What is the Shape of a Black Hole?

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It is a great honour to be invited to speak here at the twentieth the Workshop on General Relativity and Gravitation in Japan and in a year which coincides with the 60th birthdays of Takashi Nakamura and Kei-ichi Maeda. Other's will want to say more about Takashi Nakamura, but before starting my talk proper, I would like to say a few words about my old friend Kei-chi Maeda whom I know much better, and with whom I have collaborated over the years.

It is a particular pleasure to be here in Kyoto, and to be able to participate in, among other things, a celebration of the 60th birthday of my old friend and colleague Kei-ichi Maeda, the more so because, as he told me at the celebration of my own 60'th birthday 4 years ago in Cambridge, $60 = 5 \times 12$ has a special significance in Japan.

Like me, Maeda-san was born when the world was recovering from what was probably the most destructive conflict in human history and faced even greater dangers. Sixty years on, things don't seem quite so gloomy despite the many challenges the world no faces. This is due in part to the fact that at the macro-level the people of our planet increasing see themselves as part of a single global community. To me Maeda-san's outstanding scientific career is a brilliant illustration of that at the micro-level. He has tirelessly travelled the world, and talked and collaborated at all levels with our own micro-community, and contributed to its international organisations.

That is how I first met him, during what I think was the first of his many visits to England. However it was in France where our most productive collaboration took place. We were both visiting the Observatoire de Paris-Meudon. Kei-ichi was a post-doc and I was on sabbatical. I still remember with great fondness the discus ions about black holes and membranes in the kitchen of the house in Reuil-Malmaison where I and my family were staying. I at least, felt we were just uncovering the tip of a vast iceberg, but I little suspected how vast that iceberg would prove to be. Only too soon (at least for me) Kei-ichi had to depart for Tokyo

But our work * was not cut short despite the absence of the internet, not least because of his energy and enthusiasm, and his truly remarkable ability to grasp and remember the essential points of any discussion and fill in the details later .

*4) Black Holes and Membranes in Higher Dimensional Theories with Dilaton Fields. G.W. Gibbons, (Meudon Observ. & Ecole Normale Superieure) , Kei-ichi Maeda, (Meudon Observ. & Tokyo U.) . UTAP-48-87, LPTENS-87-10, Mar 1987. 55pp. Published in Nucl.Phys.B298:741,1988.

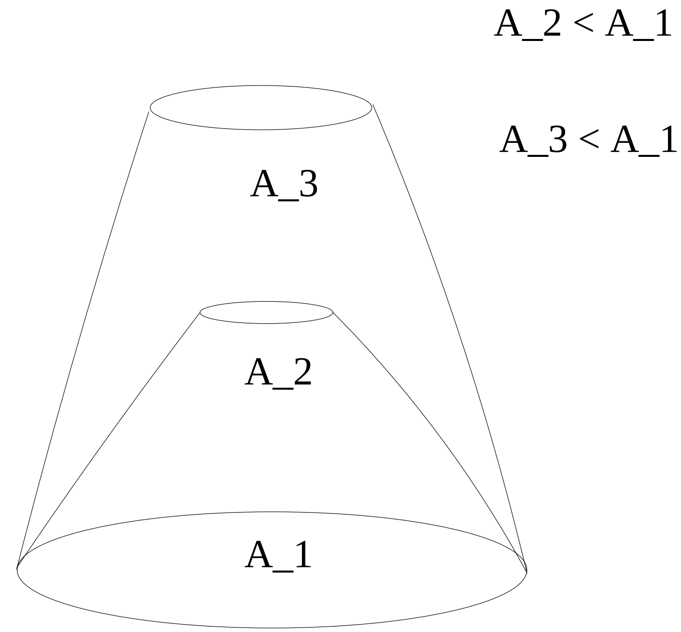
It is to the subject of black holes, in both four and higher dimensions, to which Kei-ichi Maeda has contributed so much, that I want to turn to in this talk. In particular, I want to revisit some very old questions in the light of recent developments. The basic problem is:

How do we describe and characterise the geometry of horizons, and what are their general geometrical properties?

