Stress Tensor Correlators of Various Black Hole Vacua in Two Dimensions

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2012 Asia Pacific School/Workshop on Cosmology and Gravitation (March 4, 2012)

Outline

I. Introduction
II. Black hole vacua and mode functions
III. The renormalized stress tensor
IV. Stress tensor correlators and fluctuations
V. Discussions

I. Introduction

1. Fluctuations of the stress energy tensor can backreact onto the background spacetime. In the theory of stochastic gravity, this is in the form of a stochastic force on the right hand side of the Einstein equation.
2. Fluctuations of Hawking radiation (Wu and Ford (1999)): How big are they?
3. Fluctuations of the quantum field near the horizon: Are they divergent?
4. Sizable fluctuations might induce instability and invalidate the semi-classical approximation.
5. Renormalization is needed to obtain finite quantities. Here we adopt the point-splitting method.
6. There are a lot of simplifications in two dimensions. In particular, two dimensional spacetimes are all conformal to the Minkowski spacetime.

II. Black hole vacua and mode functions

In two dimensions, take the black hole metric as

$$
\begin{aligned}
ds^2 &= -\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2 \\
&= \left(1-\frac{2M}{r}\right)\left(-dt^2+dx^2\right)
\end{aligned}
$$

where

$$
x = r + 2M\ln\left(\frac{r}{2M}-1\right)
$$

is the tortoise coordinate.

It is conformal to Minkowski spacetime. Using the null coordinates, $u = t - x$ and $v = t + x$, one has

$$ds^2 = \left(1 - \frac{2M}{r}\right) du\, dv$$

The mode functions for a massless minimally coupled scalar are just

$$\frac{1}{\sqrt{4\pi\omega}} e^{-i\omega u} \quad ; \quad \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega v}$$

This is the Schwarzschild coordinates.

One can also use the Kruskal coordinates,

$$U = -4Me^{-u/4M} \quad ; \quad V = 4Me^{v/4M}$$

then the metric becomes

$$ds^2 = \frac{2M}{r} e^{-r/2M} dU\, dV$$

which is well-defined (like Minkowski) at the horizon, $r = 2M$. The mode functions are

$$\frac{1}{\sqrt{4\pi\omega}} e^{-i\omega U} \quad ; \quad \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega V}$$

Choosing different mode functions corresponds to choosing different vacua:

Boulware vacuum,

$$\frac{1}{\sqrt{4\pi\omega}}e^{-i\omega u} \text{ and } \frac{1}{\sqrt{4\pi\omega}}e^{-i\omega v}$$

Hartle-Hawking vacuum,

$$\frac{1}{\sqrt{4\pi\omega}}e^{-i\omega U} \text{ and } \frac{1}{\sqrt{4\pi\omega}}e^{-i\omega V}$$

Unruh vacuum,

$$\frac{1}{\sqrt{4\pi\omega}}e^{-i\omega U} \text{ and } \frac{1}{\sqrt{4\pi\omega}}e^{-i\omega v}$$

III. The renormalized stress tensor

For a massless minimally coupled scalar field ϕ, the stress tensor

$$T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\rho \phi \nabla_\rho \phi$$

$\langle T_{\mu\nu}(x) \rangle$ is divergent. Point-splitting regularization,

$$\begin{aligned} \langle T_{\mu\nu}(x) \rangle &= \lim_{x' \to x, x'' \to x} \frac{1}{2} \left(g_\mu{}^{\alpha'} g_\nu{}^{\beta''} + g_\mu{}^{\beta''} g_\nu{}^{\alpha'} + g_{\mu\nu} g^{\alpha' \beta''} \right) \\ &\qquad \nabla_{\alpha'} \nabla_{\beta''} G^+(x', x'') \end{aligned}$$

where $G^+(x', x'') = \langle \phi(x') \phi(x'') \rangle$ is the Wightman function. Usually one take $x' = x + \epsilon$ and $x'' = x - \epsilon$ along a geodesic with ϵ the geodesic distance. The limit means that $\epsilon \to 0$.

In this two dimensional setting, the renormalized stress tensor was given by Davies and Fulling (1977)

$$\langle T_{\mu\nu}\rangle_{\rm ren} = \theta_{\mu\nu} + \frac{1}{48\pi} R g_{\mu\nu}$$

where the state dependent tensor

$$\begin{aligned}
\theta_{uu} &= -\frac{1}{12\pi}\left(C^{1/2}\partial_u^2 C^{-1/2}\right)\\
\theta_{vv} &= -\frac{1}{12\pi}\left(C^{1/2}\partial_v^2 C^{-1/2}\right)\\
\theta_{uv} &= 0
\end{aligned}$$

and the Ricci scalar

$$R = -\frac{4}{C}\left[\frac{\partial_u\partial_v C}{C} - \frac{(\partial_u C)(\partial_v C)}{C^2}\right]$$

