Restricted Gravity

—A New Approach to Quantum Gravity—

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- **Problems of spin-two graviton**

1. The metric is a classical concept which allows precise mesurement, but quantum gravity requires a quantum field which requires intrinsic fuzziness — Geroch.

2. The metric can not describe the gravitational coupling to fermions

$$(\bar{\psi}\gamma^a \partial_\mu \psi) \times e_a^\mu.$$

This tells that the tetrad (4 spin-one fields e_a^μ) is more fundamental than the metric. So we need a new paradigm for quantum gravity.

- **Motivation**

1. Is Einstein's theory the simplest possible generally invariant theory?
   Yes?.....No!

2. What is the simpler theory?
   Restricted gravity which describes the core dynamics of Einstein's theory.

3. How can we obtain such gravity?
   Making Abelian projection to Einstein's theory.

4. How can we describe the graviton in this theory?
   By a spin-one Abelian gauge field.

Quantum gravity

- **Plan**

1. Treat Einstein's theory as a gauge theory of Lorentz group. Make the Abelian projection to decompose the connection to the restricted part and the valence part.
2. Remove the valence part to separate the core dynamics of Einstein's theory. Obtain the restricted gravity.
3. Express the restricted gravity by an Abelian gauge theory, and show that the graviton can be described by a massless spin-one gauge field.
4. Recover Einstein's theory adding the valence part. Establish the Abelian dominance in Einstein's theory.

Example: Restricted QCD

A) Abelian decomposition

- Let $(\hat{n}_1, \hat{n}_2, \hat{n}_3 = \hat{n})$ be an orthonormal basis, and select $\hat{n}$ to be the Abelian (i.e., the color) direction. Make the Abelian projection

$$D_\mu \hat{n} = \partial_\mu \hat{n} + g\vec{A}_\mu \times \hat{n} = 0. \qquad (\hat{n}^2 = 1)$$

$$\vec{A}_\mu \rightarrow \hat{A}_\mu = A_\mu \hat{n} - \frac{1}{g}\hat{n} \times \partial_\mu \hat{n}. \qquad (A_\mu = \hat{n} \cdot \vec{A}_\mu)$$

- With this we have the Abelian (Cho-Faddeev-Niemi or Cho-Duan-Ge) decomposition

$$\vec{A}_\mu = A_\mu \hat{n} - \frac{1}{g}\hat{n} \times \partial_\mu \hat{n} + \vec{X}_\mu, \qquad (\hat{n} \cdot \vec{X}_\mu = 0).$$

- Under the infinitesimal gauge transformation

$$\delta \vec{A}_\mu = \frac{1}{g} D_\mu \vec{\alpha}, \qquad \delta \hat{n} = -\vec{\alpha} \times \hat{n},$$

we have

$$\delta \hat{A}_\mu = \frac{1}{g} \hat{D}_\mu \vec{\alpha}, \qquad \delta \vec{X}_\mu = -\vec{\alpha} \times \vec{X}_\mu .$$

1. $\hat{A}_\mu$ has the full SU(2) gauge degrees of freedom, and forms an SU(2) connection space by itself.
2. $\vec{X}_\mu$ transforms covariantly.

B) Restricted QCD (RCD)

- $\hat{A}_\mu$ is essentially Abelian, but has a dual structure

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + g\hat{A}_\mu \times \hat{A}_\nu = (F_{\mu\nu} + H_{\mu\nu})\hat{n},$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$
$$H_{\mu\nu} = -\frac{1}{g}\hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}) = \partial_\mu C_\nu - \partial_\nu C_\mu,$$
$$C_\mu = \frac{1}{g}\hat{n}_1 \cdot \partial_\mu \hat{n}_2.$$

So $\hat{F}_{\mu\nu}$ is described by two Abelian potentials, the "electric" A_μ and the "magnetic" C_μ.

- Let $\vec{C}_\mu = -\frac{1}{g}\hat{n} \times \partial_\mu \hat{n}$ and find

$$\vec{H}_{\mu\nu} = \partial_\mu \vec{C}_\nu - \partial_\nu \vec{C}_\mu + g\vec{C}_\mu \times \vec{C}_\nu = H_{\mu\nu}\hat{n}.$$

  Moreover, $\vec{C}_\mu$ with $\hat{n} = \hat{r}$ describes precisely the Wu-Yang monopole, where $\hat{n}$ represents the non-Abelian monopole topology $\Pi_2(S^2)$.

- Define the restricted QCD by

$$\mathcal{L}_{RCD} = -\frac{1}{4}\hat{F}_{\mu\nu}{}^2.$$

  It has the full non-Abelian gauge invariance and thus inherits all topological properties of QCD, but is much simpler than QCD.