By "horizon" I shall mean "apparent horizon" or " outermost closed marginally trapped" $D-2$ dimensional hypersurface S in a D dimensional asymptotically flat or asymptotically Anti-De-Sitter spacetime whose energy momentum tensor satisfies the Dominant Energy Condition.

Closed Trapped and Marginally Trapped Surfaces

The area of a closed trapped 2-surface decreases in both the inward *and* the outward directions if pushed to the future along its two lightlike normals



The many exact solutions of supergravity and Kaluza-Klein type theories in diverse dimensions now available satisfy this condition and provide a copious supply of examples for formulating and testing conjectures

In many cases, especially if $D \leq 5$ and the cosmological constant $\Lambda = 0$, powerful solution generating techniques, now often thought of as T and S dualities, are available because of the high degree of symmetry of the space of fields, which is not infrequently a *symmetric space*

In recent months, Mirjam Cvetic, Chris Pope and I , with some initial help from a Summer Student,Thiti Sirithanakorn (“We”) have been availing ourselves of these opportunities.

Note that supergravity solutions are useful in this way even if one is not interested in supergravity *per se*.

What is meant by Shape ?

By "shape " I shall mean *intrinsic* geometrical properties, such as the area A which determined by the induced Riemannian metric g. The *extrinsic* geometry is determined by its being marginally trapped

I shall also be interested in how the shape is related to dynamical quantities such as the total energy E (i.e. ADM or Abbott-Deser mass), total angular momentum J, or total angular momenta J_i, $i = 1, 2, \ldots, [\frac{D-2}{2}]$ or electric charge Q or charges Q_a.

The Penrose Inequality

The best known and best investigated example of what I have in mind is variously called the Penrose, Cosmic Censorship, or Isoperimetric Inequality for Black Holes In four dimensions*

$$\boxed{A \leq 16\pi E^2} \tag{1}$$

*we set Newton's $G = 1$

Bekenstein-Hawking Entropy

Physically, since the black hole entropy

$$\boxed{S = \frac{1}{4}A}\,, \tag{2}$$

this is the statement that for fixed energy, the Schwarzschild, or the Kottler solution has the largest possible entropy. However (2) is only rigourously established for *stationary*, i.e. time independent black holes.

All known examples are consistent with this, and in the four-dimensional *time symmetric* , or so-called *Riemannian* case there are rigourous proofs for asymptotically flat initial data due to Huisken and Ilmanen and by Bray. In higher dimensions we have work by Barrabes and Frolov and Gibbons and Holzegel and by Bray. These provide partial results showing that

$$\boxed{\frac{A}{\mathcal{A}_{-2}} \leq \left(\frac{16\pi E}{(D-2)\mathcal{A}_{D-2}}\right)^{\frac{D-2}{D-3}}} \tag{3}$$

Bray's results on the Riemannian case are valid up to $D = 8$, this seems to be related to the the failure of regularity of minimal surfaces which also been encountered in Brane theory *

*The Bernstein Conjecture, Minimal Cones, and Critical Dimensions. Gary W, Kei-ichi Maeda and Umpei Miyamoto Class.Quant.Grav.26:185008,2009. e-Print: arXiv:0906.0264 [hep-th]

Two other measures of shape

If $D = 4$, in addition to the area the metric g on S gives rise to two other important measures of the shape

- The length $l(S, g)$ of the shortest non-trivial closed geodesic

- Birkhoff's invariant $\beta(S, g)$

There is an obvious generalisation of $l(S, g)$ to higher dimensions the generalisation of $\beta(S, g)$ is much less so and not unique.

