A New Approach to Equation of Motion for a fast-moving particle

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Why we need Fast motion approximation

- Equation of Motion(EOM) plays an important role in Gravitational Wave (GW) Astronomy
- Slow-motion (post-Newtonian) approximation has been developed up to 3.5 pN order
- 4 pN approximation seems impossible at the moment and even if we have it, PN series converges very slowly
- There are interesting GW sources with high velocities

Fast-motion approximation will be necessary

There has been development in this field such as Mino, Sasaki and Tanaka, and others

However actual calculation of self and radiation reaction force needs special treatment (separation of divergent self-field) and is difficult for arbitrary motion

New method of fast motion approximation without any divergence and easy to calculate radiation reaction will be very useful

The problem we want to study

- **A small charged particle (BH) moving in an arbitrary external gravitational and electromagnetic fields.**

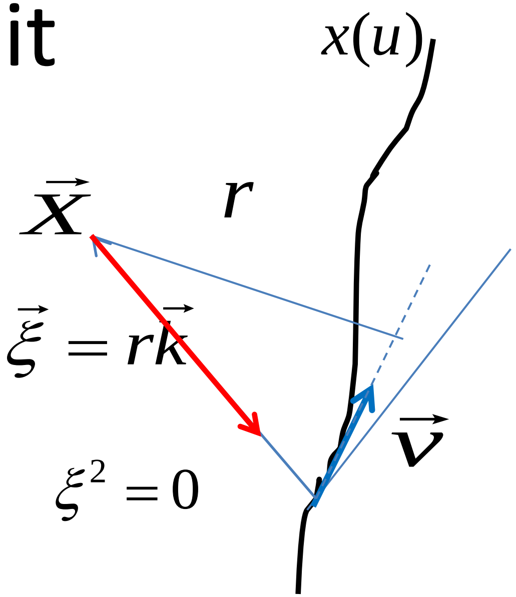
We do not want to have divergences anywhere in the derivation and EOM.

We have learned that infinities appears by considering delta function type source, and we have also learned that no infinities appear by considering point particle limit in post-Newtonian case.

We would like to consider point particle limit, but the limit will be taken along null direction on this case

We will use a (future) null coordinate with vertex at the particle

Point particle limit



We want to consider the following situation and taking a limit r -> 0 along the null direction

$x(u)$: a timelike world-line with 4-vel. v and 4-accelaration a

We also consider the situation where a small particle produces a small perturbation on a background except very small region around the particle(body zone)

This may be possible by shrinking the boundary of the body zone as m and assuming the following scaling

$$m,\ e \sim \varepsilon^2 \quad as \quad r \sim \varepsilon$$

The boundary always stays in the far zone viewed from the body, thus far zone expansion is possible at the boundary

Background space-time

A solution of Einstein-Maxwell equations

We consider the following form of the metric in the coordinate (x,y, r, u)

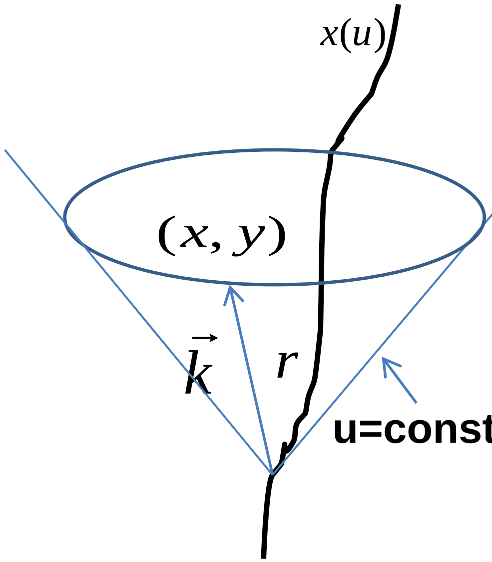
$$ds^2 = -\frac{r^2}{p^2}(e^\alpha ch\beta\, dx + e^\alpha sh\beta\, dy + adu)(e^\alpha sh\beta\, dx + e^\alpha ch\beta\, dy + bdu) + 2dudr + c\, du^2$$

$$g_{\mu\nu}k^\mu k^\nu = 0,$$

$$k_\mu = \frac{dx^\mu}{dr} = u,_\mu$$

Without a massive particle, the spacetime is regular on a timelike (r=0) world-line and in a nbd. of the world-line