C) Abelian dominance

- Find

$$\vec{F}_{\mu\nu} = \hat{F}_{\mu\nu} + (\hat{D}_\mu \vec{X}_\nu - \hat{D}_\nu \vec{X}_\mu) + g\vec{X}_\mu \times \vec{X}_\nu,$$

$$\mathcal{L}_{QCD} = -\frac{1}{4}\vec{F}_{\mu\nu}^2 = -\frac{1}{4}\hat{F}_{\mu\nu}^2 - \frac{g}{2}\hat{F}_{\mu\nu}\cdot(\vec{X}_\mu \times \vec{X}_\nu)$$
$$-\frac{1}{4}(\hat{D}_\mu \vec{X}_\nu - \hat{D}_\nu \vec{X}_\mu)^2 - \frac{g^2}{4}(\vec{X}_\mu \times \vec{X}_\nu)^2.$$

So QCD can be viewed as RCD made of $\hat{A}_\mu$ which has the valence gluons as colored source.

- The valence gluons play no role in confinement, because they are the colored source which have to be confined.

D) Monopole dominance

- The Abelian projection separates the monopole potential gauge independently.
- The one-loop effective action of QCD shows that the monopole condensation plays the central role in color confinement.
- The monopole dominance in the color confinement has been confirmed by recent KEK lattice calculations based on Abelian projection.

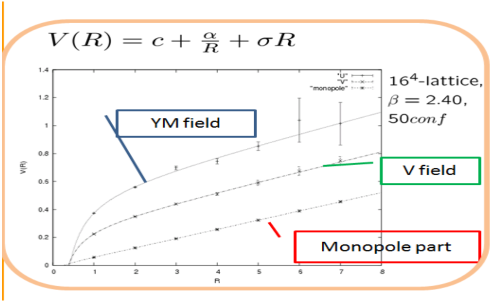

Figure: The monopole dominance based on Abelian projection in lattice QCD.

A) Vacuum potential

- Impose the vacuum isometry

$$\forall_i \; D_\mu \hat{n}_i = (\partial_\mu + g\vec{A}_\mu \times)\; \hat{n}_i = 0,$$

$$\forall_i \; [D_\mu, D_\nu]\; \hat{n}_i = \vec{F}_{\mu\nu} \times \hat{n}_i = 0 \quad \Rightarrow \quad \vec{F}_{\mu\nu} = 0.$$

- Construct the most general vacuum potential

$$\vec{A}_\mu \rightarrow \hat{\Omega}_\mu = C_\mu^{\;\;k}\; \hat{n}_k = -\frac{1}{2g}\epsilon_{ij}^{\;\;\;k}\; (\hat{n}_i \cdot \partial_\mu \hat{n}_j)\; \hat{n}_k.$$

- With S^3 compactification of R^3, we have the multiple vacua $|n\rangle$ classified by the Hopf invariant $\Pi_3(S^3) \simeq \Pi_3(S^2)$ which represents the knot topology of $\hat{n} = \hat{n}_3$,

$$n = -\frac{g^3}{96\pi^2} \int \epsilon_{\alpha\beta\gamma}\epsilon_{ijk} C_\alpha^i C_\beta^j C_\gamma^k d^3x. \qquad (\alpha, \beta, \gamma = 1, 2, 3)$$

- With $\hat{\Omega}_\mu$, the restricted potential $\hat{A}_\mu$ admits further decomposition

$$\hat{A}_\mu = \hat{\Omega}_\mu + \vec{B}_\mu, \qquad \vec{B}_\mu = (A_\mu + \widetilde{C}_\mu)\ \hat{n},$$
$$\delta\hat{\Omega}_\mu = \frac{1}{g} {D_\mu}^{(0)} \vec{\alpha}, \quad \delta\vec{B}_\mu = -\vec{\alpha} \times \vec{B}_\mu, \quad ({D_\mu}^{(0)} = \partial_\mu + g\ \hat{\Omega}_\mu \times).$$

So $\hat{\Omega}_\mu$ (just like $\hat{A}_\mu$) forms its own SU(2) connection space.

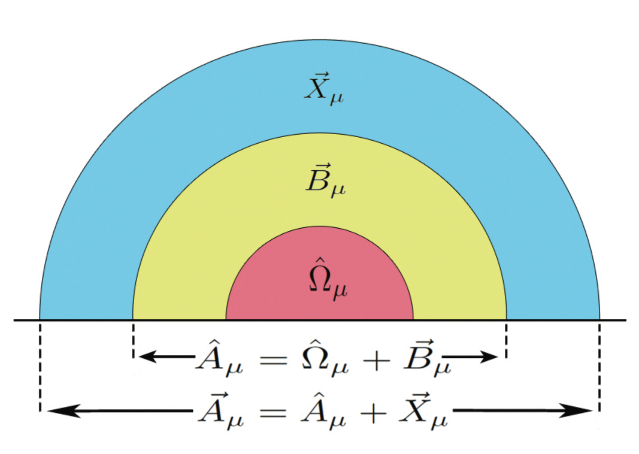

Figure: The structure of non-Abelian connection space: It has two proper subspaces made of the restricted potentials $\hat{A}_\mu$ and the vacuum potentials $\hat{\Omega}_\mu$ which form their own non-Abelian connection spaces.