Antipodal Isometries

The horizons of all known isolated black holes in all dimensions admit an *antipodal isometry*, that is fixed point free involution I preserving the metric. E.g. $I : (\theta, \phi) \rightarrow (\pi - \theta, \phi + \pi)$ for Kerr-Newman. If $D = 4$, and this is assumed, then Pu showed, by passing to $S/I \equiv RP^2$,

$$\boxed{l(S, g) \leq \sqrt{\pi A}} \tag{4}$$

Assuming the Cosmic-Censorship Inequality (1) we obtain

$$\boxed{l(S, g) \leq 4\pi E} \tag{5}$$

which smells like the *Thorne's Hoop Conjecture* (see later)

A new conjecture

I conjecture that if $D = 4$, then

$$\boxed{l(S, g) \leq 4\pi E} \tag{6}$$

always holds regardless of whether S, g admits an antipodal isometry

- We have checked it on all examples known to us.

Higher Dimensions

In higher dimensions there is no theorem analogous to Pu's theorem. However, if S admits an antipodal isometry I we may bound $l(S,g)$ above by estimating the distance $d(x,Ix)$ between points $x \in S$ and their antipodes Ix. by a topological argument there must a shortest geodesic homotopic to the projection of the curve joining x and Ix on S/I. We have found if $D=5$ and even dimensional cases examined that

$$\boxed{l(S,g) \leq 2\pi \Big(\frac{A}{\mathcal{A}_{D-2}} \Big)^{\frac{1}{D-2}}} \tag{7}$$

$$\boxed{l(S,g) \leq \Big(\frac{16\pi^{D-2} E}{(D-2)\mathcal{A}_{D-2}} \Big)^{\frac{1}{D-3}}} \tag{8}$$

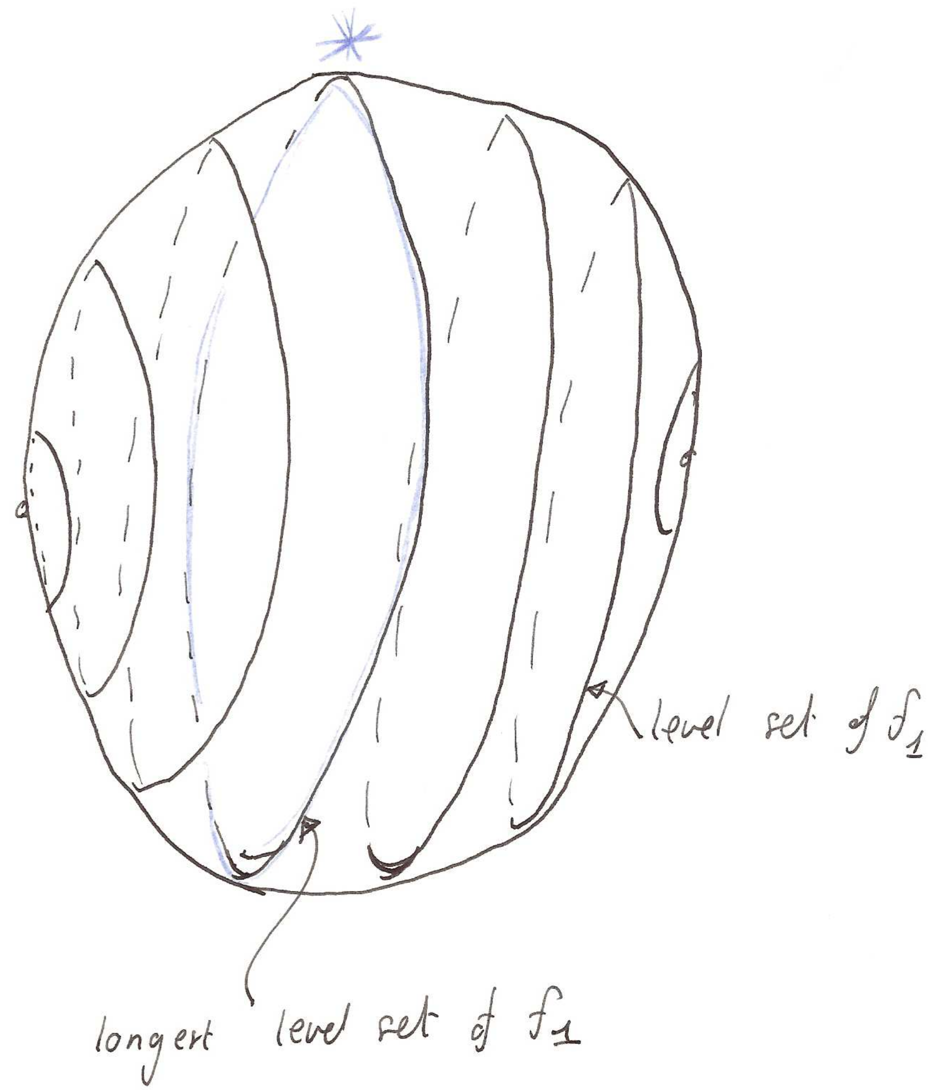
For odd $D \geq 7$ our results are inconclusive.