$$g = \eta + O(r^2)$$



Therefore we assume our metric approaches to the flat metric at a timelike curve X=x(u) written in the following form(Newman-Unti,1962)

$$ds^2 = \eta_{\alpha\beta} dX^{\alpha} dX^{\beta}$$

$$= -r^2 P_0^{-2}(dx^2 + dy^2) + 2dudr + (1 - 2h_0 r)du^2$$

where

$$\varsigma^{\mu} = \left(1 + \frac{1}{4}(x^2 + y^2), -x, -y, -1 + \frac{1}{4}(x^2 + y^2) \right)$$

$$P_0 = v_{\mu} \varsigma^{\mu}$$

$$h_0 = a_{\mu} k^{\mu} = \frac{\partial}{\partial u}(\log P_0)$$

$v = \dfrac{dx}{du}$: 4-velocity of the worlfdline r=0

$$a = \frac{dv}{du} :$$

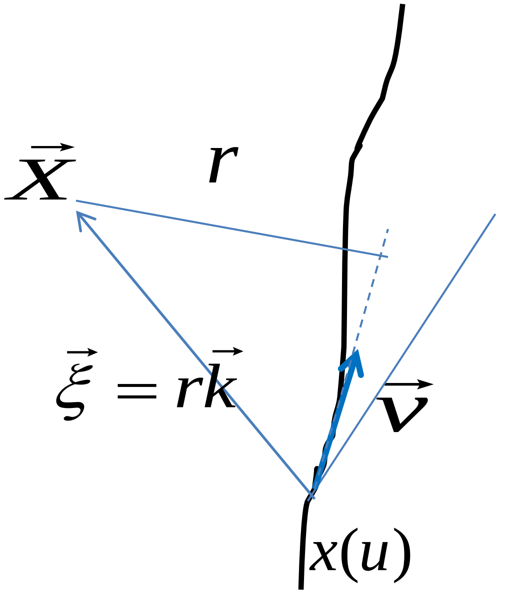
Derivation

$$\xi^{\mu} = X^{\mu} - x^{\mu}(u): \quad \xi_{\mu}\xi^{\mu} = 0$$

$$r \equiv v_{\mu}\xi^{\mu} \geq 0$$

$$0 = \xi_{\mu}(\delta X^{\mu} - \delta x^{\mu}) = \xi_{\mu}(\delta X^{\mu} - v^{\mu}\delta u)$$

$$\Rightarrow \quad u,_{\mu} \equiv \frac{\partial u}{\partial X^{\mu}} = \frac{1}{r}\xi_{\mu} \equiv k^{\mu}$$



$$\frac{\partial k^{\mu}}{\partial r} = k^{\mu},_{\nu} k^{\nu} = 0, \quad \frac{\partial k^{\mu}}{\partial u} = k^{\mu},_{\nu} v^{\nu} = -h_0 k^{\mu},$$

$$h_0 \equiv a_{\mu}k^{\mu}$$

$$k^{\mu} = P_0^{-1}(u)\varsigma^{\mu}(x,y) \quad h_0 = P_0^{-1}\frac{\partial P_0}{\partial u}$$

$$\varsigma^{\mu} = \left(1 + \frac{1}{4}(x^2 + y^2), -x, -y, -1 + \frac{1}{4}(x^2 + y^2)\right)$$

Coordinate transformation relating the Cartesian (X) and curvilinear (u,r,x,y)

$$X^{\mu} = x^{\mu}(u) + rP_0^{-1}\varsigma^{\mu}$$

Then

$$dX^{\mu} = dx^{\mu} + r\varsigma^{\mu}dP_0^{-1} + k^{\mu}dr + rP_0^{-1}d\varsigma^{\mu}$$

$$= (v^{\mu} - rP_0^{-1}h_0\varsigma^{\mu})du + k^{\mu}dr + rP_0^{-1}\left(\frac{\partial\varsigma^{\mu}}{\partial x}dx + \frac{\partial\varsigma^{\mu}}{\partial y}dy\right)$$

$$ds^2 = \eta_{\alpha\beta}dX^{\alpha}dX^{\beta}$$

$$= -r^2P_0^{-2}(dx^2 + dy^2) + 2dudr + (1 - 2h_0r)du^2$$

Small r expansion around r=0

$$ds^2 = -\frac{r^2}{p^2}(e^{\alpha} ch\beta\, dx + e^{\alpha} sh\beta\, dy + a du)(e^{\alpha} sh\beta\, dx + e^{\alpha} ch\beta\, dy + b du) + 2 du dr + c\, du^2$$