B) Vacuum tunneling

- The multiple vacua $|n\rangle$ are physically (as well as topologically) inequivalent, but are unstable under the quantum fluctuation. They are connected by the vacuum tunneling through the instantons.

- The vacuum tunneling assures the existence of the θ-vacuum in QCD

$$|\Omega\rangle = \sum_n e^{in\theta}\, |n\rangle.$$

- The SU(2) results directly applies to Einstein's theory because SU(2) is the rotation subgroup of Lorentz group.

Einstein's Theory: Gauge Theory of Lorentz Group

- Einstein's theory can be viewed as a gauge theory of Lorentz group, and the local Lorentz invariance assures the general invariance.
- In the presence of spinor field one must have the local Lorentz invariance. This necessitates a gauge theory of Lorentz group, where the tetrad (not the metric) plays the fundamental role.
- Constructing a gauge theory of Lorentz group is a natural way to rediscover Einstein's theory.

- Introduce a coordinate basis ∂_μ and an orthonormal basis e_a

$$[\partial_\mu,\ \partial_\nu] = 0, \qquad [e_a,\ e_b] = f_{ab}{}^c\ e_c,$$
$$e_a = e_a{}^\mu\ \partial_\mu, \quad \partial_\mu = e_\mu{}^a\ e_a. \quad (\mu, \nu; a, b, c = 0, 1, 2, 3)$$

Let $J_{ab} = -J_{ba}$ be the generators of Lorentz group,

$$[J_{ab},\ J_{cd}] = \eta_{ac} J_{bd} - \eta_{bc} J_{ad} + \eta_{bd} J_{ac} - \eta_{ad} J_{bc}$$
$$= f_{ab,cd}{}^{mn}\ J_{mn},$$

where $\eta_{ab} = diag\ (-1, 1, 1, 1)$ is the Minkowski metric.

- With the 3-dimensional rotation and boost generators L_i and K_i we have

$$[L_i,\ L_j] = \epsilon_{ijk} L_k, \qquad [L_i,\ K_j] = \epsilon_{ijk} K_k,$$
$$[K_i,\ K_j] = -\epsilon_{ijk} L_k.$$

- Notice that

1. The Lorentz group is non-compact, so that the invariant metric is indefinite.
2. The Lorentz group has the well-known invariant tensor ϵ_{abcd} which allows the dual transformation.
3. The Lorentz group has rank two, so that it has two commuting Abelian subgroups.

- Remember that

1. In the gauge formalism of Einstein's theory the spin connection ${\omega_\mu}^{ab}$ corresponds to the gauge potential ${\Gamma_\mu}^{ab}$, and the curvature tensor ${R_{\mu\nu}}^{ab}$ corresponds to the field strength ${F_{\mu\nu}}^{ab}$.

2. In Einstein's theory the metric $g_{\mu\nu}$ propagates, but in gauge theory the potential ${\Gamma_\mu}^{ab}$ propagates.

3. The Einstein-Hilbert action is linear in ${R_{\mu\nu}}^{ab}$ $(R = {e_a}^\mu {e_b}^\nu {R_{\mu\nu}}^{ab})$, but in gauge theory the Yang-Mills action is quadratic in ${F_{\mu\nu}}^{ab}$ $(F^2 = {F_{\mu\nu}}^{ab} {F_{\mu\nu}}^{ab})$.

- Let p^{ab} $(p^{ab} = -p^{ba})$ be an adjoint representation of Lorentz group

$$\delta_{ab}\ p^{cd} = -\frac{1}{2} f_{ab,mn}{}^{cd}\ p^{mn}.$$

We can denote p^{ab} by a sextet $\mathbf{p}$ made of two triplets $\vec{m}$ and $\vec{e}$,

$$\mathbf{p} = \frac{1}{2}\ p_{ab}\mathbf{I}^{ab} = \begin{pmatrix} \vec{m} \\ \vec{e} \end{pmatrix},\quad p^{ab} = \mathbf{p}\cdot\mathbf{I}^{ab} = \frac{1}{2} p^{cd} I_{cd}{}^{ab},$$

$$I_{cd}{}^{ab} = \left(\delta_c{}^a \delta_d{}^b - \delta_c{}^b \delta_d{}^a\right),$$

where $\vec{m}$ is the magnetic (rotation) part and $\vec{e}$ is the electric (boost) part of $\mathbf{p}$.

- Lorentz group has two maximal Abelian subgroups, A_2 made of L_3 and K_3 and B_2 made of $(L_1 + K_2)/\sqrt{2}$ and $(L_2 - K_1)/\sqrt{2}$. In both cases the magnetic isometry is described by two, not one, commuting sextet vector fields which are dual to each other.

- Let one of the isometry vector be $\mathbf{p}$

$$D_\mu \mathbf{p} = (\partial_\mu + \mathbf{\Gamma}_\mu \times)\ \mathbf{p} = 0.$$

This automatically assures that $\tilde{\mathbf{p}}$ also becomes an isometry,

$$D_\mu \tilde{\mathbf{p}} = (\partial_\mu + \mathbf{\Gamma}_\mu \times)\ \tilde{\mathbf{p}} = 0.$$

- The isometry is described by two Casimir invariants α and β,

$$\alpha = \mathbf{p} \cdot \mathbf{p} = \vec{m}^2 - \vec{e}^2, \qquad \beta = \mathbf{p} \cdot \tilde{\mathbf{p}} = 2\vec{m} \cdot \vec{e},$$

  and we can always choose (α, β) to be either $(\pm 1, 0)$ or $(0, 0)$.