Birkhoff's Invariant and Thorne's Hoop Conjecture

To connect with Thorne's Hoop conjecture we turn to the Birkhoff invariant $\beta(S,g)$ in $D=4$ spacetime dimensions. We consider a "foliation", "sweep out" or "slicing " of S by S^1 leaves $f = \text{constant} = c$ whose length or circumference is $l(S,g,f,c)$ with two point-like leaves. E.G. $f = \cos\theta\,, l = r_+ \sin\theta$ for Kerr-Newman. Let $\beta(S,g,f)$ be the *maximum* circumference for that choice of slicing

$$\beta(S,g,f) = \max_c \quad l(S,g,f,c) \tag{9}$$

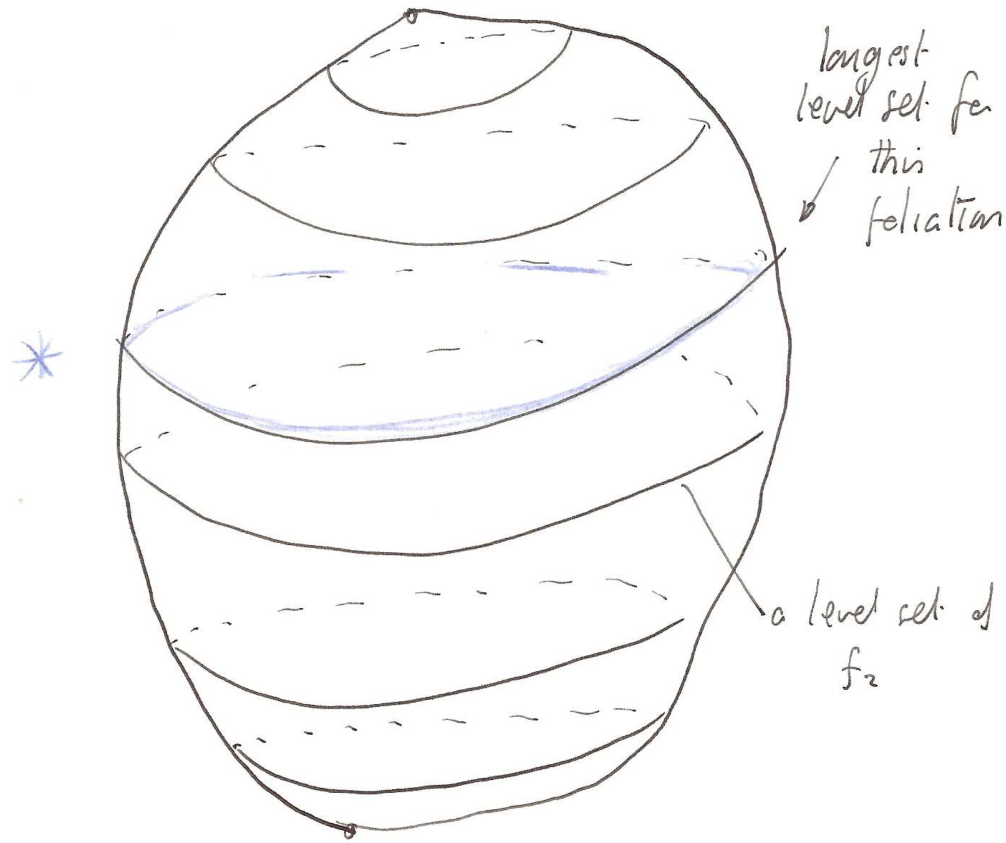
A sweep out of S by circles S^1



Now *minimize* over all choices of slices (e.g. "over all ways of passing a hoop or an elastic band over the horizon") and define

$$\beta(S,g) = \min_f \quad \beta(S,g,f) = \min_f \max_c l(S,g,f,c) \qquad (10)$$