$$\to -r^2 P_0^{-2}(dx^2 + dy^2) + 2 du dr + (1 - 2h_0 r) du^2 \qquad r \to 0$$

By solving Einstein-Maxwell equation order by order

$$p = P_0(1 + q_2 r^2 + q_3 r^3 + ...)$$

$$\alpha = \alpha_2 r^2 + \alpha_3 r^3 + ...,$$

$$\beta = \beta_2 r^2 + \beta_3 r^3 + ...,$$

$$a = a_1 r \ + a_2 r^2 + ...,$$

$$b = b_1 r + b_2 r^2 + ...,$$

$$c = 1 - h_0 r + c_2 r^2 + ...$$

Similary background Maxwell field $A = L dx + M dy + K du$

$$L = r^2 L_2 + r^3 L_3 + ...,$$

$$M = r^2 M_2 + r^3 M_3 + ...,$$

$$K = r\ K_1 + r^2 K_2 + ...,$$

BH perturbation of Background

$$ds^2 = g_{\mu\nu}(\alpha,\beta,a,b,p,c)dx^\mu dx^\nu$$

$$p = P_0(1 + q_2 r^2 + q_3 r^3 + ...)$$

$$\alpha = \alpha_2 r^2 + \alpha_3 r^3 + ...,$$

$$\beta = \beta_2 r^2 + \beta_3 r^3 + ...,$$

$$a = a_1 r \;\; + a_2 r^2 + ...,$$

$$b = b_1 r + b_2 r^2 + ...,$$

$$c = 1 - h_0 r + c_2 r^2 + ...$$

Introduce a small charged BH on the background

$$p = \hat{P}_0(1 + \hat{q}_2 r^2 + \hat{q}_3 r^3 + ...)$$

$$\alpha = \hat{\alpha}_2 r^2 + \hat{\alpha}_3 r^3 + ...,$$

$$\beta = \hat{\beta}_2 r^2 + \hat{\beta}_3 r^3 + ...,$$

$$a = \frac{\hat{a}_{-1}}{r} + \hat{a}_0 + \hat{a}_1 r \;\; + \hat{a}_2 r^2 + ...,$$

$$b = \frac{\hat{b}_{-1}}{r} + \hat{b}_0 + \hat{b}_1 r + \hat{b}_2 r^2 + ...,$$

$$c = \frac{e^2}{r^2} - \frac{2(m + 2\hat{f}_{-1})}{r} + \hat{c}_0 + \hat{c}_1 r + c_2 r^2 + ...$$

where

$$\hat{a}_{-1} = \hat{a}_0 = O(m,e) \equiv O_1,$$

$$\hat{b}_{-1} = \hat{b}_0 = O_1, \;\; \hat{f}_{-1} = O_2$$

$$\hat{A} - A = O_1$$

$$\hat{P}_0 = P_0 \;\; (1 + \underset{O_1}{Q_1} \; + \; \underset{O_2}{Q_2} \; + \; O_3 \;)$$

All of these coefficients are chosen in order to satisfy the Reisnner-Nordstrom black hole limit, and higher-order terms are determined by solving Einstein-Maxwell field equations by order by order in r

$$A = Ldx + Mdy + Kdu$$

RN BH as r->0

$$L = r^2 L_2 + r^3 L_3 + ...,$$
$$M = r^2 M_2 + r^3 M_3 + ...,$$
$$K = r\ K_1 + r^2 K_2 + ...,$$

$\Rightarrow$

$$L = \hat{L}_0 + r^2 \hat{L}_2 + r^3 \hat{L}_3 + ...,$$
$$M = \hat{M}_0 + r^2 \hat{M}_2 + r^3 \hat{M}_3 + ...,$$
$$K = \frac{(e + \hat{K}_{-1})}{r} + \hat{K}_0 + r\ \hat{K}_1 + r^2 \hat{K}_2 + ...,$$

$$\hat{L}_0 = \hat{M}_0 = O_1,$$
$$\hat{K}_{-1} = O_2, \quad \hat{K}_0 = O_1$$

We assume that A is predominantly the Lienard-Wiechert form near r=0

$$A = A_\mu dX^\mu = \frac{e v_\mu dX^\mu}{r} = e\left(d(\log r) + \left(\frac{1}{r} - h_0\right)du\right) \to e\left(\frac{1}{r} - h_0\right)du$$

$$r = v_\nu (X^\nu - x^\nu(u)) \to r,_\mu = v_\mu - (1 - r h_0) k_\mu$$
$$\to v_\mu dX^\mu = dr - (1 - r h_0) du$$

