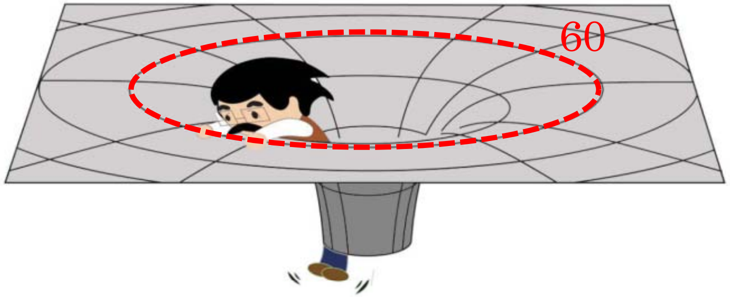
Dimensional Reduction



I. introduction

II. Spacetime symmetry

III. Dynamics of singular hypersurface

IV. Inversion

V. Summary

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I. INTRODUCTION

Dimension D

- spacetime dimension D=4, or 10 in superstring

- # of degree of freedom

    - phase space of a particle D=6

    - # of dynamical variables

Dimensional Reduction

The basic equations in high dimensions : complicated.

- nonlinear
- coupled
- partial differential

↓ reduction

solvable

I focus on "gravity"

II. SPACETIME SYMMETRY

Assume spacetime symmetry

Existence of a Killing vector

↓ dimensional reduction

Lower-dimensional effective theory

- Kaluza-Klein theory
- Higher-dimensional unified theory (supergravity)

Symmetry of an internal space

→ gauge interactions in 4D

Kaluza-Klein cosmology

- **Cosmological dimensional reduction** A. Chodos & S. Detweiler (1980)

5D Kasner solution

$$ds^2 = -dt^2 + a^2(t)dx^2 + b^2(t)dy^2$$

$$a = t^{\frac{1}{2}} \quad b = t^{-\frac{1}{2}}$$

our 3 space : expanding, 5th direction : contracting

⇨ **dynamically explain the large 3 space** b (volume modulus) : time dependent

→ time dependent G_N

$$^{(4)}M_{PL}^2 = \frac{1}{8\pi G_N}$$

$$S = \frac{(^{(D)}M_{PL})^{D-2}}{2} \int dX^D \sqrt{-g} R$$

$$\approx \frac{(^{(D)}M_{PL})^{D-2}}{2} \int dy^n \sqrt{^{(n)}g} \times \int dx^4 \sqrt{-^{(4)}g}\, ^{(4)}R$$

$$= \int dx^4 \frac{(^{(4)}M_{PL})^2}{2} \sqrt{-^{(4)}g}\, ^{(4)}R$$

$$^{(4)}M_{PL} = b(t)^{n/2} (^{(D)}M_{PL})^{(D-2)/2}$$

observational constraint

$$\left|\frac{\dot{G}_N}{G_N}\right| \leq (0.2 \pm 0.4) \times 10^{-11} \text{year}^{-1}$$ Viking Project (1983)

$$\leq (-0.06 \pm 0.2) \times 10^{-11} \text{year}^{-1}$$ binary pulsar (1996)

→ Stabilization of volume modulus

How to find the present universe

N=2, D=6 Kaluza-Klein supergravity

KM & Nishino 1985

$$\mathcal{L}_{BG} = \mathcal{L}_g + \mathcal{L}_v \,, \tag{1}$$

where

$$\mathcal{L}_g = (\sqrt{-g}/4\kappa^2) R \tag{2}$$

and

$$\mathcal{L}_v = -\tfrac{1}{2}\sqrt{-g}\,[g^{MN}\partial_M\varphi\,\partial_N\varphi + (1/2\kappa)e^{\sqrt{2}\kappa\varphi}F_{PQ}F^{PQ} + (g_0^2/\kappa^3)e^{-\sqrt{2}\kappa\varphi}] \,. \tag{3}$$

compactification $M_4 \times S^2$ ← internal space

↑ our world

Size b(t) : small & "stabilize"

scale factor: a(t) large & inflation

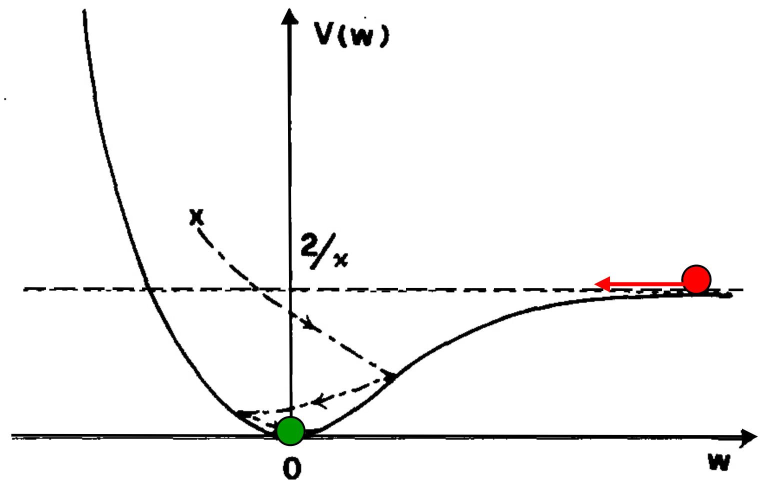
transient inflation to standard Big Bang