Another sweep out of S^2 by circles S^1



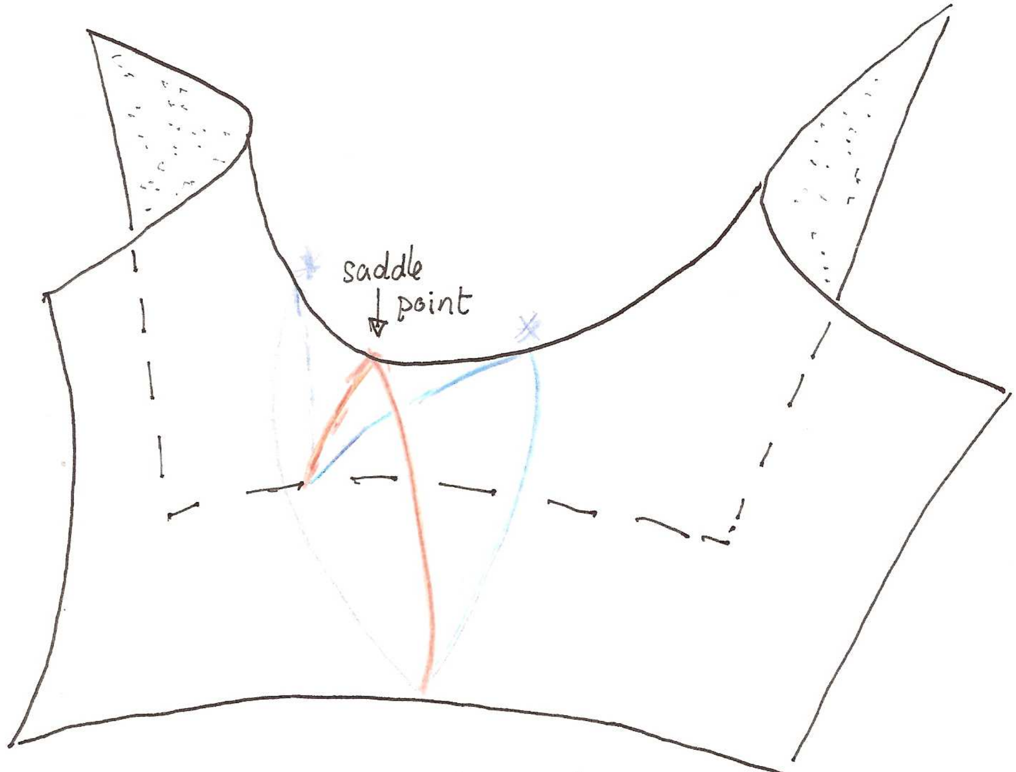
$$\beta \leq \beta(f_2) < \beta(f_1)$$

In 1917* Birkhoff used this "Mountain Pass Method" method to prove that very metric on S^2 admits at least one non-trivial closed geodesic whose length is no greater than $\beta(S,g)$, thus

$$\boxed{l(S,g) \leq \beta(S,g)\,.} \tag{11}$$

*G. D. Birkhoff, Dynamical systems with two degrees of freedom *Trans. Amer. Math. Soc.* **18** (1917)

Mountain Pass Method



To locate the saddle point
minimise the maximum height of all
curves crossing the pass

Another conjecture

In the spirit, if not perhaps not precisely the letter, of Thorne's Hoop conjecture I have recently conjectured that *

$$\boxed{\beta(S,g) \leq 4\pi E\,.} \tag{12}$$

This implies the previous inequality

$$\boxed{l(S,g) \leq 4\pi E\,.} \tag{13}$$

*Birkhoff's invariant and Thorne's Hoop Conjecture. G.W. Gibbons. e-Print: arXiv:0903.1580 [gr-qc]

Obtaining an upper bound for $\beta(S, g)$ merely entails estimating the maximum circumference of a conveniently chosen foliation. In all cases we have examined (with or without a negative Λ term) the conjecture has been verified. These now extend to rotating asymptotically flat black ho;es with four distinct charges and rotating ADS black holes with two charges set equal. Further evidence comes from collapsing shells.