Boulware vacuum η, $C = 1 - 2M/r$.

$$
\begin{aligned}
T^{\eta}_{uu} &= \frac{1}{24\pi M^2}\left(\frac{3M^4}{2r^4} - \frac{M^3}{r^3}\right) = T^{\eta}_{vv} \\
T^{\eta}_{uv} &= \frac{1}{24\pi M^2}\left(1 - \frac{2M}{r}\right)\left(-\frac{M^3}{r^3}\right) \\
T^{\eta}_{tt} &= \frac{1}{24\pi M^3}\left(\frac{7M^4}{r^4} - \frac{4M^3}{r^3}\right) \sim \frac{1}{r^3} \quad \text{as } r \to \infty \\
T^{\eta}_{tr} &= 0 \\
T^{\eta}_{rr} &= \frac{1}{24\pi M^2}\left(1 - \frac{2M}{r}\right)^{-2}\left(-\frac{M^4}{r^4}\right) \sim \frac{1}{r^4} \quad \text{as } r \to \infty
\end{aligned}
$$

In a local frame, $T^{\eta}_{\hat{t}\hat{t}}$ and $T^{\eta}_{\hat{r}\hat{r}} \sim (1 - 2M/r)^{-1}$ as $r \to 2M$.

Hartle-Hawking vacuum ν, $C = 2Me^{-r/2M}/r$,

$$\begin{aligned}
T^{\nu}_{uu} &= \frac{1}{24\pi M^2}\left(\frac{3M^4}{2r^4} - \frac{M^3}{r^3} + \frac{1}{32}\right) = T^{\nu}_{vv} \\
T^{\nu}_{uv} &= \frac{1}{24\pi M^2}\left(1 - \frac{2M}{r}\right)\left(-\frac{M^3}{r^3}\right) = T^{\eta}_{uv} \\
T^{\nu}_{tt} &= \frac{1}{24\pi M^2}\left(\frac{7M^4}{r^4} - \frac{4M^3}{r^3} + \frac{1}{16}\right) \sim \frac{\pi}{6}\left(\frac{1}{8\pi M}\right)^2 \quad \text{as } r \to \infty \\
T^{\nu}_{tr} &= 0 \\
T^{\nu}_{rr} &= \frac{1}{24\pi M^2}\left(1 - \frac{2M}{r}\right)^{-2}\left(-\frac{M^4}{r^4} + \frac{1}{16}\right) \\
&\sim \frac{\pi}{6}\left(\frac{1}{8\pi M}\right)^2 \quad \text{as } r \to \infty
\end{aligned}$$

This corresponds to a thermal gas with temperature $T = 1/8\pi M$.

In a local frame, as $r \to 2M$,

$$
\begin{aligned}
T^{\nu}_{\hat{t}\hat{t}} &\sim -\frac{1}{96\pi M^2} \\
T^{\nu}_{\hat{r}\hat{r}} &\sim \frac{1}{96\pi M^2}
\end{aligned}
$$

The stress tensor is finite in this near horizon limit.
The Hartle-Hawking vacuum is defined with respect to the Kruskal coordinates which are well-defined at the horizon.

Unruh vacuum ξ

$$
\begin{aligned}
T^{\xi}_{uu} &= T^{\nu}_{uu} = \frac{1}{24\pi M^2}\left(\frac{3M^4}{2r^4} - \frac{M^3}{r^3} + \frac{1}{32}\right) \\
T^{\xi}_{uv} &= T^{\eta}_{uv} = \frac{1}{24\pi M^2}\left(1 - \frac{2M}{r}\right)\left(-\frac{M^3}{r^3}\right) \\
T^{\xi}_{vv} &= T^{\eta}_{vv} = \frac{1}{24\pi M^2}\left(\frac{3M^4}{2r^4} - \frac{M^3}{r^3}\right) \\
T^{\xi}_{tr} &= \frac{1}{24\pi M^2}\left(1 - \frac{2M}{r}\right)^{-1}\left(-\frac{1}{32}\right) \\
&\sim -\frac{\pi}{12}\left(\frac{1}{8\pi M}\right)^2 \quad \text{as } r \to \infty
\end{aligned}
$$

This represents an out-going flux of Hawking radiation with temperature $T = 1/8\pi M$.