- The A_2 isometry has $(\pm 1, 0)$, so that it can be called the rotation-boost (or non-lightlike) isometry. But the B_2 isometry has $(0, 0)$, so that it can be called the null (or lightlike) isometry.

A) A_2 isometry

- Express the A_2 isometry by

$$\mathbf{l} = \mathbf{l}_3 = \begin{pmatrix} \hat{n} \\ 0 \end{pmatrix}, \qquad \tilde{\mathbf{l}} = \mathbf{k}_3 = \begin{pmatrix} 0 \\ -\hat{n} \end{pmatrix},$$

$$D_\mu\, \mathbf{l} = 0, \qquad D_\mu\, \tilde{\mathbf{l}} = 0,$$

and find $(\alpha, \beta) = (1, 0)$. Find the restricted connection $\hat{\mathbf{\Gamma}}_\mu$ of A_2

$$\begin{aligned} \hat{\mathbf{\Gamma}}_\mu &= \Gamma_\mu\, \mathbf{l} - \widetilde{\Gamma}_\mu\, \tilde{\mathbf{l}} - \mathbf{l} \times \partial_\mu \mathbf{l} \\ &= \Gamma_\mu\, \mathbf{l} - \widetilde{\Gamma}_\mu\, \tilde{\mathbf{l}} - \frac{1}{2}(\mathbf{l} \times \partial_\mu \mathbf{l} - \tilde{\mathbf{l}} \times \partial_\mu \tilde{\mathbf{l}}), \end{aligned}$$

$$\Gamma_\mu = \mathbf{l} \cdot \mathbf{\Gamma}_\mu, \quad \widetilde{\Gamma}_\mu = \tilde{\mathbf{l}} \cdot \mathbf{\Gamma}_\mu.$$

- The restricted field strength $\hat{\mathbf{R}}_{\mu\nu}$ of A_2 is given by

$$\begin{aligned}\hat{\mathbf{R}}_{\mu\nu} &= \partial_\mu \hat{\mathbf{\Gamma}}_\nu - \partial_\nu \hat{\mathbf{\Gamma}}_\mu + \hat{\mathbf{\Gamma}}_\mu \times \hat{\mathbf{\Gamma}}_\nu \\ &= (\Gamma_{\mu\nu} + H_{\mu\nu})\ \mathbf{l} - (\widetilde{\Gamma}_{\mu\nu} + \widetilde{H}_{\mu\nu})\ \tilde{\mathbf{l}},\end{aligned}$$

$$\Gamma_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu, \quad H_{\mu\nu} = -\mathbf{l} \cdot (\partial_\mu \mathbf{l} \times \partial_\nu \mathbf{l}) = \partial_\mu \widetilde{C}_\nu - \partial_\nu \widetilde{C}_\mu,$$
$$\widetilde{\Gamma}_{\mu\nu} = \partial_\mu \widetilde{\Gamma}_\nu - \partial_\nu \widetilde{\Gamma}_\mu, \qquad \widetilde{H}_{\mu\nu} = -\tilde{\mathbf{l}} \cdot (\partial_\mu \mathbf{l} \times \partial_\nu \mathbf{l}) = 0,$$

so that we have

$$\hat{R}_{\mu\nu}{}^{ab} = (\Gamma_{\mu\nu} + H_{\mu\nu})\ l^{ab} - \widetilde{\Gamma}_{\mu\nu}\ \tilde{l}^{ab}.$$

- With this the full connection of Lorentz group is given by

$$\boldsymbol{\Gamma}_\mu = \hat{\boldsymbol{\Gamma}}_\mu + \mathbf{Z}_\mu, \quad \mathbf{l} \cdot \mathbf{Z}_\mu = \tilde{\mathbf{l}} \cdot \mathbf{Z}_\mu = 0,$$

where $\mathbf{Z}_\mu$ is the valence connection.

- The corresponding field strength $\mathbf{R}_{\mu\nu}$ (the curvature tensor) is written as

$$\mathbf{R}_{\mu\nu} = \partial_\mu \boldsymbol{\Gamma}_\nu - \partial_\nu \boldsymbol{\Gamma}_\mu + \boldsymbol{\Gamma}_\mu \times \boldsymbol{\Gamma}_\nu = \hat{\mathbf{R}}_{\mu\nu} + \mathbf{Z}_{\mu\nu},$$
$$\mathbf{Z}_{\mu\nu} = \hat{D}_\mu \mathbf{Z}_\nu - \hat{D}_\nu \mathbf{Z}_\mu + \mathbf{Z}_\mu \times \mathbf{Z}_\nu,$$
$$\hat{D}_\mu = \partial_\mu + \hat{\boldsymbol{\Gamma}}_\mu \times .$$

