PERIODIC ORBIT APPROACH TO PROLATE DOMINANCE IN NUCLEAR DEFORMATION

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October 26, 2010 @YITP workshop on LACM and related topics

1. Introduction
   – Prolate dominance in nuclear ground-state deformations
   – Shell structures and periodic orbit bifurcations

2. The $r^\alpha$ potential model
   – Scaling properties and Fourier transformation technique
   – Spin degree of freedom

3. Semiclassical origin of prolate dominance
   – Bridge orbit bifurcations and their roles
   – Effect of spin-orbit coupling

4. Summary
1. Introduction

- **Prolate dominance in nuclear ground-state deformations**
  - Most of the nuclear ground state deformations are prolate
  - Stronger preference to prolate shapes in heavier nuclei
    (limited number of oblate nuclei are found mostly in light regions)

**Keys to understand above global properties**

- Deformation is essentially determined by **shell effect**
- **Prolate-oblate asymmetry** in deformed shell structures
- Change of asymmetry due to the size of nucleus
  ⇝ **surface character** of the single-particle potential
- Effect of spin-orbit coupling
  Interplay between spin-orbit coupling and surface diffuseness
Various approaches to understand prolate dominance

  Semiclassical periodic orbit theory
  Prolate-oblate asymmetry in deformed shell structure
  ⇔ Asymmetry in Classical periodic orbits

  Systematic Nilsson-Strutinsky calc.
  ⇒ Strong correlation between \( L S \) and \( L^2 \) terms
  Indicate the importance of spin-orbit coupling for true understandings of prolate dominance

  Prolate-oblate asymmetry in Nilsson level splittings
  ⇔ Surface character of single-particle potential
Origin of asymmetric level splittings

High $l$ orbits become lower as the surface becomes sharper

Strong correlation between $l_s$ and $l^2$ terms

Contour plot of “proportion of prolate deformation” $R_p = \frac{N_p}{N_p + N_o}$

Nuclear deformations and shell structures

Nuclear deformations

↔ Shell structures in single-particle spectra of deformed system

Distinct magic numbers

Single-particle level density

\[ g(e) = \bar{g}(e) + \delta g(e) \]

Energy (mass) of nucleus

\[ E(N) = \bar{E}(N) + \delta E(N) \]

Stable nuclei: small \( \delta E(N) \)

↔ Low level density at Fermi energy \( e_F \): small \( \delta g(e_F) \)
Semiclassical theory of shell structure

Trace formula for level density

\[ g(E) \sim g_0(E) + \sum_r A_r(E) \cos \left( \frac{S_r(E)}{\hbar} - \nu_r \right) \], \quad S_r(E) = \int_r p \cdot d\mathbf{r} \]

Oscillating part \( \delta g \) \( \Leftrightarrow \) contribution of classical periodic orbits

Energy scale \( \delta E \) of the level density oscillation: \( \delta E \approx \frac{2\pi h}{T_r} \)

GROSS shell structure (large \( \delta E \)) \( \Leftrightarrow \) SHORT periodic orbits (small \( T_r \))

Deformed shell structure: Level density minima along

\[ \frac{S_r(E, \beta)}{\hbar} - \nu_r = (2n + 1)\pi \quad \ldots \quad \text{constant-action curves in } (\beta, E) \text{ plane} \]

Constant-action curves in cavity system

\[ S_r \left( \frac{E, \mu}{\hbar} \right) = k L_r(\mu) = (2n + 1)\pi + \nu_r, \quad E \propto k^2 \propto \frac{1}{L_r(\mu)^2} \]

Formation of fission isomer

K.A. and M.Brack, in preparation
Periodic orbit bifurcation

Effect of PO bifurcation to shell structure

\[ g(E) \sim \int \sqrt{D} e^{iS(q)/\hbar} dq \]

Illustration of bifurcation (Example: “pitchfork” bifurcation)

Periodic orbit: stationary point \( S'(q) = 0 \)
Bifurcation: zero curvature \( S''(q) = 0 \)

\[ \rightarrow \] continuous family of quasi-stationary point around PO:
local quasi-periodic orbit family
coherent contribution to the path integral

\[ \rightarrow \] enhancement of \( A_r \)
\[ \rightarrow \] Growth of shell effect