scalar field in effective 4D spacetime

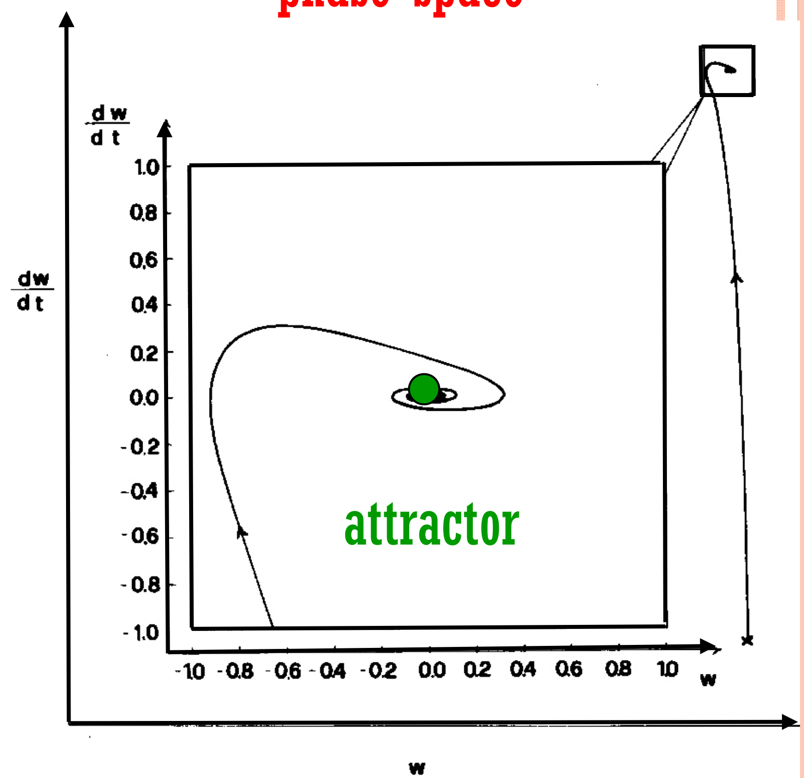
$w = \ln b$

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{w}^2 + V(w)\right]$$

effective potential



phase space



- inflation a: extremely large
- $\Lambda=0$ b: small & static

Our universe is obtained as an attractor !

Dynamical dimensional reduction in the phase space

[(2+1)+1] formalism

KM, Sasaki, Nakamura, Miyama ('80)

4D spacetime with one Killing vector

Dynamics of axi-symmetric system ⇒ rotational Killing vector

⇓

R. Geroch
J. Math. Phys. **12**, 918 (1971)

A Method for Generating Solutions of Einstein's Equations

APPENDIX:
A THREE-DIMENSIONAL FORMALISM
FOR SPACE-TIMES WITH ONE KILLING VECTOR

3D Einstein equations with additional fields

⇓

an "electro-magnetic" like field + a scalar field

(KK reduction from 4D to 3D)

Apply the ADM formalism to discuss the dynamics

III. Dynamics of shingular hypersurface

codimension-one singular hypersurface with symmetry

cf B. Carter's talk

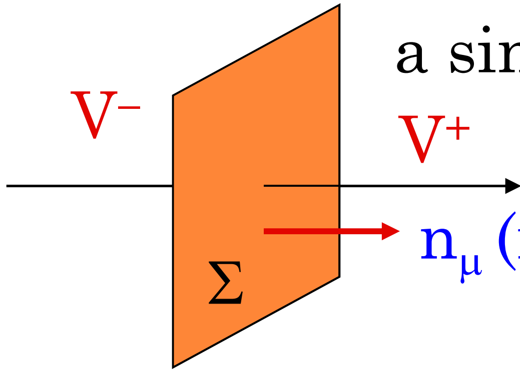
- dynamics of spherically symmetric thin shell in 4D
  - a toy model of gravitational collapse
  - dynamics of a void in FLRW universe

- brane world in 5D

Israel's approach

W.Israel, Nuovo Cimento 44B, 1 (66)

motion of a thin wall (or a thin shell)



a singular hypersurface Σ

n_μ (normal vector)

The projection tensor onto Σ $\quad h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$

The extrinsic curvature of Σ $\quad K_{\mu\nu} = h_\mu{}^\rho h_\nu{}^\sigma n_{\rho;\sigma}$

Gauss-Coddazzi eqs. + Einstein eqs.

$$ {}^{(3)}R + K_{\mu\nu}K^{\mu\nu} - K^2 = -2G_{\mu\nu}n^\mu n^\nu = -16\pi G T_{\mu\nu}n^\mu n^\nu $$

$$ D_\nu K_\mu{}^\nu - D_\mu K = G_{\rho\sigma}h_\mu{}^\rho n^\sigma = 8\pi G T_{\rho\sigma}h_\mu{}^\rho n^\sigma $$

$$ {}^{(3)}R^\mu{}_\nu - \frac{\partial}{\partial n}K^\mu{}_\nu - K K^\mu{}_\nu = R^\rho{}_\sigma h^\mu{}_\rho h^\nu{}_\sigma = 8\pi G \left(T^\rho{}_\sigma - \frac{1}{2}T\delta^\rho{}_\sigma \right) h^\mu{}_\rho h^\nu{}_\sigma $$