B) B_2 isometry

- Express the B_2 isometry by

$$\mathbf{j} = \frac{e^\lambda}{\sqrt{2}}(\mathbf{l}_1 + \mathbf{k}_2) = \frac{e^\lambda}{\sqrt{2}} \begin{pmatrix} \hat{n}_1 \\ \hat{n}_2 \end{pmatrix},$$

$$\tilde{\mathbf{j}} = \frac{e^\lambda}{\sqrt{2}}(\mathbf{l}_2 - \mathbf{k}_1) = \frac{e^\lambda}{\sqrt{2}} \begin{pmatrix} \hat{n}_2 \\ -\hat{n}_1 \end{pmatrix},$$

$$D_\mu \mathbf{j} = 0, \quad D_\mu \tilde{\mathbf{j}} = 0,$$

where λ is an arbitrary function. Find $(\alpha, \beta) = (0, 0)$.

- Let

$$\mathbf{k} = \frac{e^{-\lambda}}{\sqrt{2}}(\mathbf{l}_1 - \mathbf{k}_2), \qquad \tilde{\mathbf{k}} = -\frac{e^{-\lambda}}{\sqrt{2}}(\mathbf{l}_2 + \mathbf{k}_1),$$
$$\mathbf{l} = -\mathbf{j} \times \tilde{\mathbf{k}}, \qquad \tilde{\mathbf{l}} = \mathbf{j} \times \mathbf{k}.$$

- With this find the restricted connection $\hat{\mathbf{\Gamma}}$ of B_2

$$\begin{aligned} \hat{\mathbf{\Gamma}}_\mu &= \Gamma_\mu \, \mathbf{j} - \widetilde{\Gamma}_\mu \, \tilde{\mathbf{j}} - \mathbf{k} \times \partial_\mu \mathbf{j} \\ &= \Gamma_\mu \, \mathbf{j} - \widetilde{\Gamma}_\mu \, \tilde{\mathbf{j}} - \frac{1}{2}(\mathbf{k} \times \partial_\mu \mathbf{j} - \tilde{\mathbf{k}} \times \partial_\mu \tilde{\mathbf{j}}) \end{aligned}$$

$$\Gamma_\mu = \mathbf{k} \cdot \mathbf{\Gamma}_\mu, \quad \widetilde{\Gamma}_\mu = \tilde{\mathbf{k}} \cdot \mathbf{\Gamma}_\mu.$$

- The restricted curvature tensor $\hat{\mathbf{R}}_{\mu\nu}$ of B_2 is given by

$$\hat{\mathbf{R}}_{\mu\nu} = \partial_\mu \hat{\mathbf{\Gamma}}_\nu - \partial_\nu \hat{\mathbf{\Gamma}}_\mu + \hat{\mathbf{\Gamma}}_\mu \times \hat{\mathbf{\Gamma}}_\nu$$
$$= (\Gamma_{\mu\nu} + H_{\mu\nu})\mathbf{j} - (\widetilde{\Gamma}_{\mu\nu} + \widetilde{H}_{\mu\nu})\tilde{\mathbf{j}},$$
$$\Gamma_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu, \quad \widetilde{\Gamma}_{\mu\nu} = \partial_\mu \widetilde{\Gamma}_\nu - \partial_\nu \widetilde{\Gamma}_\mu,$$
$$H_{\mu\nu} = -\mathbf{k} \cdot (\partial_\mu \mathbf{j} \times \partial_\nu \mathbf{k} - \partial_\nu \mathbf{j} \times \partial_\mu \mathbf{k}) = \partial_\mu H_\nu - \partial_\nu H_\mu,$$
$$\widetilde{H}_{\mu\nu} = -\tilde{\mathbf{k}} \cdot (\partial_\mu \mathbf{j} \times \partial_\nu \mathbf{k} - \partial_\nu \mathbf{j} \times \partial_\mu \mathbf{k}) = \partial_\mu \widetilde{H}_\nu - \partial_\nu \widetilde{H}_\mu.$$

- Adding the valence part $\mathbf{Z}_\mu$ to $\hat{\mathbf{\Gamma}}_\mu$ we obtain the full connection and the full curvature tensor

$$\mathbf{\Gamma}_\mu = \hat{\mathbf{\Gamma}}_\mu + \mathbf{Z}_\mu, \qquad \mathbf{k} \cdot \mathbf{Z}_\mu = \tilde{\mathbf{k}} \cdot \mathbf{Z}_\mu = 0.$$
$$\mathbf{R}_{\mu\nu} = \hat{\mathbf{R}}_{\mu\nu} + \mathbf{Z}_{\mu\nu}, \qquad \mathbf{Z}_{\mu\nu} = \hat{D}_\mu \mathbf{Z}_\nu - \hat{D}_\nu \mathbf{Z}_\mu + \mathbf{Z}_\mu \times \mathbf{Z}_\nu.$$

