Note on the equivalence of a barotropic perfect fluid with a k-essence scalar field

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Summary

- We obtain the necessary and sufficient condition for a class of noncanonical single scalar field models to be exactly equivalent to barotropic perfect fluids, under the assumption of an irrotational fluid flow.

$$\Downarrow$$

The nonadiabatic pressure perturbation in this class of scalar field systems vanishes exactly at all orders in perturbation theory and on all scales.

- The Lagrangian for this general class of scalar field models depends on both the kinetic term and the value of the field. However, after a field redefinition, it can be effectively cast in the form of a purely kinetic k-essence model.

$$P(X,\phi) = f\big(Xg(\phi)\big)$$

f, g — Arbitrary functions

$$X \equiv -\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \qquad d\varphi = \sqrt{g}\,d\phi \qquad Y = gX = -\tfrac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$$

Introduction 1

- Both scalar fields and perfect fluids are pervasive in our present models for the evolution of the Universe.

  - A barotropic perfect fluid can be defined as a perfect fluid where the pressure is a function of the energy density only (the same function for the background and the perturbations).

$$P = P(\rho)$$

- The theory of cosmological perturbations in these models has been intensively studied and is well developed! See for instance:

Perfect fluid

Linear: Kodama and Sasaki '84 Mukhanov, Feldman and Brandenberger '92

Second order: Malik and Wands '09 Bartolo *et al*. '10

Third order: D'Amico *et al*. '08 Christopherson and Malik '09

Introduction 2

Noncanonical scalar field ⇐ Brane inflation: produces large non-Gaussianity

Linear: Garriga and Mukhanov ’99

Second order: Seery and Lidsey ’05 Chen *et al*. ’07

Third order: Arroja and Koyama ’08 Chen *et al*. ’09 Arroja *et al*. ’09

Some fully nonlinear results: Lyth, Malik and Sasaki ’05 Langlois and Vernizzi ’05

➢ So when, if in any case, can we describe a scalar field by a perfect fluid and vice versa?

- One can use these known results to check for consistency when both calculations for the dual models exist or use these results in the side of the duality where calculations haven’t been done yet.

The model: k-essence/k-inflation

Armendariz-Picon *et al*. '99

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left[M_{Pl}^2 R + 2P(X,\phi)\right]$$

$P(X,\phi)$ – Pressure

ϕ – Scalar field

$X \equiv -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$

Canonical: $P(X,\phi) = X - V(\phi)$

DBI inflation: $P(X,\phi) = -\frac{1}{f(\phi)}\left(\sqrt{1-2f(\phi)X} - 1\right) - V(\phi)$

Silverstein and Tong'03

Background

Energy-momentum tensor: $T_{\mu\nu} = -2\frac{\partial P}{\partial g^{\mu\nu}} + g_{\mu\nu}P = P_{,X}\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}P$

If $\partial_\mu\phi$ is time like then it is of the perfect fluid form

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + g_{\mu\nu}P$$

$$\rho = 2XP_{,X} - P$$

Four-velocity: $u_\mu = \frac{\partial_\mu\phi}{\sqrt{2X}}$ ⟸ Assume potential flow only

$$u_\mu u^\mu = -1$$

Klein-Gordon eq.: $\nabla^\nu\left(P_{,X}\nabla_\nu\phi\right) + P_{,\phi} = 0$

FLRW universe: $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$

Friedmann eqs.: $3H^2 = \rho_0$

$$\dot{\rho}_0 = -3H\left(\rho_0 + P_0\right)$$

background

Linear perturbations

3D metric: $h_{ij} = a^2 \left[(1 + 2\mathcal{R}_c)\delta_{ij} + 2\partial_i\partial_j E + 2\partial_{(i}F_{j)} \right]$

Curvature perturbation

- Neglect tensors and choose gauge such that $E, \; F_i$ are zero.

Scalar field side

Comoving time-slices: $\delta\phi = 0$

Equation of motion: $$\frac{\partial}{\partial t}\left(\frac{a^3 \epsilon}{c_{ph}^2} \frac{\partial}{\partial t}\mathcal{R}_c \right) - a\,\epsilon\,\delta^{ij}\frac{\partial^2}{\partial x^i \partial x^j}\mathcal{R}_c = 0$$

$\epsilon = -\dot{H}/H^2$

$$c_{ph}^2 = \frac{P_{0,X}}{\rho_{0,X}} = \frac{P_{0,X}}{P_{0,X} + 2X_0 P_{0,XX}}$$ $\Longrightarrow$ Speed of sound

Barotropic perfect fluid side

Comoving time-slices: $T^0{}_j = 0$

$$c_s^2 = \frac{\dot{P}_0}{\dot{\rho}_0}$$ $\Longrightarrow$ Adiabatic sound speed

➢ Same equation for $\mathcal{R}_c$ but with c_{ph}^2 replaced with c_s^2 .

Equivalence condition

$$c_{ph}^2 = \frac{P_{0,X}}{\rho_{0,X}} = \frac{P_{0,X}}{P_{0,X}+2X_0 P_{0,XX}} \Longrightarrow$$ Speed of sound

$$c_s^2 = \frac{\dot{P}_0}{\dot{\rho}_0} \Longrightarrow$$ Adiabatic sound speed

- In general these two speeds are different. Christopherson and Malik '09

When are they equal?

The condition is a second order partial differential equation for the pressure (=Lagrangian):

$$c_s^2 = c_{ph}^2 \quad \Longleftrightarrow \quad P_{0,\phi} - X_0 P_{0,X\phi} + X_0 P_{0,\phi} \frac{P_{0,XX}}{P_{0,X}} = 0$$

Dual models

$$P_{0,\phi} - X_0 P_{0,X\phi} + X_0 P_{0,\phi} \frac{P_{0,XX}}{P_{0,X}} = 0$$

Also the necessary condition for the existence of exact stationary field configurations.

