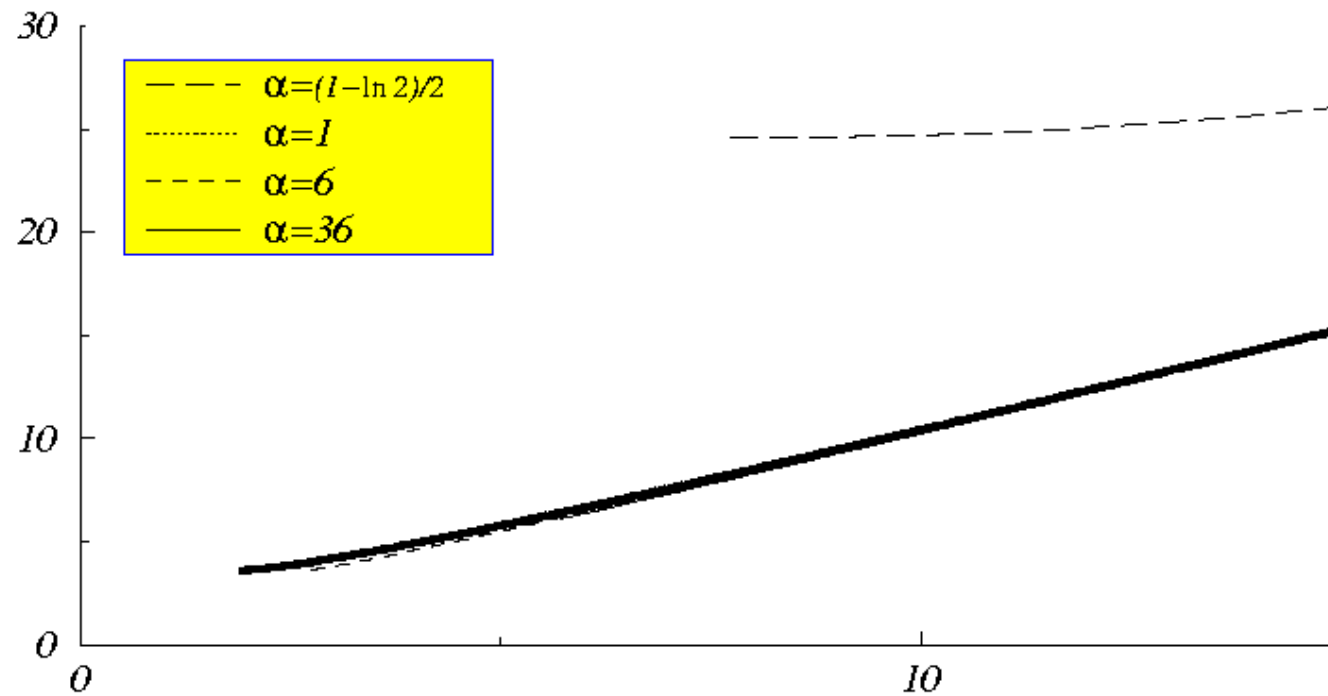


FIG. 3 – Plot of effective potential  $\Upsilon$  against  $\varrho$  with various values of ratio  $\alpha = m^2/m_*^2$  for  $|b| = 2\pi\kappa_0|N/C|$  far outside “safe” range, where no equilibrium nor oscillation is possible.

## 7-6. Stationary string states in flat background

World sheet tangent to unit static Killing vector satisfying  $\nabla_{\mu}k^{\nu} = 0$  ,  
with orthog unit spacelike tangent vector  $e^{\mu}$  satisfying  $k^{\nu}\nabla_{\nu}e^{\mu} = 0$  ,  
giving first fundamental tensor  $\eta^{\mu\nu} = -k^{\mu}k^{\nu} + e^{\mu}e^{\nu}$  and second  
fundamental tensor  $K_{\mu\nu}^{\rho} = e_{\mu}e_{\nu}K^{\rho}$  with curvature vector  
 $K^{\mu} = e^{\nu}\nabla_{\nu}e^{\mu}$  . Flow velocity  $v$  of timelike eigenvector,  
 $\bar{T}_{\nu}^{\mu}u^{\nu} = -\mathcal{U}u^{\nu}$  , specified by expression  
 $u^{\mu} = (1 - v^2)^{-1/2}(k^{\mu} + v e^{\mu})$  .

Free dynamical equation reduces to  $(\mathcal{U} - v^2\mathcal{T})K^{\rho} = 0$  .

Straight solution,  $K^{\rho} = 0$  , possible for arbitrary  $v$  , but circular (or  
more general) curved configuration must have (generically uniform)  
wiggle propagation speed,  $v^2 = \mathcal{T}/\mathcal{U}$  .