Restricted Gravity

- Introduce the Lorentz covariant 4-index metric $g_{\mu\nu}{}^{ab}$

$$\mathbf{g}_{\mu\nu} = \mathbf{g}_{\mu\nu}{}^{ab} \cdot \mathbf{I}_{ab} = e_\mu{}^a e_\nu{}^b \; \mathbf{I}_{ab},$$
$$g_{\mu\nu}{}^{ab} = (e_\mu{}^a e_\nu{}^b - e_\nu{}^a e_\mu{}^b) = e_\mu{}^c e_\nu{}^d I_{cd}{}^{ab},$$

  and find

$$\nabla_\alpha g_{\mu\nu} = 0 \quad \Longleftrightarrow \quad \mathscr{D}_\mu \mathbf{g}^{\mu\nu} = 0,$$

  where $\mathscr{D}_\mu = \nabla_\mu + \mathbf{\Gamma}_\mu \times$ is the generally and gauge covariant derivative.

- Construct the restricted gravity with $\mathbf{Z}_\mu = 0$. Use the first order formalism.

A) A_2 gravity

- Impose the A_2 isometry and put $\mathbf{Z}_\mu = 0$. Let

$$S[e_a{}^\mu, \Gamma_\mu, \widetilde{\Gamma}_\mu] = \int e \left\{ \mathbf{g}_{\mu\nu} \cdot \hat{\mathbf{R}}^{\mu\nu} + \lambda_\mu \hat{\mathscr{D}}_\nu \mathbf{g}^{\mu\nu} \right\} d^4x$$

$$= \int e \left\{ G_{\mu\nu} (\Gamma^{\mu\nu} + H^{\mu\nu}) - \tilde{G}_{\mu\nu} \widetilde{\Gamma}^{\mu\nu} + \lambda_\mu \hat{\mathscr{D}}_\nu \mathbf{g}^{\mu\nu} \right\} d^4x,$$

$$e = \mathrm{Det}\ (e_{\mu a}), \quad \hat{\mathscr{D}}_\mu = \nabla_\mu + \hat{\Gamma}_\mu \times$$

$$G_{\mu\nu} = e_\mu{}^a e_\nu{}^b\ l_{ab}, \quad \tilde{G}_{\mu\nu} = e_\mu{}^a e_\nu{}^b\ \tilde{l}_{ab},$$

$$\Gamma_{\mu\nu} + H_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad A_\mu = \Gamma_\mu + \widetilde{C}_\mu.$$

$$\widetilde{\Gamma}_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \qquad (B_\mu = \widetilde{\Gamma}_\mu).$$

- Find the Maxwell-type equation of motion of A_2 gravity

$$\nabla_\mu G^{\mu\nu} = 0, \qquad \nabla_\mu \widetilde{G}^{\mu\nu} = 0,$$
$$G_{\mu\nu}(\partial^\nu A^\rho - \partial^\rho A^\nu) - \widetilde{G}_{\mu\nu}(\partial^\nu B^\rho - \partial^\rho B^\nu) = 0,$$
$$\hat{\mathscr{D}}_\mu \mathbf{g}^{\mu\nu} = 0.$$

Notice that $G_{\mu\nu}$ admit "gravitational potential" G_μ

$$G_{\mu\nu} = \nabla_\mu G_\nu - \nabla_\nu G_\mu = \partial_\mu G_\nu - \partial_\nu G_\mu.$$

Compare this with Einstein's equation

$$R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} = 0.$$

B) B_2 gravity

- Impose the B_2 isometry and put $\mathbf{Z}_\mu = 0$. Let

$$S[e_a{}^\mu, \Gamma_\mu, \widetilde{\Gamma}_\mu] = \int e \left\{ \mathbf{g}_{\mu\nu} \cdot \hat{\mathbf{R}}^{\mu\nu} + \lambda_\mu \hat{\mathscr{D}}_\nu \mathbf{g}^{\mu\nu} \right\} d^4x$$

$$= \int e \left\{ \mathcal{J}_{\mu\nu} (\Gamma^{\mu\nu} + H^{\mu\nu}) - \widetilde{\mathcal{J}}_{\mu\nu} (\widetilde{\Gamma}^{\mu\nu} + \widetilde{H}^{\mu\nu}) + \lambda_\mu \hat{\mathscr{D}}_\nu \mathbf{g}^{\mu\nu} \right\} d^4x,$$

$$\mathcal{J}_{\mu\nu} = e_\mu{}^a e_\nu{}^b \; j_{ab}, \quad \widetilde{\mathcal{J}}_{\mu\nu} = e_\mu{}^a e_\nu{}^b \; \tilde{j}_{ab},$$

$$\Gamma_{\mu\nu} + H_{\mu\nu} = \partial_\mu K_\nu - \partial_\nu K_\mu, \qquad K_\mu = \Gamma_\mu + H_\mu,$$

$$\widetilde{\Gamma}_{\mu\nu} + \widetilde{H}_{\mu\nu} = \partial_\mu \widetilde{K}_\nu - \partial_\nu \widetilde{K}_\mu, \qquad \widetilde{K}_\mu = \widetilde{\Gamma}_\mu + \widetilde{H}_\mu.$$