Akhoury *et al*. '09

This equation can be integrated once to give:

$$X_0 P_{0,X} = A(\phi) P_{0,\phi}$$

Where $A(\phi)$ is an arbitrary function of ϕ.

Using the method of characteristics the previous equation can be further integrated to find:

$$P(X,\phi) = f\left(Xg(\phi)\right)$$

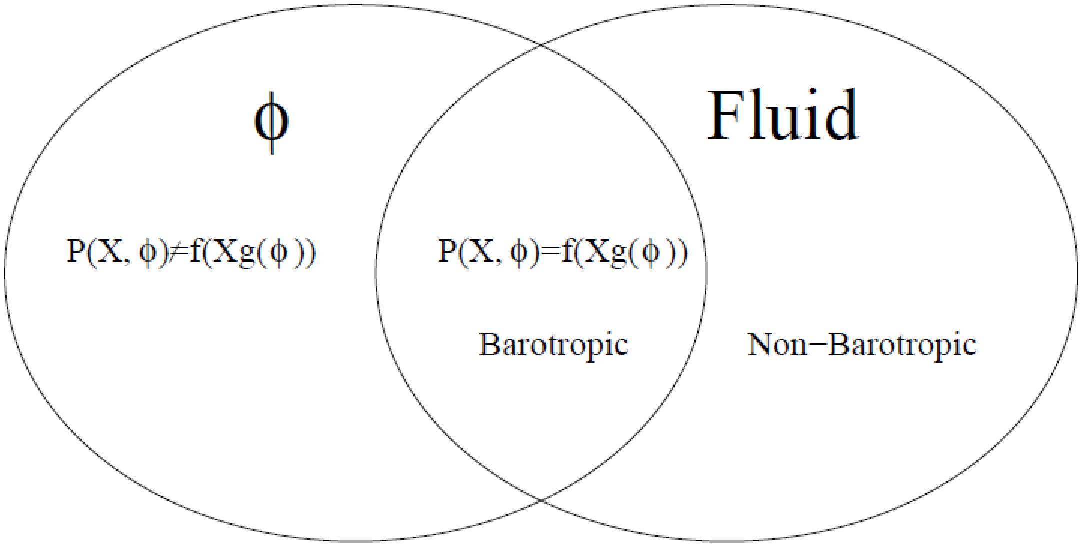
f, g – Arbitrary functions

The two speeds are equal to: $c_{ph}^2 = c_s^2 = \left(1 + 2Y\frac{f_{,YY}}{f_{,Y}}\right)^{-1}$

$$Y \equiv Xg(\phi)$$

- This is the most general class of scalar field models $P(X,\phi)$ that is exactly equivalent to a barotropic perfect fluid, under the assumption of the velocity being described by a single scalar potential.

The picture



- The left ellipse represents the set of all the models with a general Lagrangian $P(X,\phi)$ while the right ellipse represents the set of all the perfect fluids. We have shown that the intersection of these two sets corresponds to barotropic perfect fluids or scalar field models with Lagrangian $P(X,\phi)=f\big(Xg(\phi)\big)$

Examples

- All purely kinetic k-essence models, i.e. $P(X)$, are included in this class,

- The standard canonical scalar field model, $P(X,\phi) = X - V(\phi)$, it is not because:

$$c_{ph}^2 = 1$$

$$c_s^2 = -1 - \frac{\eta - 2\epsilon}{3}\,, \quad \text{with} \quad \epsilon = -\frac{\dot{H}}{H^2}\,, \quad \eta = \frac{\dot{\epsilon}}{\epsilon H}$$

$$\Downarrow$$

$$c_s^2 \neq c_{ph}^2$$

For $P = f(Y)$, the conserved number density n is

$$\rho = 2f_{,Y}Y - f(Y)$$

$$\frac{dn}{n} = \frac{d\rho}{\rho + P} = \frac{f_{,Y} + 2f_{,YY}Y}{2f_{,Y}Y}dY$$

Which can be integrated to give: $n = K f_{,Y} Y^{1/2}$

Comments

❖ For the scalar field Lagrangian $P(X,\phi)=f\big(Xg(\phi)\big)$, because both the pressure and the energy density are functions of one variable only, i.e. $Y\equiv Xg(\phi)$, it can be shown that the nonadiabatic pressure perturbation vanishes exactly to all orders in perturbations and on all scales.

This is not surprising because all models of this form are dual to barotropic (i.e. adiabatic) perfect fluids where the nonadiabatic pressure perturbation vanishes by definition.

❖ The equivalence between the dual models is independent of the background because the derived second-order differential equation is independent of the background but dependent only on the form of P as a function of X and ϕ, despite the fact that it was derived from the condition for the equivalence in linear perturbation theory.

Conclusions

- We have studied under which conditions a general k-essence scalar field is equivalent to an irrotational barotropic perfect fluid.

  We found that the condition can be written as a second-order partial differential equation for the Lagrangian of the field, that simply states that the sound speed c_{ph}^2 at which scalar perturbations propagate has to be equal to the adiabatic sound speed c_s^2.

- We have found the most general solutions: $P(X,\phi) = f\big(Xg(\phi)\big)$

- This Lagrangian depends on both the kinetic term and the value of the field. However, after a field redefinition, it can be effectively cast in the form of a purely kinetic k-essence model.

- For these scalar field models the nonadiabatic pressure perturbation vanishes exactly to all orders in perturbations and on all scales.

- ✓ These scalar fields have been called perfect scalar fields.