$\Rightarrow$ $\hat{L}_0 = \hat{M}_0 = O_2, \quad \hat{K}_0 = -e h_0 + O_2$

Derivation method of Equation of Motion

The null-hypersurfaces u=const. in the perturbed spacetime are approximately future null-cones for small r

The wave fronts can be approximated 2-spheres near the black hole

Neglecting O(r^4)-terms, the line element induced on these null hypersurfaces are given by

$$ds_0^2 = -r^2 \hat{P}_0^{-2}(dx^2 + dy^2) \qquad \hat{P}_0 = P_0(1 + Q_1 + Q_2 + O_3)$$

Necessary conditions for the 2-surfaces with the above line-elements to be smooth, non-trivial deformations of 2-spheres will be the equations of motion of the black hole.

Solve Einstein-Maxwell equation (R=T_EM) order by order in m and e

$$\Xi_{ab} \equiv \hat{R}_{ab} - T_{ab}^{EM}$$

$$\Xi_{AA} = r^{-4}O_3 + \quad + r^{-2}O_3 + r^{-1}O_2 + O(r^0)$$

$$_{(-2)}\Xi_{AA} = 2\hat{\Delta}\log\hat{P}_0 - 4\hat{P}_0^{\,4}\left(\frac{\partial \hat{M}_0}{\partial x} - \frac{\partial \hat{L}_0}{\partial y}\right)\left(\frac{\partial \hat{M}_2}{\partial x} - \frac{\partial \hat{L}_2}{\partial y}\right) - 3\hat{P}_0^{\,2}\left[\frac{\partial}{\partial x}\left(\hat{P}_0^{\,-2}\hat{a}_{-1}\right) + \frac{\partial}{\partial y}\left(\hat{P}_0^{\,-2}\hat{b}_{-1}\right)\right]$$

$$+ 4e\hat{K}_1 + 4e\hat{K}_1\hat{K}_{-1} - 8e\left(\hat{a}_{-1}\hat{L}_2 + \hat{b}_{-1}\hat{M}_2\right) - 2\hat{c}_0 + 12e^2\hat{q}_2 - \frac{1}{2}\hat{P}_0^{\,-2}\left(\hat{a}_{-1}^{\,2} + \hat{b}_{-1}^{\,2}\right)$$

$$\Rightarrow \hat{c}_0 = 1 + \Delta Q_1 + 2Q_1 + 8eF_{\mu\nu}k^{\mu}v^{\nu} + O_2$$

$$\Xi_{34} = r^{-4}O_4 + r^{-3}O_3 + r^{-2}O_2 + r^{-1}O_2 + O(r^0)$$

$$_{(1)}M^4 = -2\hat{P}_0^{\,-2}\hat{K}_1 + 4\hat{P}_0^{\,-2}\left(\hat{a}_{-1}\hat{L}_2 + \hat{b}_{-1}\hat{M}_2 - e\hat{q}_2\right) + 2\left(\frac{\partial \hat{L}_2}{\partial x} + \frac{\partial \hat{M}_2}{\partial y}\right)$$

$$\Rightarrow \hat{K}_1 = P_0^{\,2}\left(\frac{\partial \hat{L}_2}{\partial x} + \frac{\partial \hat{M}_2}{\partial y}\right) + O_1 = F_{\mu\nu}k^{\mu}v^{\nu} + O_1$$

$$M^4 = \quad rO_2 + O(r^2)$$

Equation governing the perturbation of the 2-sphere around BH

$$_{(-2)}\Xi_{44} = \frac{1}{2}\Delta(\Delta Q_1 + 2Q_1) + 6ma_\mu p^\mu - 6eF_{\mu\nu}p^\mu v^\nu + O_2$$

where $\Delta \equiv P_0^2\left(\partial_x^2 + \partial_y^2\right)$ Laplacian on the 2-sphere

$$p^\mu = h^\mu_\nu k^\nu, \quad h^\mu_\nu = \delta^\mu_\nu - v^\mu v_\nu$$

$$\Delta p^\mu + 2p^\mu = 0$$

$$\frac{1}{2}\Delta(\Delta Q_1 + 2Q_1) + 3ma_\mu \Delta p^\mu - 3eF_{\mu\nu}v^\nu \Delta p^\mu = O_2$$

$$\Rightarrow \quad \Delta Q_1 + 2Q_1 = 6ma_\mu p^\mu - 6eF_{\mu\nu}p^\mu v^\nu + A(u) + O_2$$

$$\Rightarrow \quad (ma_\mu - eF_{\mu\nu}v^\nu)p^\mu = O_2, \quad \forall p^\mu$$