Deformed shell structures \( \Leftrightarrow \) Bifurcations of short PO
2. The $r^\alpha$ potential model

- **Mean field potential**
  - Potential depth $W_0 \approx 50\text{MeV}$
  - Radius $R \approx r_0 A^{1/3}$ ($r_0 \approx 1.2 \text{ fm, } A: \text{ mass number}$)
  - Surface diffuseness $a \approx 0.5 \text{ fm}$

Woods-Saxon potential

Light nuclei ($R \sim a$) $\sim$ Harmonic oscillator

Heavy nuclei ($R \gg a$) $\sim$ Square well $\sim$ Infinite well
Approximation of Woods-Saxon potential

Woods-Saxon potential $\approx$ simpler potential with $r^\alpha$ radial dependence:

$$V(r) = -\frac{W_0}{1 + \exp\left(\frac{r - R(\theta, \varphi)}{a}\right)} \approx -W_0 + U_0 \left(\frac{r}{R(\theta, \varphi)}\right)^\alpha$$

fitting of the potential

quantum spectra
Scaling properties
The \( r^\alpha \) potential model

\[
H(p, r) = \frac{p^2}{2m} + U_0 \left( \frac{r}{R(\theta, \varphi)} \right)^\alpha
\]

Scaling property of classical Hamiltonian

\[
H \left( c^{1/2} p, c^{1/\alpha} r \right) = cH(p, q)
\]

Classical EOM is invariant under scaling transformation

\[
p \to c^{1/2} p, \quad r \to c^{1/\alpha} r, \quad t \to c^{-\left(\frac{1}{2} - \frac{1}{\alpha}\right)} t \quad \text{as} \quad E \to cE
\]

Phase space profile is independent of energy

Same periodic orbits in any energy
Fourier Transformation technique for scaling system

Action integral along a periodic orbit

\[ S_r(E) = \oint_{r, E} p \cdot dr = \left( \frac{E}{U_0} \right)^{\frac{1}{2} + \frac{1}{\alpha}} \oint_{r, E=U_0} p \cdot dr \equiv \mathcal{E} \hbar \tau_r \]

scaled energy \( \mathcal{E} = \left( \frac{E}{U_0} \right)^{\frac{1}{2} + \frac{1}{\alpha}} \), scaled period \( \tau_r = \frac{1}{\hbar} \oint_{r, E=U_0} p \cdot dr \)

Trace Formula for scaled energy level density

\[ g(\mathcal{E}) = g(E) \frac{dE}{d\mathcal{E}} = g_0(\mathcal{E}) + \sum_r A_r(\mathcal{E}) \cos (\mathcal{E} \tau_r - \nu_r) \]

Fourier Transform of level density

\[ F(\tau) = \int d\mathcal{E} \ e^{i\tau \mathcal{E}} g(\mathcal{E}) = \sum_n e^{i\tau \mathcal{E}_n}, \quad \mathcal{E}_n = (E_n/U_0)^{\frac{1}{2} + \frac{1}{\alpha}} \] (quantum)

\[ \sim F_0(\tau) + \pi \hbar \sum_r e^{i\nu_r} \tilde{A}_r \delta(\tau - \tau_r) \] (semiclassical)

\( F(\tau) \) ... peaks at periodic orbits \( \tau = \tau_r \) with height proportional to \( A_r \)

Information on PO out of quantum energy spectrum
**Spin-orbit coupling**

Introduction of spin-coupling in the $r^\alpha$ potential

$$H = \frac{p^2}{2m} + U_0 \left( \frac{r}{R} \right)^\alpha - 2\kappa \mathbf{B} \cdot \mathbf{s}, \quad \mathbf{B} = \frac{1}{m} \nabla \left( \frac{r}{R} \right)^{1+\frac{\alpha}{2}} \times \mathbf{p}$$

Classical spin canonical variable $\Leftrightarrow$ SU(2) coherent state path integral

$$\mathbf{s} = (s \sin \vartheta \cos \varphi, s \sin \vartheta \sin \varphi, s \cos \vartheta), \quad (p_s, q_s) = (s \cos \vartheta, \varphi)$$