Σ : δ–functional singularity

Integrating the equations from V^- to V^+ $\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} dn(*)$

The energy-mementum tensor on Σ

$$S_{\mu\nu} = \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} dn T_{\rho\sigma} h_\mu{}^\rho h_\nu{}^\sigma$$

$$[X]^\pm = X^+ - X^- \quad \text{and} \quad \{X\}^\pm = X^+ + X^-$$

$$[K^\mu{}_\nu]^\pm = -8\pi G\left(S^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu S\right)$$

$$\tilde{K}_{\mu\nu} S^{\mu\nu} = [T^{\mu\nu} n_\mu n_\nu]^\pm$$

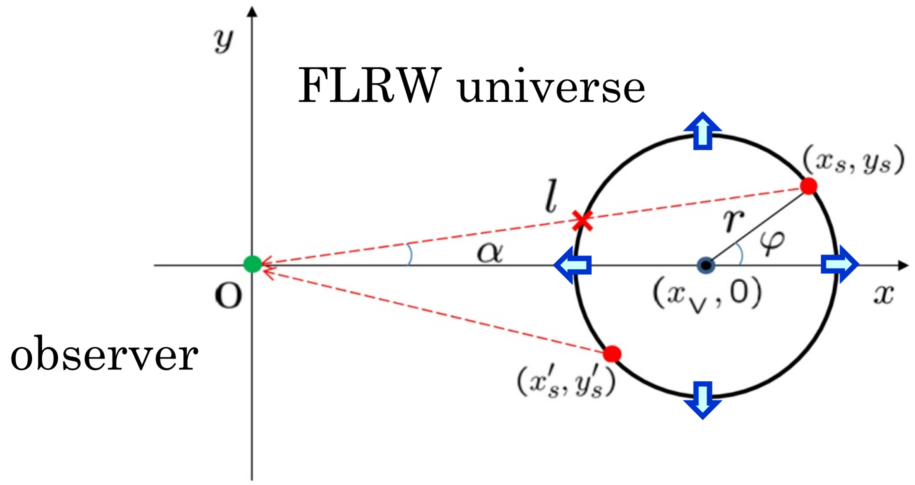
$$D_\nu S_\mu{}^\nu = -[T^{\rho\sigma} n_\rho h_{\mu\sigma}]^\pm$$

Israel's junction condition

$$\tilde{K}_{\mu\nu} = \frac{1}{2}\{K_{\mu\nu}\}^\pm$$

⇒ dynamical equation for a shell

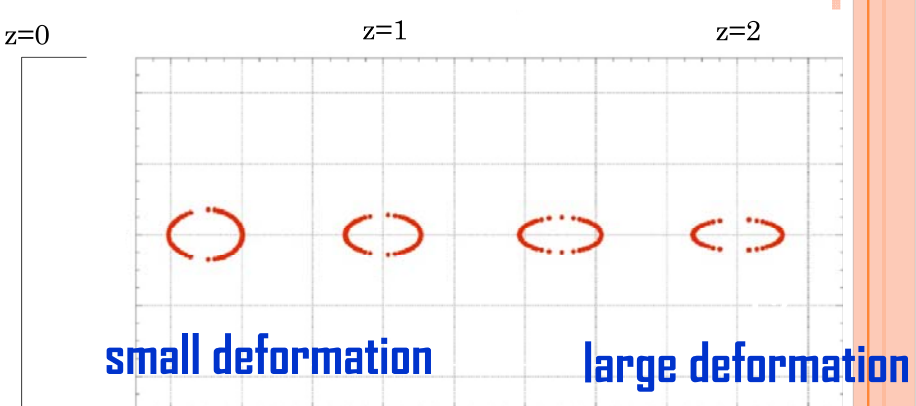
- dynamics of spherically symmetric shell in 4D
  - a toy model of gravitational collapse
  - dynamics of a void in FLRW universe



KM, Sakai, Triay ('10)

peculiar velocity

deformation in a redshift space



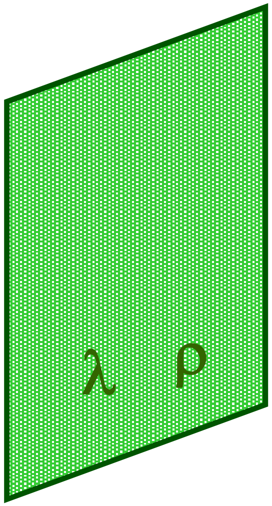
brane world in 5D

brane cosmology in RS II model

5D spacetime

***V* (bulk)**

Λ (<0)



our 3 space

brane : isotropic & homogeneous

infinitesimally thin

maximally symmetric space

Einstein equations

$$^{(5)}G_{AB} = \kappa_5^2 \, {}^{(5)}T_{AB} + \text{brane}$$

inhomogeneous & time dependent

On the brane

- **Israel's junction condition**

$$\left[K^A{}_B\right]^\pm = -\kappa^2\left[S^A{}_B - \frac{1}{3}\delta^A{}_B S\right]^\pm$$