Same equation is obtained by taking angular average over a small sphere around r=0

Q_1 is l=0 or l=1 trivial perturbation, thus we can take Q_1=0

In the next order

Induced metric on the two-sphere around BH

$$ds_0^2 = -r^2 \hat{P}_0^{-2}(dx^2 + dy^2) \qquad \hat{P}_0 = P_0(1 + Q_1 + Q_2 + O_3)$$

$$_{(-2)}\Xi_{44} = \frac{1}{2}\Delta(\Delta Q_2 + 2Q_2) + A_{l=0} + A_{l=1} + A_{l=2} + O_3$$

l=0 spherical harmonic

l=1 spherical harmonic

l=2 spherical harmonic

$$A_{l=0} = -4\dot{G} - \frac{16}{3}e^2 F_\mu{}^\rho F_{\rho\nu} v^\mu v^\nu$$

$$A_{l=1} = 6ma_\mu p^\mu - 6eF_{\mu\nu}p^\mu v^\nu - 4e^2 h_\mu^\nu \dot{a}_\nu p^\mu - 8me\dot{F}_{\mu\nu}p^\mu v^\nu - 8e^2 h_\mu^\lambda F_\lambda{}^\rho F_{\rho\nu} p^\mu p^\nu + 12Ga_\mu p^\mu$$
$$-12e^2 K(u) F_{\mu\nu} p^\mu v^\nu - 3(e^2 U(u) + 2mX(u))\frac{\partial}{\partial x}(\log P_0) - 3(e^2 V(u) + 2mY(u))\frac{\partial}{\partial y}(\log P_0)$$

$$A_{l=2} = \chi_{\mu\nu}(u) k^\mu k^\nu$$

Provided A_0=O_3, this equation can be integrated without introduction of directional singularities

$$\dot{G} = -\frac{4}{3} e^2 F_\mu^{\ \rho} F_{\rho\nu} v^\mu v^\nu$$

Eqn. for Q_2

$$\frac{1}{2}\Delta(\Delta Q_2 + 2Q_2) + \frac{1}{2}\Delta A_{l=1} + \frac{1}{6}\Delta A_{l=2} = O_3$$

$$\Rightarrow \quad \Delta Q_2 + 2Q_2 = A_1 + \frac{1}{3} A_2 + O_3$$

Or, taking angular average over a small sphere around r=0

For Q2 to be free of directional singularities we must have $A_1 = O_3$ which gives us the following EOM

$$ma_\mu = eF_{\mu\nu}v^\nu + \frac{2}{3}e^2 h^\nu_\mu \dot{a}_\nu + \frac{4}{3}e^2 h^\lambda_\mu F^\rho_\lambda F_{\rho\nu}v^\nu + T_\mu + O_3$$

where

$$T_\mu = \frac{e}{m}\left\{ \omega^\nu_\rho F_{\nu\mu} - \omega^\nu_\mu F_{\nu\rho} - 2GF_{\mu\rho} \right\} v^\rho$$

with

$$\dot{\omega}_{\mu\nu} = 2e^2 K(u) F_{\mu\nu} + \Omega_{\mu\nu}(u)$$

$$\dot{G} = -\frac{4}{3}e^2 F^\rho_\mu F_{\rho\nu} v^\mu v^\nu$$

Remaining perturbation

$$\Delta Q_2 + 2Q_2 = -\frac{1}{18}\Delta A_2 + O_3 \rightarrow Q_2 = -\frac{1}{12}A_2$$

Smooth non-trivial l=2 perturbations of the wave fronts near the black hole

$$\Delta Q_2 + 6Q_2 = 0$$

Problem?

Does not coincide with DeWitt-Breme

(a charged particle on a fixed curved background and no EM background) .

