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Gauge invariance of Boltzmann equation with polarization

Atsushi Naruko (YITP)

Collaborators : Misao Sasaki (YITP),
David Wands, Kazuya Koyama, Cyril Pitrou (ICG,UK)

Polarization

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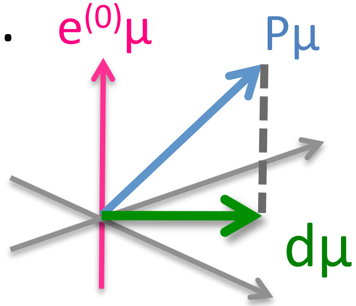
- In order to express the polarization of photon, we introduce the tensor valued distribution function $F_{\mu\nu}$.

$$F_{\mu\nu} \sim A_\mu A_\nu$$

A_μ = vector potential in EM
$F_{\mu\nu}$ ≠ field strength

- We usually express the polarization on the 2d surface, we define the screen projection $S_{\mu\nu}$ which is orthogonal to observer (= $e_\mu^{(0)}$) and the direction of photon (= d_μ).



$$S_{\mu\nu} \equiv g_{\mu\nu} + \underline{e_\mu^{(0)} e_\nu^{(0)}} - \underline{d_\mu d_\nu}$$

- We define $f_{\mu\nu}$ as the projection of $F_{\mu\nu}$ onto this screen.

$$f_{\mu\nu} \equiv S_\mu{}^\alpha S_\nu{}^\beta F_{\alpha\beta} \equiv I S_{\mu\nu} + P_{\mu\nu} + V \epsilon_{\rho\mu\nu\sigma} e^\rho_{(0)} d^\sigma$$

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Boltzmann equation with polarization

- Boltzmann equation with polarization

$$\frac{\mathcal{D}F_{\alpha\beta}}{\mathcal{D}\lambda}(x^\mu, q, n^i) = P^\mu \underline{F_{\alpha\beta;\mu}} + \frac{dq}{d\lambda}\frac{\partial F_{\alpha\beta}}{\partial q} + \frac{dn^i}{d\lambda}\frac{\partial F_{\alpha\beta}}{\partial n^i} = C_{\alpha\beta}$$

Covariant derivative

↓

$$\begin{aligned}\frac{\mathcal{D}I}{\mathcal{D}\lambda} &= \frac{q}{a^2N}\frac{\partial I}{\partial\eta} + \frac{q}{a^2}\left(n^i - \delta^{ik}h_{kj}n^j - \frac{N^i}{N} + \frac{3}{2}\delta^{ij}h_{jk}\delta^{kl}h_{lm}n^m\right)\frac{\partial I}{\partial x^i} \\ &+ \frac{q^2}{a^2}\Big[-\alpha_{,i}n^i + \delta_{jk}N^k{}_{,i}n^in^j - \dot{h}_{ij}n^in^j + \alpha(\alpha_{,i}n^i - \delta_{jk}N^k{}_{,i}n^in^j + \dot{h}_{ij}n^in^j) \\ &\quad + \alpha_{,i}h_{kj}\delta^{ik}n^j + (N^k{}_{,i} - \delta_{im}\delta^{kl}N^m{}_{,l})h_{jk}n^jn^k + N^kh_{ij,k}n^in^j + 2\delta^{kl}\dot{h}_{il}h_{kj}n^in^j\Big]\frac{\partial I}{\partial q} \\ &+ \frac{q}{a^2}(n^in^j - \delta^{ij})\{N_{,j} - \delta_{kl}N^l{}_{,j}n^k + \dot{h}_{jk}n^k + (h_{jl,k} - h_{kl,j})n^kn^l\}\frac{\partial I}{\partial n^i}\end{aligned}$$