Collapsing Shells and Convex Bodies

This is a class of examples * in which a shell of null matter collapses at the speed of light in which the apparent horizon S may be thought of as a convex body isometrically embedded in Euclidean space E^3. In this case one has

$$8\pi M_{\text{ADM}} \geq \int_S H dA \,, \tag{14}$$

where $H = \frac{1}{2}(\frac{1}{R_1} + \frac{1}{R_2})$ is the mean curvature and R_1 and R_2 the principal radii of curvature of S and dA is the area element on S. The right hand side is called the total mean curvature and it was shown

*G. W. Gibbons, Collapsing Shells and the Isoperimetric Inequality for Black Holes, *Class. Quant. Grav.* **14** (1997) 2905 [arXiv:hep-th/9701049].

by Álvarez Paiva [†] that in this case that

$$\beta(g) \leq \frac{1}{2} \int_S H dA\,. \tag{15}$$

[†] J .C. Álvarez Paiva, Total mean curvature and closed geodesics. *Bull. Belg. Math. Soc. Simon Stevin* **4** (1997) 373–377.

Combining Álvarez Paiva's (15) with (14) establishes the conjecture in this case.

• Surprisingly, perhaps my conjecture holds up even if in grossly non-asymptotically flat situations with a magnetic field. However, as we see, this also leads to a puzzle.

In fact the proof is close to the ideas of Tod. If $\mathbf{n}$ is a unit vector we define the height function on $S \subset E^3$ by

$$h = \mathbf{n}.\mathbf{x}\,, \qquad \mathbf{x} \in \mathbf{S}\,. \tag{16}$$

Let $S_{\mathbf{n}}$ be the orthogonal projection of the body S onto a plane with unit normal $\mathbf{n}$ and let $C(\mathbf{n}) = l(\partial S_{\mathbf{n}})$ be the perimeter of $S_{\mathbf{n}}$. Then

$$\beta(g) \le \beta(h) \le C(\mathbf{n})\,. \tag{17}$$

Now

$$\int_S H dA = \frac{1}{2\pi} \int_{S^2} C(\mathbf{n}) d\omega\,, \tag{18}$$

where $d\omega$ is the standard volume element on the round two-sphere S^2 of unit radius. Thus averaging (17) over S^2 and using (18) gives (15).

Asymptotically-Melvin black holes

were first constructed using a Harrison transformation in Einstein-Maxwell theory, by Ernst and in an explicit form by Ernst and Wild . The metric is

$$ds_4^2 = F^2\Big\{-\Big(1-\frac{2m}{r}\Big)dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + r^2 d\theta^2\Big\} + \frac{r^2 \sin^2\theta}{F^2} d\phi^2 \,, \qquad (19)$$

with

$$F = 1 + \frac{B^2}{4} r^2 \sin^2\theta \,, \qquad (20)$$

where B is the applied magnetic field. If $m = 0$ we get the Melvin solution, whilst if instead $B = 0$ we get the Schwarzschild solution. The energy with respect to the Melvin background is given by $E = m$ and the horizon, which is located at $r = 2m$.