$S^A{}_B$: energy-momentum tensor of matter field on a brane

- $\mathbf{Z_2}$ symmetry

Friedmann equation

$$H^2 + \frac{k}{a^2} = \frac{{}^{(4)}\Lambda}{3} + \frac{8\pi G_N}{3}\rho + \frac{\kappa_5^4}{36}\rho^2 + \frac{C}{a^4}$$

$${}^{(4)}\Lambda = \frac{1}{2}\left(\Lambda + \frac{\kappa_5^4}{6}\lambda^2\right) \qquad 8\pi G_N = \frac{\kappa_5^4}{6}\lambda$$

${}^{(4)}\Lambda = 0$: tuning condition in RS brane model

We recover the conventional Friedmann eq at late stage ($\rho \to 0$)

New effects in the early stage

brane cosmology

Effective four dimensional approach

Shiromizu,KM,Sasaki ('00)

- **The "Einstein" equations on the brane world**

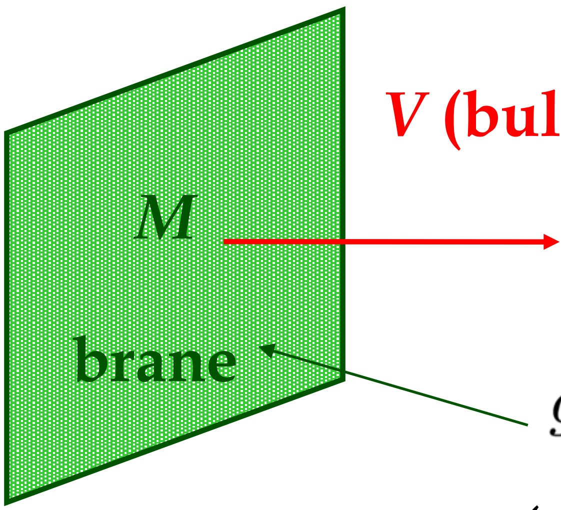
5D Einstein gravity with Λ + 4D matter on the brane

$${}^{(5)}g_{AB}$$

$$g_{\mu\nu}$$

⇒ effective 4D "Einstein" eqs on the brane

V **(bulk)**



n^A: normal (spacelike)

$$g_{AB} = {}^{(5)}g_{AB} - n_A n_B$$

(projection: induced metric on M)

(1) Gauss equation

$$^{(4)}R^A{}_{BCD} = {}^{(5)}R^M{}_{NRS}\, g^A{}_M g^N{}_B g^R{}_C g^S{}_D + K^A{}_C K_{BD} - K^A{}_D K_{BC}\,,$$

$\Rightarrow$ Einstein tensor $\quad {}^{(4)}G_{\mu\nu} = {}^{(4)}R_{\mu\nu} - \frac{1}{2}\,{}^{(4)}R^{(4)}g_{\mu\nu}$

+ 5D Einstein eqs $\quad {}^{(5)}G_{AB} = \kappa_5^2\,{}^{(5)}T_{AB}$

$$G_{\mu\nu} = \frac{2\kappa_5^2}{3}\left[{}^{(5)}T_{RS}g_\mu{}^R g_\nu{}^S + g_{\mu\nu}\left({}^{(5)}T_{RS}n^R n^S - \frac{1}{4}\,{}^{(5)}T\right)\right]$$
$$+KK_{\mu\nu} - K_\mu{}^\lambda K_{\nu\lambda} - \frac{1}{2}g_{\mu\nu}(K^2 - K^{\alpha\beta}K_{\alpha\beta}) - E_{\mu\nu}$$

$${}^{(5)}T_{AB} = -\,\Lambda\,{}^{(5)}g_{AB}$$ (5D cosmological constant)

$$E_{\mu\nu} = {}^{(5)}C_{ARBS}n^A n^B g_\mu{}^R g_\nu{}^S \sim {}^{(5)}C_{5\mu5\nu}$$ (5D object)

$K_{\mu\nu}$

- Israel's junction condition on the brane

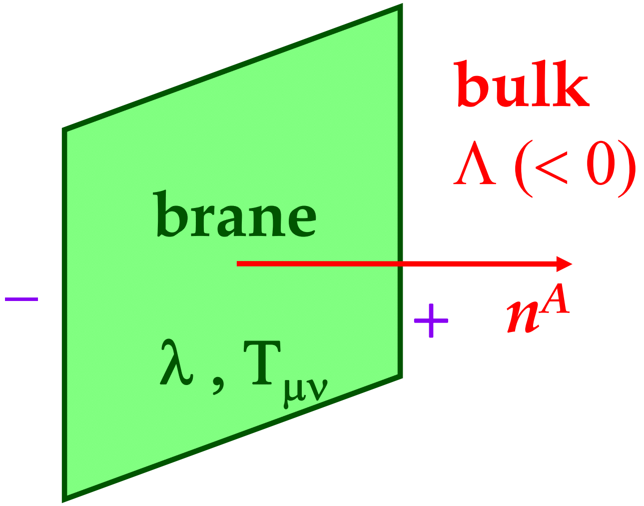
$$[g_{\mu\nu}]_\pm = 0 \qquad [X]_\pm = X_+ - X_-$$

$$[K_{\mu\nu}]_\pm = -\kappa_5^2 \left(S_{\mu\nu} - \frac{1}{3} g_{\mu\nu} S \right)$$