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Boltzmann equation with polarization

$$\frac{\mathcal{D}V}{\mathcal{D}\lambda} = \frac{q}{a^2 N}\frac{\partial V}{\partial \eta} + \frac{q}{a^2}(n^i - \delta^{ik}h_{kj}n^j - N^i)\frac{\partial V}{\partial x^i} + \frac{q^2}{a^2}\left(-\alpha_{,i}n^i + \delta_{jk}N^k{}_{,i}n^i n^j - \dot{h}_{ij}n^i n^j\right)\frac{\partial V}{\partial q}$$
$$+ \frac{q}{a^2}(n^i n^j - \delta^{ij})\{N_{,j} - \delta_{kl}N^l{}_{,j}n^k + \dot{h}_{jk}n^k + (h_{jl,k} - h_{kl,j})n^k n^l\}\frac{\partial V}{\partial n^i}$$

$$S_0{}^\mu S_i{}^\nu \frac{\mathcal{D}P_{\mu\nu}}{\mathcal{D}\lambda} = (N^j - \delta_{lm}N^l n^m n^j)(\delta_i{}^k - \delta_{in}n^n n^k)(\dot{P}_{jk} + n^p P_{jk,p} - 2\mathcal{H}P_{jk})$$

$$S_i{}^\mu S_j{}^\nu \frac{\mathcal{D}P_{\mu\nu}}{\mathcal{D}\lambda} = \frac{q}{a^2}\perp_i^k\perp_j^l\left\{P^{(2)}_{kl,0} + n^m P^{(2)}_{kl,m} - 2\mathcal{H}P^{(2)}_{kl} - \alpha(P_{kl,0} - 2\mathcal{H}P_{kl}) - (\delta^{mn}h_{np}n^p + N^m)P_{kl,m}\right\}$$
$$- \frac{q}{a^2}\perp_i^k\perp_j^l\, n^n\delta^{mp}\{(h_{pk,n} + h_{pn,k} - h_{kn,p})P_{ml} + (h_{pl,n} + h_{pn,l} - h_{nl,p})P_{km}\}$$
$$- \frac{q}{2a^2}\perp_i^k\perp_j^l\,\{(N^m{}_{,l} + 2\delta^{mn}\dot{h}_{nl} - \delta^{mn}\delta_{lp}N^p{}_{,n})P_{km} + (N^m{}_{,k} + 2\delta^{mn}\dot{h}_{nk} - \delta^{mn}\delta_{kp}N^p{}_{,n})P_{lm}\}$$
$$+ \frac{q^2}{a^2}\perp_i^k\perp_j^l\left(-\alpha_{,m}n^m + \delta_{np}N^p{}_{,m}n^m n^n - \dot{h}_{mn}n^m n^n\right)\frac{\partial P_{kl}}{\partial q}$$
$$+ \frac{q}{a^2}\perp_i^k\perp_j^l\,(n^m n^n - \delta^{mn})\{N_{,n} - \delta_{pq}N^q{}_{,n}n^p + \dot{h}_{np}n^p + (h_{nq,p} - h_{pq,n})n^p n^q\}\frac{\partial P_{kl}}{\partial n^m}$$
$$+ \frac{q}{a^2}\{\perp_i^k\,\delta_{jd}(\delta^{de}n^l - \delta^{le}n^d)h_{ef}n^f + \delta_{ia}(\delta^{ab}n^k - \delta^{kb}n^a)h_{bc}n^c\perp_j^l\}(\dot{P}_{kl} + n^g P_{kl,g} - 2\mathcal{H}P_{kl})$$

$$\perp_i^j = \delta_i{}^j - n_i n^j$$

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Gauge invariance of Boltzmann eq.

- In order to show the gauge invariance of the derived equation, we calculate the gauge transformation rule for metric, momentum and distribution function.
- In the gauge transformation rule for distribution function, we find the new terms which express the change of screen.

$$f_{\mu\nu} \equiv \underbrace{S_\mu{}^\alpha S_\nu{}^\beta}_{\text{screen}} \underbrace{F_{\alpha\beta}}_{\text{tensor}}$$

$$\tilde{f}_{\alpha\beta} = \underbrace{f_{\alpha\beta} - \mathcal{L}_\xi f_{\alpha\beta} - \delta q \frac{\partial f_{\mu\nu}}{\partial q} - \delta n^i \frac{\partial f_{\mu\nu}}{\partial n^i}}_{\text{Usual part = tensor part}} - \underbrace{\frac{a^2}{q} T_{,i} (P_\alpha f^i{}_\beta + f_\alpha{}^i P_\beta)}_{\text{change of screen}}$$

- We can show the gauge invariance of the Boltzmann equation.