- Find the Maxwell-type equation of motion of B_2 gravity

$$\nabla_\mu \mathcal{J}^{\mu\nu} = 0, \qquad \nabla_\mu \widetilde{\mathcal{J}}^{\mu\nu} = 0,$$
$$\mathcal{J}_{\mu\nu} \left(\partial^\nu K^\rho - \partial^\rho K^\nu\right) - \widetilde{\mathcal{J}}_{\mu\nu} \left(\partial^\nu \widetilde{K}^\rho - \partial^\rho \widetilde{K}^\nu\right) = 0,$$
$$\hat{\mathscr{D}}_\mu \mathbf{g}^{\mu\nu} = 0,$$

where $\mathcal{J}_{\mu\nu}$ admit "gravitational potential" $\mathcal{J}_\mu$

$$\mathcal{J}_{\mu\nu} = \nabla_\mu \mathcal{J}_\nu - \nabla_\nu \mathcal{J}_\mu = \partial_\mu \mathcal{J}_\nu - \partial_\nu \mathcal{J}_\mu.$$

Again compare this with Einstein's equation.

- Notice that

1. Restricted gravity is generally invariant, but simpler than Einstein's gravity.
2. It describes a Maxwell-type Abelian (dual) core dynamics of Einstein's gravity, with massless spin-one graviton.
3. It inherits all topological properties of Einstein's gravity.
4. Restricted gravity and Einstein's gravity have identical vacuum.

Abelian Dominance

Topology of Vacuum Space-time

- How can one obtain the most general vacuum space-time?

  Solving "the vacuum Einstein's equation"

  $$R_{\mu\nu} - \frac{1}{2} R \; g_{\mu\nu} = 0$$

  will not help, because we need the vacuum of quantum gravity (the flat space-time)

  $$R_{\mu\nu\rho}{}^{\sigma} = 0.$$

- Impose the vacuum isometry and construct the most general vacuum connection. Classify the classical vacua using the isometry.

- Let

$$\mathbf{l}_i = \begin{pmatrix} \hat{n}_i \\ 0 \end{pmatrix}, \quad \mathbf{k}_i = \begin{pmatrix} 0 \\ \hat{n}_i \end{pmatrix} = -\tilde{\mathbf{l}}_i,$$

$$\hat{n}_1 \times \hat{n}_2 = \hat{n}_3, \qquad (i = 1, 2, 3)$$

and impose the vacuum isometry (the maximal isometry)

$$^{\forall_i}\ D_\mu \mathbf{l}_i = 0, \qquad ^{\forall_i}\ D_\mu \mathbf{k}_i = 0.$$

Notice that

$$D_\mu \mathbf{l}_i = 0, \qquad \Longleftrightarrow \qquad D_\mu \mathbf{k}_i = 0.$$

- Let

$$\mathbf{p} = \begin{pmatrix} \vec{m} \\ \vec{e} \end{pmatrix}, \quad \mathbf{\Gamma}_\mu = \begin{pmatrix} \vec{A}_\mu \\ \vec{B}_\mu \end{pmatrix},$$

and find in 3-d notation $D_\mu \mathbf{p} = 0$ is written as

$$D_\mu \vec{m} = \vec{B}_\mu \times \vec{e}, \quad D_\mu \vec{e} = -\vec{B}_\mu \times \vec{m}.$$

- So the vacuum isometry ${}^{\forall_i}\ D_\mu \mathbf{l}_i = 0$ (and ${}^{\forall_i}\ D_\mu \mathbf{k}_i = 0$) is written as

$$^{\forall_i}\ D_\mu \hat{n}_i = \vec{B}_\mu \times \hat{n}_i, \quad D_\mu \hat{n}_i = -\vec{B}_\mu \times \hat{n}_i,$$

or equivalently

$$^{\forall_i}\ D_\mu \hat{n}_i = 0, \quad \vec{B}_\mu = 0\ !$$

- Obtain the most general vacuum connection

$$\mathbf{\Gamma}_\mu \to \mathbf{\Omega}_\mu = \begin{pmatrix} \hat{\Omega}_\mu \\ 0 \end{pmatrix}$$

$$\hat{\Omega}_\mu = -\frac{1}{2}\epsilon_{ijk}(\hat{n}_i \cdot \partial_\mu \hat{n}_j)\hat{n}_k.$$

This tells that the flat space-time has $\Pi_3(S^2)$ topology of the SU(2) QCD vacuum.

- This is nothing but the topology of $\Pi_3(SO(3,1)) \simeq \Pi_3(SO(3))$.