$S_{\mu\nu}$: 4D energy-momentum

$$S_{\mu\nu} = -\lambda g_{\mu\nu} + T_{\mu\nu}$$

tension　　matter



- Z_2 symmetry

$$K^+_{\mu\nu} = -K^-_{\mu\nu} = -\frac{\kappa_5^2}{2} \left(S_{\mu\nu} - \frac{1}{3} g_{\mu\nu} S \right)$$

(1) The effective "Einstein" equations in 4D

$$G_{\mu\nu} = -{}^{(4)}\Lambda g_{\mu\nu} + 8\pi G_N T_{\mu\nu} + \kappa_5^4 \pi_{\mu\nu} - E_{\mu\nu}$$

$${}^{(4)}\Lambda = \frac{1}{2}\left({}^{(5)}\Lambda + \frac{1}{6}\kappa_5^4\lambda^2\right)$$: **4D cosmological constant**

$$G_N = \frac{\kappa_5^4\lambda}{48\pi}$$: Newton's gravitational constant

new terms

$$\pi_{\mu\nu} = -\frac{1}{4}T_{\mu\alpha}T_\nu{}^\alpha + \frac{1}{12}TT_{\mu\nu} + \frac{1}{8}g_{\mu\nu}T_{\alpha\beta}T^{\alpha\beta} - \frac{1}{24}g_{\mu\nu}T^2$$

$$E_{\mu\nu} = {}^{(5)}C_{ARBS}\, n^A n^B g_\mu{}^R g_\nu{}^S$$ **(nonlocal)**

windows to see extra-dimemsions

(2) **Codazzi equation**

$$D_N K_M^N - D_M K = {}^{(5)}R_{RS}\, n^S g^R_{\ M} = 0$$

(5D Einstein eqs with a cosmological constant)

$$\Rightarrow \quad \boxed{D_\nu T_\mu^{\ \nu} = 0}$$

The energy-momentum on the brane is conserved

- Dynamics of a thin brane in 5D Schwatzschild AdS spacetime

Krauss ('00), Ida ('00)

IV. Inversion

4D object ⟸ higher-dimensional spacetime

non-trivial solutions   simple extension

time-dependent black hole solutions

black holes in "our" world

asymptotically flat vacuum spacetime

+electromagnetic field

Schwarzschild BH

Reissner-Nordtrom BH

Kerr BH

Kerr-Newman BH

Time dependent exact solutions ?

Difficult to find them

black holes in the Universe

No solution has been found

Except for Schwarzschild-de Sitter, RN dS & Kastor-Traschen sol.

BH from higher dimensions

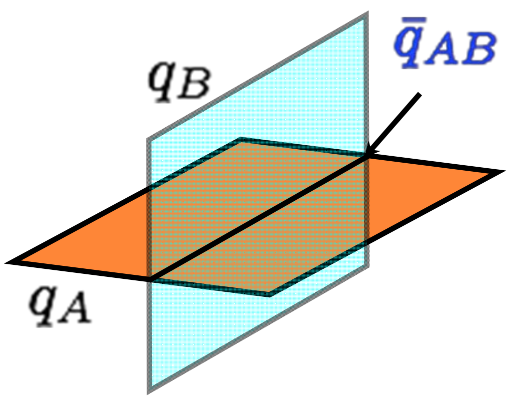
D-dimensional effective action

$$S = \frac{1}{16\pi G_D}\int d^D X\sqrt{-g}\left[\mathcal{R} - \frac{1}{2}(\nabla\varphi)^2 - \sum_A \frac{1}{2\cdot n_A!}e^{a_A\varphi}F^2_{\mathbf{n_A}}\right]$$

φ dilaton $F_{\mathbf{n_A}}$ n_A form fields A: type of branes (2-brane, 5-brane etc)

Stationary spacetime with branes

Branes in some dimensions → gravitational sources



compactification

BHs (Black objects) in 4 or 5 dim

♯branes ~ charges

Solution in D-dim spacetime

$$ds_D^2 = \left[(1+f)\prod_A H_A^{\frac{2}{\Delta_A}[D-d-q_A(d-2)]}\right]^{-\frac{1}{d-2}} d\bar{s}_d^2$$

$$+\prod_A H_A^{-2\frac{D-q_A-3}{\Delta_A}}(1+f)\left[dx_1 - \frac{1}{1+f}\left(f\,dt - \frac{\mathcal{A}}{2}\right)\right]^2 + \sum_{\alpha=2}^{p}\prod_A H_A^{-2\frac{\delta_{\alpha A}}{\Delta_A}} dx_\alpha^2$$

$$\delta_A^\alpha = \begin{cases} D-q_A-3 \\ -(q_A+1) \end{cases} \text{for} \begin{cases} x^\alpha \text{ belonging to } q_A\text{-brane} \\ \text{otherwise} \end{cases}.$$

$$\partial^2 H_A = 0$$ harmonic function on R^4

$$\partial^j \mathcal{F}_{ij} = 0$$ $\mathcal{A}_i$: vector harmonic function on R^4 $\quad \mathcal{F} = d\mathcal{A}$

$$\partial^2 f = \frac{\beta}{2}\prod_A H_A^{-\frac{2(D-2)}{\Delta_A}} \mathcal{F}_{ij}^2$$:Poisson equation (Laplace eq. for β=0)

$$\beta = 1 - \sum_{A'} \frac{(D-2)}{\Delta_{A'}}$$ (constant)

compactification

Einstein frame in *d*-dimensions

$$d\bar{s}_d^2 = -\Xi^{d-3}\left(dt + \frac{\mathcal{A}}{2}\right)^2 + \Xi^{-1}\sum_{i=1}^{d-1} dz_i^2$$

$$\Xi \equiv (1+f)^{-1/(d-2)} \prod_A H_A^{-\frac{2(D-2)}{(d-2)\Delta_A}}$$

D=11, supergravity, d=5 $M2 \perp M5$

	t	x^1	x^2	x^3	x^4	x^5	x^6	z^1	z^2	z^3	z^4
M2	○	○	○								
M5	○	○		○	○	○	○				
W	○	○									

