Quantum Stress Tensor Fluctuations and Cosmology - Part 2

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Some features of stress tensor fluctuations:

Subtle correlations and anticorrelations

Non-Gaussian probability distribution

Larger on smaller scales

Effects in inflationary cosmology: density fluctuations and gravity waves

Stress tensor fluctuations during inflation lead to expansion fluctuations during and after inflation. These in turn lead to density fluctuations.

These fluctuations are in addition to the usual, nearly scale invariant and Gaussian fluctuations coming from quantum fluctuations of the inflaton field.

Inflationary expansion followed by reheating and a radiation dominated universe

$$a(\eta) = \frac{1}{1 - H\eta}, \qquad \eta_0 < \eta < 0, \quad \eta_0 = \text{conformal time when inflation begins}$$

$$a(\eta) = 1 + H\,\eta, \qquad \eta > 0,$$

$t_R =$ reheating time in comoving time

$$a(t) = \mathrm{e}^{H(t - t_R)}, \quad t \le t_R\,,$$

$$a(t) = \sqrt{1 + 2H(t - t_R)}, \qquad t \ge t_R.$$

Density Fluctuations - A Kinematic Model

K.W. Ng, C.H. Wu &LF, PRD 75 103502 (2007)

Basic idea: Use the Raychuadhuri equation to find expansion fluctuations along geodesics, assuming Ricci tensor flucutuations driven by a conformal field, such as the EM field

$$\frac{d\theta}{d\lambda} = -R_{\mu\nu}u^\mu u^\nu - \frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu}$$

$$R_{\mu\nu} = 8\pi(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)$$

Assume $\sigma_{\mu\nu} = \omega_{\mu\nu} = 0$, so that

$$\frac{d\theta}{d\lambda} = -R_{\mu\nu}u^{\mu}u^{\nu} - \frac{1}{3}\theta^2$$

Let $\theta = \theta_0 + \theta_1$, where $\theta_0 = 3\dot{a}/a$, and

$$\frac{d\theta_1}{dt} = -\left(R_{\mu\nu}u^{\mu}u^{\nu}\right)_q - \frac{2}{3}\theta_0\theta_1$$

$$\Rightarrow \quad \theta_1(t) = -a^{-2}(t)\int_{t_0}^{t} dt'\, a^2(t')\,\left(R_{\mu\nu}u^{\mu}u^{\nu}\right)_q$$

Initial condition: $\theta_1 = 0$ at the beginning of inflation

$$\langle\theta(\eta_1)\,\theta(\eta_2)\rangle - \langle\theta(\eta_1)\rangle\langle\theta(\eta_2)\rangle =$$
$$a^{-2}(\eta_1)\,a^{-2}(\eta_2)\int_{\eta_0}^{\eta_1} d\eta\,a^{-1}(\eta)\int_{\eta_0}^{\eta_2} d\eta'\,a^{-1}(\eta')\,\mathcal{E}(\Delta\eta, r)$$

$\mathcal{E}(\Delta\eta, r)$ = flat space energy density correlation function

Conformally invariant fields:

$$C^{RW}_{\mu\nu\alpha\beta}(x,x') = a^{-4}(\eta)\,a^{-4}(\eta')\,C^{flat}_{\mu\nu\alpha\beta}(x,x')$$

Stress tensor correlation function

$$C_{\mu\nu\alpha\beta}(x,x') = \langle T_{\mu\nu}(x)\,T_{\alpha\beta}(x')\rangle - \langle T_{\mu\nu}(x)\rangle\langle T_{\alpha\beta}(x')\rangle$$

(conformal anomaly cancels)

Conservation law for a perfect fluid:

$$\dot{\rho} + \theta(\rho + p) = 0$$

Effect of expansion fluctuations on redshifting after reheating

Let $p = w\rho$ and integrate the conservation law to find the density fluctuations:

$$\left\langle \left(\frac{\delta\rho}{\rho} \right)^2 \right\rangle = (1+w)^2 \int_0^{\eta_s} d\eta_1 a(\eta_1) \int_0^{\eta_s} d\eta_2 a(\eta_2)$$
$$(\langle \theta(\eta_1)\, \theta(\eta_2) \rangle - \langle \theta(\eta_1) \rangle \langle \theta(\eta_2) \rangle)$$