If $\gamma = m\,|B|$, the horizon metric is

$$ds^2 = 4m^2\Big\{(1+\gamma^2\sin^2\theta)^2 d\theta^2 + \frac{\sin^2\theta}{(1+\gamma^2\sin^2\theta)^2}d\phi^2\Big\}, \tag{21}$$

Remarkably, these are the same as in the absence of the magnetic field.

$$\beta(g) \le \max_\theta \frac{4\pi\, E\sin\theta}{1+\gamma^2\sin^2\theta} \tag{22}$$

If $\gamma \le 1$, the circumference $C(\theta)$ has a single maximum at the equator $\theta = \frac{\pi}{2}$, the maximum value being $\frac{4\pi E}{1+\gamma^2} \le 4\pi E$. If $\gamma \ge 1$, the horizon is dumb-bell shaped and has two maxima with $\gamma\sin\theta = 1$, the maximum value being $\frac{2E\pi}{\gamma} < 4\pi E$. Thus the conjecture is always satisfied.

The solutions for a black hole immersed in a magnetic field in Einstein-Maxwell-Dilaton theory have been given by Yazadjiev. The conjecture continues to hold.

Problems with Thorne's Hoop conjecture?

This was that * Horizons form when and only when a mass E gets compacted onto a region whose circumference in EVERY direction is $C \leq 4\pi E$. The capitalization "EVERY "was intended to emphasis the fact that while the collapse of oblate shaped bodies, the circumferences are all roughly equal, in the prolate case, a the collapse of a long almost cylindrically shaped body whose girth was never the less small would not necessarily produce a horizon. However the polar circumference is

$$C_p = 4E \int_0^\pi (1 + \gamma^2 \sin^2 \theta)\, d\theta = 4\pi E(1 + \frac{1}{2}\gamma^2) \geq 4\pi E\,. \tag{23}$$

This would seem to contradict Thorne's "in all directions" formulation of the Hoop Conjecture.

Nonspherical Gravitational Collapse: A Short Review in Magic without Magic ed. J Klauder (San Francisco: Freeman) (1972)

Isometric Embeddings

One way of visualising two dimensional surfaces is to globally embed them isometrically into three dimensional Euclidean space E^3. If the Gauss curvature is everywhere positive, then by a theorem of Weyl and Pogorelov this is always possible and the the embedding is unique. Thus no ambiguity results from the such “inflexible” embeddings. Contrary to a statement by Ernst and Wild, the horizon of the Ernst Wild black hole can be globally isometrically embedded into Euclidean space even though its Gauss curvature can become negative near the waist of the “dumb bell”. However despite being prolate, the Gaussian curvature K of the horizon of Kerr-Newman black hole is

$$K = \frac{({r_+}^2 + a^2)(r_+^2 - 3a^2 \cos^2\theta)}{({r_+}^2 + a^2\cos^2\theta)^3}, \tag{24}$$

K can become negative at the poles $\theta = 0, \pi$ and this precludes a global isometric embedding into E^3 as discovered by Smarr * .

Frolov has pointed out that one may globally embed into four dimensions al Euclidean space E^4 , but this is probably not unique. However a theorem of Pogorelov guarantees a *unique* isometric embedding into three dimensional hyperbolic space H^3. This may be easily achieved using the upper half space model for H^3 . †

*L. Smarr, Surface Geometry of Charged Rotating Black Holes, *Phys Rev* **D 7** (1973) 289

†Global embedding of the Kerr black hole event horizon into hyperbolic 3-space. G.W. Gibbons, C.A.R. Herdeiro and C. Rebelo Phys.Rev.D80:044014,2009. e-Print: arXiv:0906.2768

Hyper-Hoops

The analogue of a “hoop” is a “ hyper-hoop”, a bag or surface of one less dimension than the horizon which can be “dragged” over it. Thus we have a sweep out or foliation by a one parameter family of $D-3$ dimensional surfaces, each of which has an area. In any given foliation f we set

$$\beta(S,g,f) = \max_c A_{D-3}(f^{-1}(c)) \tag{25}$$

and define

$$\beta(S,g) = \min_f \beta(S,g,f) \tag{26}$$

•Such sweep-outs have been used by mathematicians to construct minimal surfaces via the mountain pass method.