EOM in the spin part

$$\dot{s} = \{H, s\}_{\text{P.B.}} = \frac{\partial H}{\partial q_s} \frac{\partial s}{\partial p_s} - \frac{\partial H}{\partial p_s} \frac{\partial s}{\partial q_s} = -2\kappa \mathbf{B} \times \mathbf{s}$$

$$|s| = \hbar/2 \rightarrow \text{no scaling in general orbits}$$

orbital plane perpendicular to spin

$$\Rightarrow \quad \mathbf{B} \parallel \mathbf{s} \quad \Rightarrow \quad \dot{s} = 0 \text{ (frozen spin)} \ldots \text{ scaling}$$
Single-particle level diagram

$\alpha = 3.0, \kappa = 0.05$

$\alpha = 5.0, \kappa = 0.05$
3. Semiclassical origin of prolate dominance

- **Periodic orbits in $r^\alpha$ model**
  
  $r^\alpha$ potential with spheroidal deformation

  \[
  H_\beta = \frac{p^2}{2m} + U_0 \left( \frac{r}{R(\theta, \beta)} \right)^\alpha, \quad R(\theta, \beta) = \frac{R_0}{\sqrt{e^{-\frac{4}{3}\beta} \cos^2 \theta + e^{\frac{2}{3}\beta} \sin^2 \theta}}
  \]

  Spheroidal deformation parameter $\beta$: \( R_z/R_\perp = e^\beta \)

  - spherical: $\beta = 0$
  - superdeformed (axis ratio 2:1): $\beta = \pm \log 2 \approx \pm 0.7$

  Non-integrable except for the cases $\alpha = 2$ (HO) and $\alpha = \infty$ (cavity)

  Simple periodic orbits at normal deformations ($\beta \lesssim 0.3$)

  - isolated diametric orbit along symmetry axis
  - degenerate diameter orbit in equatorial plane
  - isolated circular orbit in equatorial plane
  - degenerate oval orbit in meridian plane

  ... **Bridge orbit** over two diametric orbits
- **Bridge orbit bifurcations**

  \[ \alpha = 2 \text{ (HO)} \ldots \text{degenerate family at } \beta = 0 \]

  \[ \alpha > 2 \ldots \text{appearance of bridge orbits over two diameters} \]

Derivation of normal form for bridge orbit bifurcation was carried out by K.A. and M. Brack

☞ *J. of Phys. A41* (2008), 385207

“Length” of the bridge grows as increasing \( \alpha \)

⇒ Increasing significance of bridge orbit for larger \( \alpha \) (larger \( A \))
Fourier transformation of quantum level density

- Nice quantum-classical correspondence
- Large Fourier amplitude along bridge orbits
  ... significant contribution to the level density

Increasing prolate-oblate asymmetry for larger $\alpha$ (larger $A$)
Oscillating level density and Shell energy

- Valley lines of level density ↔ constant-action curves of bridge orbit
- Similar valley structure in shell energy

Prolate-oblate asymmetry of bridge orbit ... origin of prolate dominance
Effect of spin-orbit coupling

Splitting of bridge orbits

- diameters $\rightarrow$ A,B
- bridge $\rightarrow$ C,D

Bifurcation diagram of the bridge orbits

\[
\beta = 0.2 \quad \beta = 0.26 \quad \beta = 0.3
\]
Fourier transforms of level density

- Still excellent quantum-classical correspondence
- Bifurcations of lower branch make new shell structure
  ⇒ prolate dominance extinguish at $\kappa = 0.025$, revive at $\kappa = 0.05$
Single-particle level diagram

$\kappa = 0.025$

$\kappa = 0.05$
Oscillating level density

$\kappa = 0.025$

$\kappa = 0.05$
4. Summary

Semiclassical analysis of deformed $r^\alpha$ potential model with spin-orbit coupling

- Bridge orbit bifurcation play important role in deformed shell structures
- Prolate-oblate asymmetry of bridge orbit is the origin of prolate dominance
- Correlation between spin-orbit coupling and surface diffuseness can be explained as the bridge orbit bifurcation effect