IV. Stress tensor correlators and fluctuations

Define the correlation,

$$\Delta T^2_{\mu\nu\alpha'\beta'}(x,x') = \langle T_{\mu\nu}(x) T_{\alpha'\beta'}(x')\rangle - \langle T_{\mu\nu}(x)\rangle\langle T_{\alpha'\beta'}(x')\rangle$$

Using point-splitting regularization, one arrives at the expression

$$\begin{aligned}
& \Delta T^2_{\mu\nu\alpha'\beta'}(x,x') \\
= \; & \left[\nabla_\mu \nabla_{\alpha'} G^+(x,x')\right]\left[\nabla_\nu \nabla_{\beta'} G^+(x,x')\right] \\
& + \left[\nabla_\mu \nabla_{\beta'} G^+(x,x')\right]\left[\nabla_\nu \nabla_{\alpha'} G^+(x,x')\right] \\
& - g_{\mu\nu}\left[\nabla_\rho \nabla_{\alpha'} G^+(x,x')\right]\left[\nabla^\rho \nabla_{\beta'} G^+(x,x')\right] \\
& + g_{\alpha'\beta'}\left[\nabla_\mu \nabla_{\sigma'} G^+(x,x')\right]\left[\nabla^\nu \nabla^{\sigma'} G^+(x,x')\right] \\
& + \frac{1}{2} g_{\mu\nu} g_{\alpha'\beta'}\left[\nabla_\rho \nabla_{\sigma'} G^+(x,x')\right]\left[\nabla^\rho \nabla^{\sigma'} G^+(x,x')\right]
\end{aligned}$$

In the Schwarzschild coordinates (Boulware vacuum),

$$G^{+}(x, x') = -\frac{1}{4\pi}\ln(\Delta u \Delta v)$$

The nonzero correlators are

$$\begin{aligned}
\left(\Delta T^2_{uuu'u'}\right)^{\eta} &= \frac{1}{8\pi^2(\Delta u)^4} \\
\left(\Delta T^2_{vvv'v'}\right)^{\eta} &= \frac{1}{8\pi^2(\Delta v)^4}
\end{aligned}$$

They are well-defined when x and x' are non-coincident. Here we consider only non-null separation.

Similarly in the Kruskal coordinates (Hartle-Hawking vacuum),

$$\left(\Delta T^2_{UUU'U'}\right)^{\nu} = \frac{1}{8\pi^2(\Delta U)^4}$$

$$\left(\Delta T^2_{VVV'V'}\right)^{\nu} = \frac{1}{8\pi^2(\Delta V)^4}$$

In the Unruh vacuum,

$$\left(\Delta T^2_{UUU'U'}\right)^{\xi} = \frac{1}{8\pi^2(\Delta U)^4}$$

$$\left(\Delta T^2_{vvv'v'}\right)^{\xi} = \frac{1}{8\pi^2(\Delta v)^4}$$

To study the fluctuations we have to take the coincident limit $x' \to x$ which is divergent.
We again use the point-splitting regularization and we obtain

$$
\begin{aligned}
& \left(\Delta T^2_{\mu\nu\alpha\beta}(x)\right)_{\rm ren} \\
= & \left(\theta_{\mu\alpha}\theta_{\nu\beta} + \theta_{\mu\beta}\theta_{\nu\alpha} - g_{\mu\nu}\theta_{\alpha\rho}\theta^{\rho}_{\beta} - g_{\alpha\beta}\theta_{\mu\rho}\theta^{\rho}_{\nu} + \frac{1}{2} g_{\mu\nu} g_{\alpha\beta}\theta_{\rho\sigma}\theta^{\rho\sigma}\right) \\
& + \frac{R}{48\pi}\left(g_{\mu\alpha}\theta_{\nu\beta} + g_{\mu\beta}\theta_{\nu\alpha} + g_{\nu\alpha}\theta_{\mu\beta} + g_{\nu\beta}\theta_{\mu\alpha}\right. \\
& \qquad \left. -2 g_{\mu\nu}\theta_{\alpha\beta} - 2 g_{\alpha\beta}\theta_{\mu\nu} + g_{\mu\nu} g_{\alpha\beta}\theta^{\rho}_{\rho}\right) \\
& + \left(\frac{R}{48\pi}\right)^2 \left(g_{\mu\alpha} g_{\nu\beta} + g_{\mu\beta} g_{\nu\alpha} - g_{\mu\nu} g_{\alpha\beta}\right)
\end{aligned}
$$

For the Boulware vacuum, we have

$$
\begin{aligned}
(\Delta T^2_{tttt})^{\eta}_{\mathrm{ren}} &= 4\left(\frac{1}{24\pi M^2}\right)^2\left(\frac{41M^8}{4r^8}-\frac{11M^7}{r^7}+\frac{3M^6}{r^6}\right)\\
(\Delta T^2_{rrrr})^{\eta}_{\mathrm{ren}} &= 4\left(\frac{1}{24\pi M^2}\right)^2\left(1-\frac{2M}{r}\right)^{-4}\times\\
&\qquad\left(\frac{41M^8}{4r^8}-\frac{11M^7}{r^7}+\frac{3M^6}{r^6}\right)
\end{aligned}
$$

As $r\to\infty$,

$$
(\Delta T^2_{tttt})^{\eta}_{\mathrm{ren}} \sim (\Delta T^2_{rrrr})^{\eta}_{\mathrm{ren}} \sim 12\left(\frac{1}{24\pi M^2}\right)^2\left(\frac{M^6}{r^6}\right)
$$

Note that in the same limit, $T^{\eta}_{tt}\sim 1/r^3$ and $T^{\eta}_{rr}\sim 1/r^4$.