## 7-7. Stability criterion for circular vorton states

Stability depends just on velocities  $v$  and  $v_L$ , always holds if subsonic  $v^2 \leq v_L^2$ . Monopole  $n = 0$  and dipole  $n = 1$  modes always stable, but instability may occur for higher modes,  $n \geq 2$  for which, in state with radius  $a$ , eigenfrequency  $\omega$  given for  $x = a\omega/v_+ n$ , with  $v_+ = 2v/(1+v^2)$ , by cubic:  $x^3 + b_2 x^2 + b_1 x + b_0 = 0$ , in which  $b_2 = \Gamma - 2 - \xi$ ,  $b_1 = -2\Gamma + (1 + \xi)(1 - n^{-2})$ ,  $b_0 = \Gamma(1 - n^{-2})$ , where  $\xi = \Gamma(1 - v_+^2)$ ,  $\Gamma = v_+^{-2}(v_L^2 - v^2)/(1 - v_L^2 v^2)$ . Stability criterion  $\Delta \geq 0$ , for roots all real, given by  $\Delta = b_2^2 b_1^2 + 18b_2 b_1 b_0 - 4b_1^3 - 4b_2^3 b_0 - 27b_0^2$ . Relativistic limit  $\xi \rightarrow 0$  gives  $\Delta \rightarrow 4n^{-2}(\Gamma + 1 + n^{-1})^2(\Gamma + 1 - n^{-1})^2$ , which is strictly positive almost always.

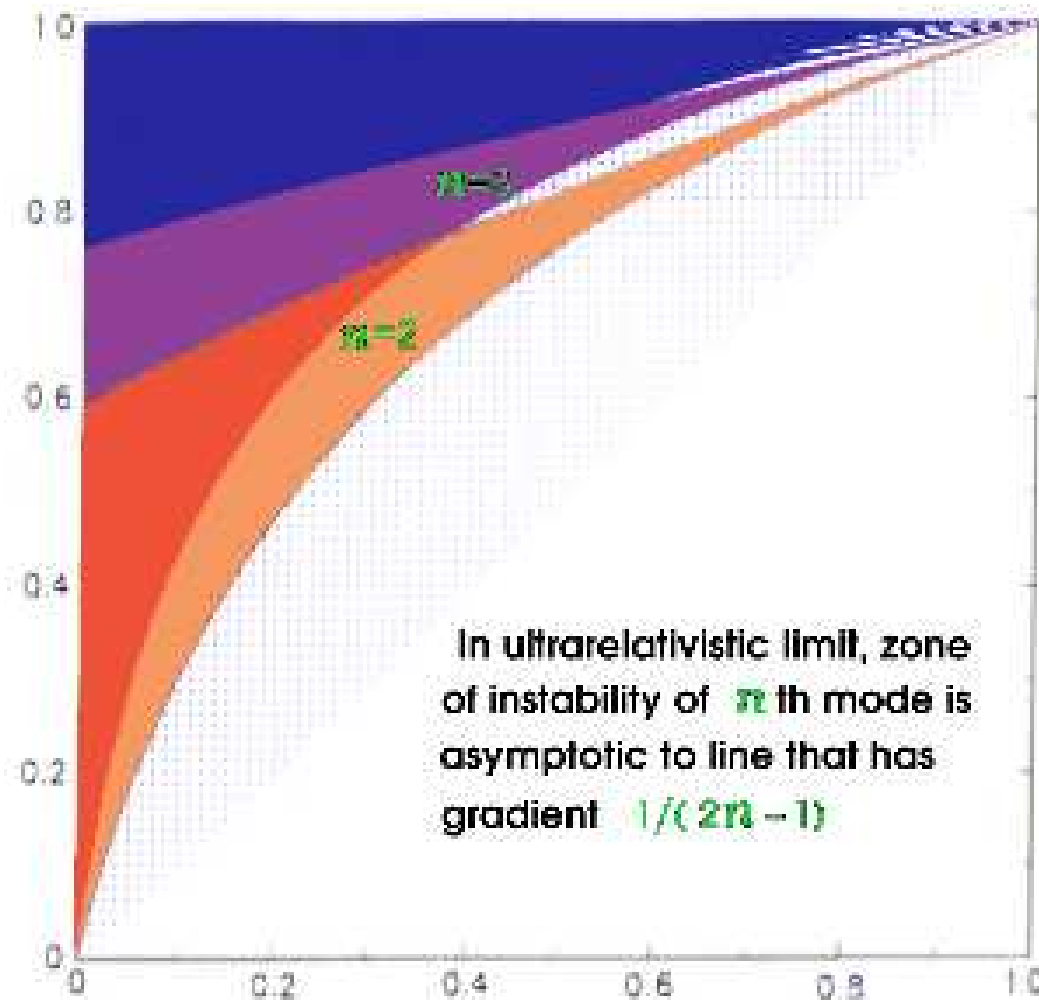


FIG. 4 – Zones of macroscopic instability of circular vorton states, as obtained in 1994 by X. Martin on plot of squared rotation (and wiggle) speed,  $v^2$ , against squared “sonic” speed  $v_L^2$ .

## Références

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