$\eta_s = \text{conformal time of last scattering}$

Density correlation function:

$$C_\rho(x_1, x_2) = \left\langle \frac{\delta\rho(x_1)}{\rho} \, \frac{\delta\rho(x_2)}{\rho} \right\rangle$$

$$= (8\,\pi)^2\, \ell_p^4\, (1+w)^2 \int_0^{\eta_s} \frac{d\eta_1}{a(\eta_1)} \int_0^{\eta_s} \frac{d\eta_2}{a(\eta_2)} \int_{\eta_0}^{\eta_1} \frac{d\eta}{a(\eta)} \int_{\eta_0}^{\eta_2} \frac{d\eta'}{a(\eta')}\, \mathcal{E}(\Delta\eta, r)$$

$\ell_p =$ Planck length

$\eta_s =$ conformal time of last scattering

(Non-Gaussian fluctuations)

Power spectrum of the density fluctuations:

$$\mathcal{P}(k) = \frac{k^3}{2\pi^2} \int d^3r\, \mathrm{e}^{-ikr}\, C_\rho(r, \eta_s)$$

Result:

$$\mathcal{P}(k) \approx \frac{16\ell_p^4 k^4}{405\,\pi}\left(-|k\eta_0|^3 + \frac{3}{\pi}|k\eta_0|^2 + \cdots\right)$$

Grows as the duration of inflation increases.

Interpret as due to non-cancellation of anti-correlated θ fluctuations.

Not scale invariant - more power on shorter scales.

II. A Dynamical Model for Density Fluctuations

S.P. Miao, K.W. Ng, R.P. Woodard, C.H. Wu &LF, PRD 82 043501 (2010)

Consider a single scalar inflaton field φ and a fluctuating conformal field stress tensor.

Basic idea:

Solve the coupled Einstein-φ equations to find $\Delta\varphi$, the fluctuation in φ induced by the stress tensor fluctuations.

$$\Box\varphi + V'(\varphi) = 0$$

$$G_{\mu\nu} = 8\pi T^{\rm total}_{\mu\nu} = 8\pi(T^{\rm infl}_{\mu\nu} + T^{\rm conf}_{\mu\nu})$$

First order perturbations:

Inflaton: $\varphi(t,\mathbf{x}) = \varphi_0(t) + \delta\varphi(t,\mathbf{x})$

Metric:

$$ds^2 = -dt^2 + 2a(t)h_{0i}(t,\mathbf{x})dtdx^i + a^2(t)\Big[\delta_{ij} + h_{ij}(t,\mathbf{x})\Big]dx^i dx^j$$

Unit normal to surfaces of constant φ

$$u^\mu = -\frac{g^{\mu\nu}\partial_\nu\varphi}{\sqrt{-g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi}} = \delta^\mu_0$$

Obtain a pair of 2nd order coupled equations for $\delta\varphi$ and $h_{\mu\nu}$

Eliminate $h_{\mu\nu}$ and obtain an equation for

$B(t,k) = \delta\varphi(t,k)/\dot{\varphi}_0$

(spatial Fourier transform)

$$\left(\partial_t^2 + 5H\,\partial_t + 6H^2 + \frac{k^2}{a^2}\right)\dot{B}(t,k) = 8\pi\,T_{tt}^{\rm conf}$$

(slow roll approximation)

Note: the expansion does not appear explicitly in this model, but is still here because $\theta_1 = \delta^{ij}\dot{h}_{ij}/2$.