Knot Topology of Vacuum Space-time

Physical Interpretation

- Consider a flat R^4 and introduce a global Cartesian coordinate basis ∂_μ $(\mu = 0, 1, 2, 3)$. Choose the Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$, and let ∂_μ are parallel to each other (i.e., let $\Gamma_{\mu\nu}{}^{\alpha} = 0$),

$$\nabla_\mu \partial_\nu = \Gamma_{\mu\nu}{}^{\alpha}\ \partial_\alpha = 0.$$

- Find the trivial connection $\Gamma_{\mu\nu}{}^{\alpha} = 0$ is metric compatible and torsionless,

$$\nabla_\alpha \eta_{\mu\nu} = 0,$$
$$C_{\mu\nu}{}^{\alpha} = \Gamma_{\mu\nu}{}^{\alpha} - \Gamma^{(0)\alpha}_{\mu\nu} = 0,$$

where $C_{\mu\nu}{}^{\alpha}$ and $\Gamma^{(0)\alpha}_{\mu\nu}$ are the contortion and the Levi-Civita connection.

- Introduce a local orthonormal frame (i.e., tetrad) e_a $(a = 0, 1, 2, 3)$

$$e_0 = e_0{}^\alpha\ \partial_\alpha = \partial_0 \quad (e_0{}^\alpha = \delta_0{}^\alpha),$$
$$e_i = e_i{}^\alpha\ \partial_\alpha = \hat{n}_i{}^\alpha\ \partial_\alpha \quad (e_i{}^\alpha = \hat{n}_i{}^\alpha \text{ with } \hat{n}_i{}^0 = 0), \quad (i = 1, 2, 3).$$

Express the trivial connection $\Gamma_{\mu\nu}{}^\alpha = 0$ in the orthonormal basis. Find the corresponding $\Gamma_\mu{}^{ab}$ becomes the vacuum connection,

$$\Gamma_\mu{}^{ab} = -\frac{\eta_{\alpha\beta}}{2}\left(e^{a\alpha}\partial_\mu e^{b\beta} - e^{b\alpha}\partial_\mu e^{a\beta}\right) = \mathbf{\Omega}_\mu{}^{ab},$$
$$\Gamma_\mu{}^{ij} = \frac{1}{2}\hat{n}^i \cdot \partial_\mu \hat{n}^j, \quad \Gamma_\mu{}^{0i} = 0,$$
$$\Rightarrow \mathbf{\Gamma}_\mu = \mathbf{\Omega}_\mu$$

- So the flat connection $\Gamma_{\mu\nu}{}^{\alpha} = 0$, in the orthonormal basis, becomes identical to the $SU(2)$ vaccum potential. This confirms that the torsionless Minkowski space-time with flat connection has a non-trivial $\Pi_3(S^2)$ topology.

- It is the tetrad (i.e., the spin structure), not the metric, which describes the knot topology of the vacuum space-time.

Knot is everywhere!

1. Non-linear sigma model (Faddeev and Niemi, Nature 1998)
2. Plasma (Faddeev and Niemi, PRL 1999)
3. Skyrme theory (Cho, PRL 2002)
4. Condensed matter
   Two-component BEC (Cho, PRA 2003)
   Two-gap SC (Babaev, PRL 2003; Cho, PRB 2004)
5. QCD
   Knot glueball (Cho, PLB 2005)
   QCD vacuum (Cho, PLB 2006)
6. Einstein's theory
   Vacuum space-time
   Knot in gravity?

- Space-time tunneling: Gravito-instantons are proposed, but never confirmed. With the tunneling, we can define "the θ-vacuum" in Einstein's theory.

- The restricted gravity could be very useful in describing the space-time of gravito-magnetic monopole.

  1. $\Pi_2(S^2)$ topology
  2. Energy quantization (cf. charge quantization)

- Reactivate the valence connection $\mathbf{Z}_\mu$ in the restricted gravity to recover the full Einstein's theory.
- Find that Einstein's gravity is nothing but the restricted gravity which has the valence connection as a gauge covariant gravitational source.
- Conclude that the restricted gravity describes the skeleton structure and the core dynamics of Einstein's theory. Establish the Abelian dominance in Einstein's theory.

Discussion

- Anatomy of Einstein's theory: Dissect and decompose it to the skeleton and the flesh. Find that the flesh (the valence connection) can not move (has no dynamical role).

- The skeleton can dance, and describes a restricted gravity which is much simpler than Einstein's gravity but has the full general invariance. Moreover it becomes Abelian.

$$g_{\mu\nu} \rightarrow G_\mu$$

$$R_{\mu\nu} - \frac{1}{2} R \; g_{\mu\nu} = 0 \Rightarrow \left(\begin{array}{c} \nabla_\mu G^{\mu\nu} = 0 \\ G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu \end{array} \right)$$

Massless spin-one graviton!

- This establishes the Abelian dominance (of a different type) in Einstein's theory.
- A_2 gravity describes Bonner and C metric, and B_2 gravity describes Einstein-Rosen-Bondi's plane wave solution.
- Knot topology of vacuum space-time and quantum tunneling: $\Pi_3(S^2)$ topology of the tetrad (spin structure)! Gravito-instantons and θ-vacuum in quantum gravity?
- Challenge: Quantize the massless spin-one graviton.

- References

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4. Y.M. Cho, S. H. Oh, and S.W. Kim, gr-qc/1102.3490 (2011).
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