For the Hartle-Hawking vacuum, we have

$$
\begin{aligned}
\left(\Delta T^2_{tttt}\right)^{\nu}_{\mathrm{ren}} &= 4\left(\frac{1}{24\pi M^2}\right)^2\left(\frac{41M^8}{4r^8} - \frac{11M^7}{r^7} + \frac{3M^6}{r^6}\right. \\
&\qquad \left. + \frac{3M^4}{32r^4} - \frac{M^3}{16r^3} + \frac{1}{1024}\right) \\
\left(\Delta T^2_{rrrr}\right)^{\nu}_{\mathrm{ren}} &= \left(1 - \frac{2M}{r}\right)^{-4}\left(\Delta T^2_{tttt}\right)^{\nu}_{\mathrm{ren}}
\end{aligned}
$$

As $r \to \infty$, we have

$$(\Delta T^2_{tttt})^{\nu}_{\mathrm{ren}} \sim (\Delta T^2_{rrrr})^{\nu}_{\mathrm{ren}} \sim \frac{1}{256}\left(\frac{1}{24\pi M^2}\right)^2$$

Since in the same limit,

$$T^{\nu}_{tt} \sim T^{\nu}_{rr} \sim \frac{1}{16}\left(\frac{1}{24\pi M^2}\right)$$

Hence, we have

$$\frac{\sqrt{(\Delta T^2_{tttt})^{\nu}_{\mathrm{ren}}}}{T^{\nu}_{tt}} \sim \frac{\sqrt{(\Delta T^2_{rrrr})^{\nu}_{\mathrm{ren}}}}{T^{\nu}_{rr}} \sim 1$$

As $r \to 2M$, in a local frame,

$$\left(\Delta T^2_{\hat{t}\hat{t}\hat{t}\hat{t}}\right)^{\nu}_{\mathrm{ren}} \sim \left(\Delta T^2_{\hat{r}\hat{r}\hat{r}\hat{r}}\right)^{\nu}_{\mathrm{ren}} \sim \frac{1}{8}\left(\frac{1}{24\pi M^2}\right)^2$$

In the same limit,

$$T^{\nu}_{\hat{t}\hat{t}} \sim -\frac{1}{4}\left(\frac{1}{24\pi M^2}\right) \quad ; \quad T^{\nu}_{\hat{r}\hat{r}} \sim \frac{1}{4}\left(\frac{1}{24\pi M^2}\right)$$

we have again

$$\sqrt{\frac{\left(\Delta T^2_{\hat{t}\hat{t}\hat{t}\hat{t}}\right)^{\nu}_{\mathrm{ren}}}{(T^{\nu}_{\hat{t}\hat{t}})^2}} \sim \sqrt{\frac{\left(\Delta T^2_{\hat{r}\hat{r}\hat{r}\hat{r}}\right)^{\nu}_{\mathrm{ren}}}{(T^{\nu}_{\hat{r}\hat{r}})^2}} \sim \sqrt{2}$$

For the Unruh vacuum,

$$\begin{aligned}(\Delta T_{trtr}^2)^{\xi}_{\rm ren} &= 2\left(\frac{1}{24\pi M^2}\right)^2\left(1-\frac{2M}{r}\right)^{-2}\times \\ &\quad \left(\frac{9M^8}{2r^8}-\frac{6M^7}{r^7}+\frac{2M^6}{r^6}+\frac{3M^4}{32r^4}-\frac{M^3}{16r^3}+\frac{1}{1024}\right)\end{aligned}$$

As $r \to \infty$,

$$(\Delta T_{trtr}^2)^{\xi}_{\rm ren} \sim \frac{1}{512}\left(\frac{1}{24\pi M^2}\right)^2$$

Hence, we have

$$\sqrt{\frac{(\Delta T_{trtr}^2)^{\xi}_{\rm ren}}{\left(T_{tr}^{\xi}\right)^2}} \sim \sqrt{2}$$

V. Discussions

1. The fluctuations of the Hartle-Hawking vacuum, for both the density and the pressure, are of order 1. The same applies to the fluctuations near the horizon.
2. Fluctuations of the Hawking flux in the Unruh vacuum are also of order 1.
3. The results show that fluctuations are sizable and they might induce passive spacetime metric fluctuation to invalidate the semi-classical approximation. This is true even for static spacetimes.
4. Results in two dimensions should only be taken as an indication. Much more work has to be done in four dimensions.