The lowest order: BMPV BH

5D supersymmetric rotating BH

Breckenridge, Myers, Peet and Vafa ('93)

Time dependent intersecting brane in 10D or 11D

KM, N. Ohta, K. Uzawa (09).

Simple Extension

One brane ($\tilde{A}$) can be time dependent

$$H_{\tilde{A}} \to at + H_{\tilde{A}}(z) \qquad \triangle_Z H_{\tilde{A}} = 0$$

$$H_A(z) \quad (A \neq \tilde{A}) \qquad \triangle_Z H_A = 0$$

↓ compactification

Time dependent Black Holes

Spherically symmetric 4D solution

M2M2M5M5 — 4 charges (4 branes)

	0	1	2	3	4	5	6	7	8	9	10
	t	x^1	x^2	x^3	x^4	x^5	x^6	x^7	z^1	z^2	z^3
M2	○	○	○								
M2	○			○	○						
M5	○	○		○		○	○	○			
M5	○		○		○	○	○	○			

compactification (columns 1–7)

our 3-space (columns 8–10)

$$ds^2 = -\Xi dt^2 + \Xi^{-1}\left(dr^2 + r^2 d\Omega^2\right)$$

isotropic coordinates

$$\Xi = (H_T H_S H_{S'} H_{S''})^{-1/2}$$

$$H_T = \frac{t}{t_0} + \frac{Q_T}{r} \qquad H_A = 1 + \frac{Q_A}{r}, \quad A = S, S', S''$$

same charges $Q_T = Q_S = Q_{S'} = Q_{S''} = Q$

$$ds_4^2 = -\Xi\, dt^2 + \frac{1}{\Xi}\left(dr^2 + r^2 d\Omega_2^2\right)$$

$$\Xi = \left(\frac{t}{t_0} + \frac{Q}{r}\right)^{-1/2}\left(1 + \frac{Q}{r}\right)^{-3/2}$$

- $r \to 0$

Extreme RN BH

$$ds^2 = -H^{-2}dt^2 + H^2\left(dr^2 + r^2 d\Omega^2\right) \qquad H = 1 + \frac{Q}{r}$$

$$\bar{r} = r + Q$$

$$ds^2 = -\left(1 - \frac{Q}{\bar{r}}\right)^2 dt^2 + \frac{1}{\left(1 - \frac{Q}{\bar{r}}\right)^2} d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

$r = 0\,(\bar{r} = Q)$: horizon

$$ds_4^2 = -\Xi\, d\bar{t}^2 + \frac{a^2}{\Xi}\left(dr^2 + r^2 d\Omega_2^2\right)$$

$$\frac{t}{t_0} = \left(\frac{\bar{t}}{\bar{t}_0}\right)^{4/3}$$

$$\Xi = \left(1 + \frac{Q}{a^4 r}\right)^{-1/2}\left(1 + \frac{Q}{r}\right)^{-3/2}$$

$$a = \left(\frac{\bar{t}}{\bar{t}_0}\right)^{1/3}$$

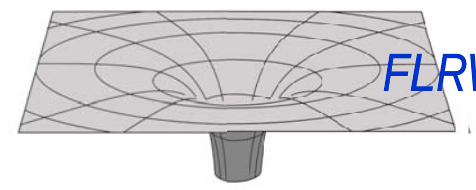
- $r \to \infty$

$$ds^2 = -d\bar{t}^2 + a^2(\bar{t})\left(dr^2 + r^2 d\Omega^2\right)$$

$$a = \left(\frac{\bar{t}}{\bar{t}_0}\right)^{1/3}$$

FLRW universe with **stiff matter**

⇨ **The BH in the Universe ?**



global structure

KM , M. Nozawa ('10)