Ours solves Einstein-Maxwell equation consistently up to O_2. We may have to go O_3

EOM with spin

Kerr solution with mass m and angular momentum (a_0,b_0,c_0) can be put in the form

$$ds_{Kerr}{}^2 = -(r^2+F^2)\,p_0^{-2}\left\{(dx+ad\Sigma)^2+(dy+bd\Sigma)^2\right\}+2dud\Sigma+c\;d\Sigma^2$$

with

$$d\Sigma = du + F_y dx - F_x dy$$

$$p_0 = 1 + 1/4(x^2+y^2)$$

$$F = p_0^{-1}\left\{a_0 x + b_0 y + c_0[1 - 1/4(x^2+y^2)]\right\}$$

$$a = \frac{p_0^2 F_y}{r^2+F^2}, \qquad b = -\frac{p_0^2 F_x}{r^2+F^2},$$

$$c = 1 - \frac{2mr}{r^2+F^2} + \frac{p_0^2(F_x^2+F_y^2)}{r^2+F^2}$$

We consider a general metric which approaches to the above form

Perturbed Space-time

$$ds^2 = -\frac{r^2}{\widehat{p}^2}(e^{\widehat{\alpha}} ch\widehat{\beta}\, dx + e^{\widehat{\alpha}} sh\widehat{\beta}\, dy + \widehat{a} d\Sigma)(e^{\widehat{\alpha}} sh\widehat{\beta}\, dx + e^{\widehat{\alpha}} ch\widehat{\beta}\, dy + \widehat{b} d\Sigma) + 2d\Sigma dr + \widehat{c}\, d\Sigma^2$$

Expansion near r=0

$$\hat{p} = \hat{P}_0(-\frac{\hat{F}^2}{2\,r^2} + 1 + \hat{q}_2 r^2 + \dots\ ) \ ,$$

$$\hat{P}_0 = P_0(1+Q_1+Q_2+O_3) \ , \qquad Q_n = O_n(n=1,2,\dots) \ ,$$

$$\begin{aligned} \hat{\alpha} &= \hat{\alpha}_2 r^2 + \hat{\alpha}_3 r^3 + \dots \ , \\ \hat{\beta} &= \hat{\beta}_2 r^2 + \hat{\beta}_3 r^3 + \dots \ , \end{aligned}$$

$$\begin{aligned} \hat{A} &= \hat{F}_y = O_1 \ , \\ \hat{B} &= -\hat{F}_x = O_1 \dots \ , \end{aligned}$$

$$\begin{aligned} \hat{a} &= \frac{\hat{P}_0^2 \hat{F}_y}{r^2} + \hat{a}_0 + \hat{a}_1 r + \dots \ , \\ \hat{b} &= -\frac{\hat{P}_0^2 \hat{F}_x}{r^2} + \hat{b}_0 + \hat{b}_1 r + \dots \ , \end{aligned}$$

$$\hat{c} = \frac{\hat{P}_0^2(\hat{F}_x^2 + \hat{F}_y^2)}{r^2} - \frac{2\,m}{r} + \hat{c}_0 + \hat{c}_1 r + \dots \ ,$$

Perturbed space-time as approximate vacuum space-time

$$\hat{R}_{33} = {}_{(-4)}\hat{R}_{33} r^{-4} + {}_{(-3)}\hat{R}_{33} r^{-3} + \dots \ ,$$

where

$${}_{(-4)}\hat{R}_{33} = \frac{1}{2}\{2\hat{F} + \hat{P}_0^2(\hat{F}_{xx} + \hat{F}_{yy})\}\{2\hat{F} - \hat{P}_0^2(\hat{F}_{xx} + \hat{F}_{yy})\}$$

$$P_0^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\hat{F} + 2\,\hat{F} \equiv \Delta\hat{F} + 2\,\hat{F} = O_2 \ ,$$

F is an l=1 spherical harmonics and can be written as

$$\hat{F} = s_i(u)\,k^i + O_2 \ , \qquad s_i\,v^i = 0 \ .$$

$${}_{(-3)}\hat{R}_{34} = 2\widehat{F}\widehat{F}_u + O_3 \quad \Rightarrow \quad \frac{ds^i}{du} = O_2$$

$${}_{(-2)}\hat{R}_{44} \quad \Rightarrow \quad ma^i = s\ R\ u\ u$$

Conclusions

- We have developed a new method for fast moving small self-gravitating particle without any divergence
- Does not coincide with DeWitt-Breme(testEM on fixed curved background) . Ours solve Einstein-Maxwell equation consistently up to O_2.
- Gravitational radiation reaction in O_4 ?