If we consider topologically spherical horizons $S \equiv S^{D-2}$, then obvious choices for "hyper-hoops" are $S^p \times S^q$, $p+q = D-3$. E.G. on a round sphere

$$ds^2 = d\theta^2 + \sin^2\theta d\Omega_p^2 + \cos^2\theta d\Omega_q^2 \tag{27}$$

If $q = 0$, we let $0 \leq \theta \leq \pi$. If $pq \neq 0$ we let $0 \leq \theta \leq \frac{\pi}{2}$.

• For Myers-Perry-AdS black holes with two unequal angular momenta $J_1 \neq J_2$ in $D = 5$, we can choose $p = q = 1$ and use the toroidal hyper-hoops swept out by the $U(1) \times U(1)$ rotational sub-group.

• For Tangherlini$_5$, these hyper-hoops are Clifford Tori.

For Clifford sweep outs we find

$$\boxed{\beta(S,g) \leq \frac{16\pi}{3} E\,.} \tag{28}$$

This agrees with some earlier numerical work of Ida and Nakao using the time symmetric initial value problem * and recent work of Yamada and Shinkai †

• Ida and Nakao also pointed out that in $D \geq 5$ some circumferences may become extremely long, and so Thorne's in all directions conjecture fails.

*Isoperimetric inequality for higher dimensional black holes. Daisuke Ida, Ken-ichi Nakao, Phys.Rev.D66:064026,2002. e-Print: gr-qc/0204082

†Black Objects and Hoop Conjecture in Five-dimensional Space-time. Yuta Yamada, Hisa-aki Shinkai, Class.Quant.Grav.27:045012,2010. e-Print: arXiv:0907.2570 [gr-qc]

Sweep outs by $S^1 \times S^{D-4}$

• If we consider the odd dimensional Kerr-AdS solutions with all angular momenta equal $J_1 = J_2 = dots = J_{[\frac{D-1}{2}]}$ then the high $SU([D-12])$ symmetry allows us to foliate by hyper-hoops with topology $S^1 \times S^{D-4}$. We find the obvious generalization of the conjecture for the Birkhoff invariant holds.

• We can also consider just one non-vanishing angular momentum. This also allows an external magnetic field. If $D = 2N+1$ is odd, we find

$$\boxed{\beta(g) \leq \frac{32\pi}{(2N-1)} (N-1)^{\frac{1}{2}(N+1)} N^{-\frac{1}{2}N} E\,,} \tag{29}$$

Conclusions

• If $D = 4$, there is good evidence and some proofs in special cases that

$$\boxed{\beta(S,g) \le 4\pi E\,.} \tag{30}$$

which implies

$$\boxed{l(S,g) \le 4\pi E\,.} \tag{31}$$

which follows if

$$\boxed{l(S,g) \le \sqrt{\pi A} \qquad \text{and} \qquad \sqrt{\pi A} \le 4\pi E\,.} \tag{32}$$

- If $D \geq 5$ and even for all examples tested

$$\boxed{l(S,g) \leq 2\pi\left(\frac{A}{\mathcal{A}_{D-2}}\right)^{\frac{1}{D-2}}} \tag{33}$$

$$\boxed{l(S,g) \leq \left(\frac{16\pi^{D-2}E}{(D-2)\mathcal{A}_{D-2}}\right)^{\frac{1}{D-3}}} \tag{34}$$

- If $D = 5$ sweep outs by 2-tori satisfy

$$\boxed{\beta(S, g) \leq \frac{16\pi}{3} E\,.} \tag{35}$$

- If $D = 2N + 1$ sweep outs by $S^1 \times S^{D-4}$ satisfy

$$\boxed{\beta(g) \leq \frac{32\pi}{(2N-1)}\,(N-1)^{\frac{1}{2}(N+1)}\,N^{-\frac{1}{2}N}\,E\,,} \tag{36}$$

• These inqualities continue to hold in the presence of external Melvin type magnetic fields in $D = 4$ and higher dimensions

• However these examples, and higher dimensional rotating black holes seem to invalidate the “in all directions” part of Thorne’s conjecture.

• If $D = 4$, Rapidly rotating horizons may not be globally isometrically embedded in Euclidean space E^3 , but they can be globally isometrically embedded in Hyperbolic space H^3