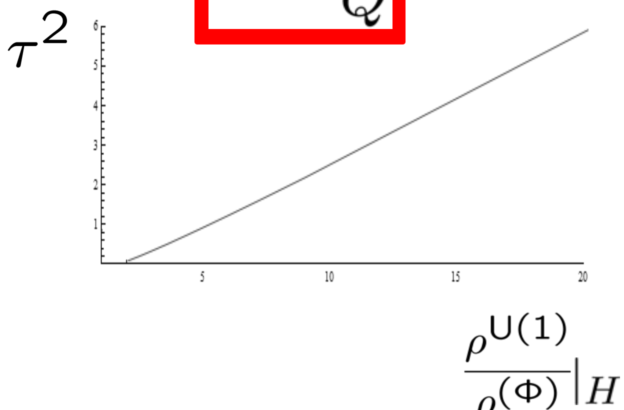
horizon radius

$$R_{\pm}^2 = \frac{\pm 1 + \sqrt{1+4\tau^2}}{2\tau} Q^2$$

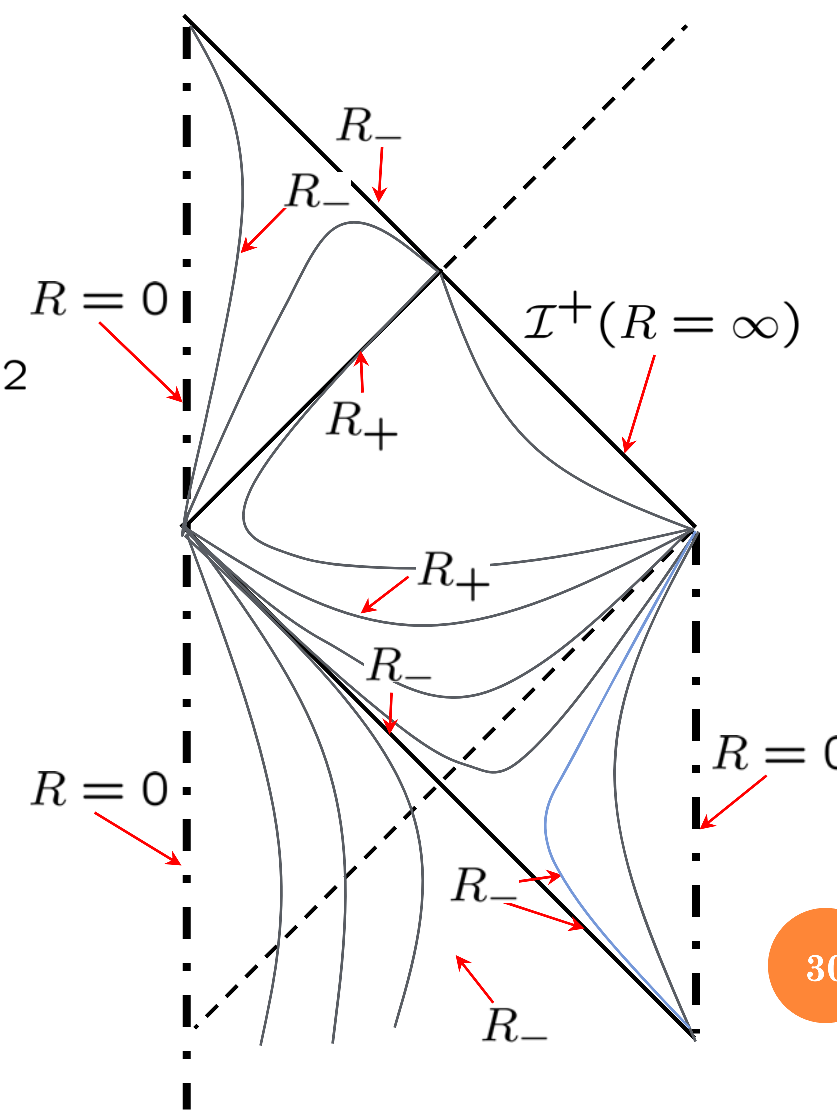
circumference radius

$$R^2 = \left(1 + \frac{tr}{Q^2}\right)^{1/2} \left(1 + \frac{r}{Q}\right)^{3/2} Q^2$$

$$\tau = \frac{t_0}{Q}$$



R = constant



The present metric plus

$$\kappa\Phi = \frac{\sqrt{6}}{4}\ln\left(\frac{H_T}{H_S}\right) \qquad \kappa A_0^{(T)} = \frac{\sqrt{2\pi}}{H_T},\ \kappa A_0^{(S)} = \frac{\sqrt{2\pi}}{H_S}$$

This is an exact solution of the Einstein-"Maxwell"-scalar system

$$\mathcal{S} = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}\mathcal{R} - \frac{1}{2}(\nabla\Phi)^2 - \frac{1}{16\pi}\sum_A e^{\lambda_A\kappa\Phi}(F_{\mu\nu}^{(A)})^2\right]$$

$$\lambda_T = \sqrt{6} \qquad \lambda_S = \lambda_{S'} = \lambda_{S''} = -\frac{\sqrt{6}}{3}$$

We find a nontrivial time-dependent 4D black hole solution from higher dimensions.

Extension

1. the universe with arbitrary expansion law

G.W. Gibbons, KM ('10)

the exact BH solution of the following system

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}R - \frac{1}{2}(\nabla\Phi)^2 - V(\Phi) - \frac{1}{16\pi}\sum_{A=T,S} g_A e^{\lambda_A \kappa\Phi}(F^{(A)}_{\mu\nu})^2\right]$$

$$V = V_0 \exp^{-\alpha\kappa\Phi} \qquad \alpha = \sqrt{\frac{2n_S}{n_T}}$$

$$\lambda_T = \sqrt{\frac{2n_S}{n_T}} \quad , \quad \lambda_S = -\sqrt{\frac{2n_T}{n_S}}$$

$$n_T + n_S = 4$$

Type	n_T	n_S	The coupling constants			Models of the Universe			$\kappa^2 V_0 t_0^2$
			α	λ_T	λ_S	p	(expansion law)	w	
I	0	4	∞	∞	0	0	(**static**)	0	0
	1	3	$\sqrt{6}$	$\sqrt{6}$	$-\sqrt{6}/3$	1/3	(**stiff matter**)	1	0
	4/3	8/3	2	2	-1	1/2	(**radiation**)	1/3	1/9
	8/5	12/5	$\sqrt{3}$	$\sqrt{3}$	$-2/\sqrt{3}$	2/3	(**dust**)	0	6/25
II	2	2	$\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	1	(**Milne**)	$-1/3$	1/2
III	3	1	$\sqrt{6}/3$	$\sqrt{6}/3$	$-\sqrt{6}$	3	(**quintessence**)	$-7/9$	3/2
	4	0	0	0	$-\infty$	∞	(**de Sitter**)	-1	3

