Poster Number 47

Construction of gauge-invariant variables for linear-order metric perturbation on some background spacetimes

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I. Introduction

- **The second order perturbation theory in general relativity has very wide physical motivation.**

  – **Cosmological perturbation theory**
    - Expansion law of inhomogeneous universe (ΛCDM v.s. inhomogeneous cosmology)
    - Non-Gaussianity in CMB (beyond WMAP)

  – **Black hole perturbations**
    - Radiation reaction effects due to the gravitational wave emission.
    - Close limit approximation of black hole - black hole collision (Gleiser, et al. (1996))

  – **Perturbation of a star (Neutron star)**
    - Rotation – pulsation coupling (Kojima 1997)

There are many physical situations to which higher order perturbation theory should be applied.

However, general relativistic perturbation theory requires very delicate treatments of "gauges".

It is worthwhile to formulate the higher-order gauge-invariant perturbation theory from general point of view.

- According to this motivation, we have been formulating the general relativistic second-order perturbation theory in a gauge-invariant manner.
  - **General framework**:
    - K.N. PTP**110** (2003), 723; *ibid.* **113** (2005), 413.
  - **Application to cosmological perturbation theory** :
    - Einstein equations (the first-order : scalar mode only):
      - K.N. PRD**74** (2006), 101301R; PTP**117** (2007), 17.
    - Equations of motion for matter fields:
      - K.N. PRD**80** (2009), 124021.
    - Consistency of the 2^{nd} order Einstein equations including all modes (perfect fluid, scalar field):
      - K.N. PTP**121** (2009), 1321.
    - Summary of current status of this formulation:
      - arXiv:1001.2612[gr-qc] (Advances in Astronomy in press.)

Our general framework of the second-order gauge invariant perturbation theory is based on a single assumption.

- metric perturbation : metric on PS : $\bar{g}_{ab}$, metric on BGS : g_{ab}

  metric expansion : $\bar{g}_{ab} = g_{ab} + \epsilon h_{ab} + \frac{1}{2}\epsilon^2 l_{ab} + O(\epsilon^3)$

- **linear order (assumption, decomposition hypothesis) :**

Suppose that the linear order perturbation h_{ab} is decomposed as

$$h_{ab} = \mathcal{H}_{ab} + \pounds_X g_{ab}$$

so that the variable $\mathcal{H}_{ab}$ and X^a are the gauge invariant and the gauge variant parts of h_{ab} , respectively.

These variables are transformed as

$$_{\mathcal{Y}}\mathcal{H}_{ab} - {}_{\mathcal{X}}\mathcal{H}_{ab} = 0 \qquad {}_{\mathcal{Y}}X^a - {}_{\mathcal{X}}X^a = \xi_1^a$$

under the gauge transformation $\Phi_\epsilon = \mathcal{X}_\epsilon^{-1} \circ \mathcal{Y}_\epsilon$.

- In cosmological perturbations, this is **almost** correct and we may choose $\mathcal{H}_{ab}$ as (longitudinal gauge, J. Bardeen (1980))

$$\mathcal{H}_{\eta\eta} = -2a^2 \overset{(1)}{\Phi}, \quad \mathcal{H}_{i\eta} = a^2 \overset{(1)}{\nu_i}, \quad \mathcal{H}_{ij} = -2a^2 \overset{(1)}{\Psi} + a^2 \overset{(1)}{\chi_{ij}},$$

Problems in decomposition hypothesis

- In cosmological perturbations, ...
  - Background metric : $g_{ab} = a^2(\eta)\left(-(d\eta)_a(d\eta)_b + \gamma_{ij}(dx^i)_a(dx^j)_b\right)$

    γ_{ij} : metric on maximally symmetric 3-space
  - **Zero-mode problem** :

$$h_{ab} =: h_{\eta\eta}(d\eta)_a(d\eta)_b + 2h_{\eta i}(d\eta)_{(a}(dx^i)_{b)} + h_{ij}(dx^i)_a(dx^j)_b$$

$$\begin{aligned} h_{\eta i} &= D_i h_{(VL)} + h_{(V)i}, \quad D^i h_{(V)i} = 0, \\ h_{ij} &= a^2 h_L \gamma_{ij} + a^2 h_{(T)ij}, \quad h_{(T)}{}^i{}_i := \gamma^{ij} h_{(T)ij} = 0, \\ h_{(T)ij} &= \left(D_i D_j - \frac{1}{3}\gamma_{ij}\Delta\right) h_{TL} + 2D_{(i}h_{(TV)j)} + h_{(TT)ij}, \\ D^i h_{(TV)i} &= 0, \quad D^i h_{(TT)ij} = 0, \quad D_i\gamma_{ij} := 0, \quad \Delta := D^i D_i. \end{aligned}$$

    - This decomposition is based on the existence of Green functions

      Δ^{-1}, $(\Delta + 2K)^{-1}$, $(\Delta + 3K)^{-1}$. K: curvature constant associated with γ_{ij}
    - In our formulation, we ignored the modes (zero modes) which belong to the kernel of the operators Δ , $(\Delta + 2K)$, $(\Delta + 3K)$.
    - **How to include these zero modes into our consideration?**
- On general background spacetime, ... --->**Generality problem** :
  - **Is the decomposition hypothesis also correct in general background spacetime?**

- In this poster presentation, ...
  - We resolve this generality problem using ADM decomposition.
    - In our proof, we assume the existence of Green functions of two derivative operators:

$$\Delta := D^i D_i, \quad \mathcal{D}_j{}^l := q_j^l \Delta + \left(1 - \frac{2}{n}\right) D_j D^l + R_j{}^l$$

----> the zero-mode problem remains and it should be examined carefully.

    - Although our explanation in this talk is for the case $\alpha = 1, \quad \beta^i = 0$, **a similar argument is applicable to the general case in which** $\alpha \neq 1, \quad \beta^i \neq 0.$

We may say that the decomposition hypothesis

$$h_{ab} = \mathcal{H}_{ab} + \pounds_X g_{ab}$$

is almost correct for the linear metric perturbation on general background spacetime.

See you at Poster 47.

Details can be seen there.