FLRW universe expansion law $a = \left(\frac{\bar{t}}{\bar{t}_0}\right)^p$ $p = \frac{n_T}{n_S}$

equation of state $P = w\rho$ $-1 \leq w \leq 1$ $4 \geq n_T \geq 1$

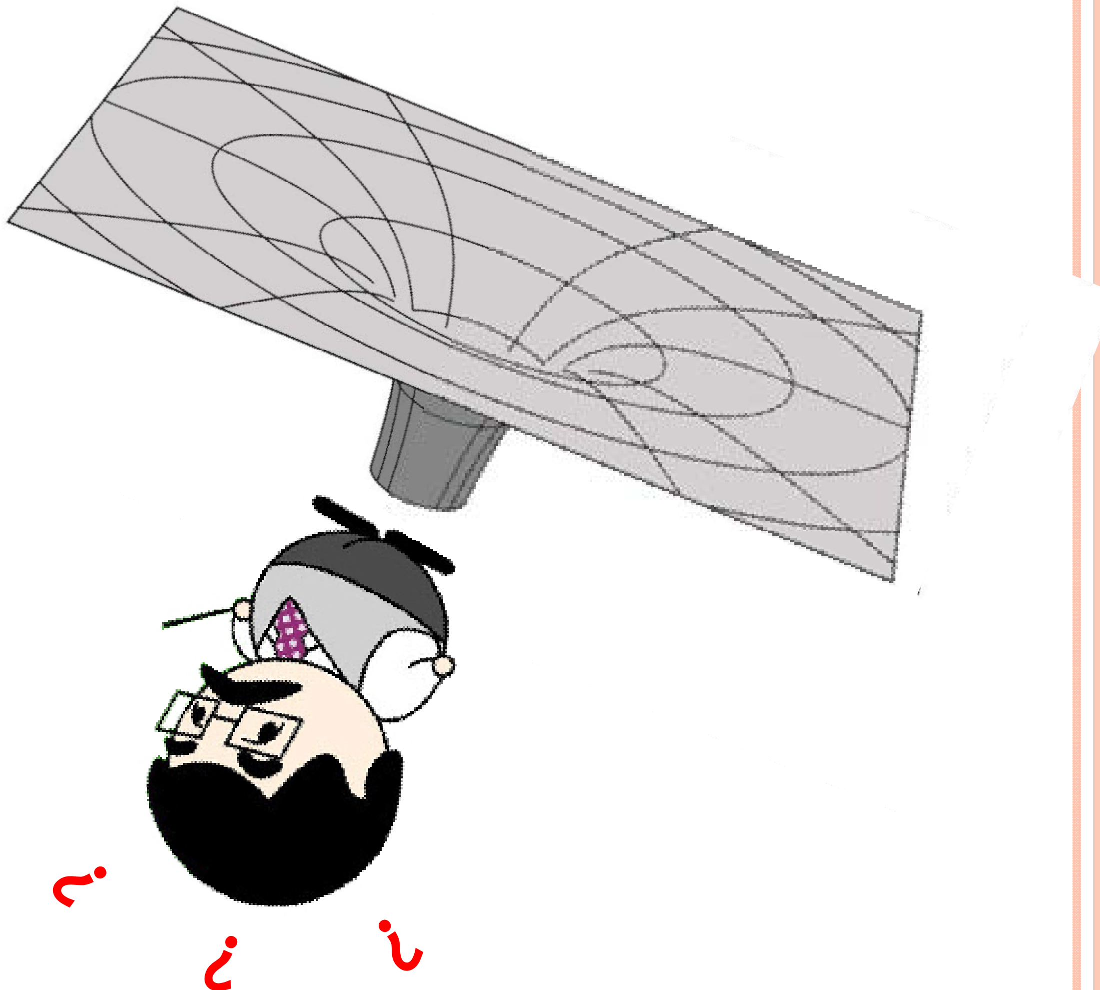
2. rotating BH in the expanding universe (5D) Nozawa, KM ('10) arXiv:1009.3688

3. collision of BHs in the contracting universe

4. thermodynamics

V. Summary (of some work in my first stage)

- **Dimensional Reduction**
  - **Spacetime symmetry**
    - Kaluza-Klein cosmology
    - [(2+1)+1] formalism
  - **Dynamics of shingular hypersurface**
    - brane world
    - void dynamics
- **Inversion**
  - **Time dependent black hole from higher dimensions**



What should I work in my second stage ?

I hope that JGRG will continue to the 60th.

Thank you for your attention
and contribution to JGRG !