Set $B = 0$ at $\eta = \eta_0$

Result for the momentum space correlation function:

$$\langle B^2 \rangle_k$$

$$= (8\,\pi)^2 \,\frac{\ell_p^4}{k^2} \int_{\eta_0}^{0} \frac{d\eta_1}{a(\eta_1)} \int_{\eta_0}^{0} \frac{d\eta_2}{a(\eta_2)} \int_{\eta_0}^{\eta_1} d\eta\, \sin[k(\eta - \eta_1)] \int_{\eta_0}^{\eta_2} d\eta'\, \sin[k(\eta' - \eta_2)]\, \mathcal{E}(\Delta\eta, k)$$

$$\sim -\frac{\ell_p^4\, H^2\, |\eta_0|^3}{480\pi^2} + \frac{\ell_p^4\, H^2\, \eta_0^2}{600\pi\, k}$$

Can calculate $\mathcal{P}_k$ from $\langle B^2 \rangle_k$ using either

the conservation law

differential redshift (Sachs-Wolfe formula)

Associated power spectrum of density perturbations:

$$\mathcal{P}_k \approx \frac{2\,\pi\,\ell_p^4 H^4}{15}\left(-|k\eta_0|^3 + \frac{4\pi}{5}|k\eta_0|^2 + \cdots\right)$$

leads to an unobservable delta function term in the spatial correlation function, so ignore this term

Power spectrum is not scale invariant and depends upon the duration of inflation.

$$\mathcal{P}_k \approx \frac{8\pi\, \ell_p^4\, H^4}{75}\, k^2\, \eta_0^2$$

It must be a small overall contribution to the total power spectrum, which leads to a constraint on the scale factor change during inflation:

$$\frac{1}{a_0} < 10^{42} \left(\frac{10^{12}\ \mathrm{GeV}}{T_R} \right)^3$$

Allows enough inflation to solve the horizon and flatness problems

Does the η_0^2 dependence contradict Weinberg's result that radiative effects in inflation should grow no faster than ln(a)?

One interpretation of our result is that it is not so much growing, as always large - the contribution of very energetic modes is preserved.

Use of transplanckian modes.

Dominant contribution comes from modes with

$$\lambda \approx 10^{-8}\,\ell_p \quad (T_R = 10^{12}\,\mathrm{GeV})$$

Is this a problem?

This effect might offer a probe of transplanckian physics in the form of a small non-Gaussian, non-scale invaraint component in the CMB.

Gravity Waves from Sress Tensor Fluctuations in Inflation

C-H Wu, J-T Hsiang, K-W Ng, &LF, PRD 84 103515 (2011)

Write $$g_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu}$$

background metric

metric perturbation - tensor modes

Impose the transverse tracefree (TT) gauge:

$$h^{\mu\nu}{}_{;\nu} = 0 \qquad h = h^{\mu}_{\mu} = 0 \qquad h_{\mu\nu}\, u^{\mu} = 0$$

covariant derivative on the background

a timelike vector; here the comoving obverver 4-velocity

Result: tensor modes in a spatially flat universe behave as massless scalars

Lifshitz 1946

$$\Box_S \, h^\mu_\nu = 0$$

scalar wave operator

Consequences:

classically stability

gravitons are equivalent to a pair of massless scalar fields

$$\Box_S \, \varphi = \varphi^{;\nu}{}_{;\nu} = 0$$

Generation of gravity waves by a source:

$$\Box_S h_\mu{}^\nu = -16\pi\, G\, S_\mu{}^\nu$$

$S_\mu{}^\nu$ = the transverse-tracefree part of the source stress tensor

(most easily defined by the use of a projection operator in momentum space)

Semiclassical theory: gravity couples to the renormalized expectation value of a quantum matter field

(effects of stress tensor fluctuations ignored)

Effects on gravity wave modes during inflation from a conformal field (e.g., EM)

[J.T. Hsiang, D.S. Lee, H.L. Yu &LF, PRF 82, 084027 (2011)]

A modification to gravity wave modes proportional to the expansion factor S during inflation:

$$h_\mu{}^\nu \to h_\mu{}^\nu + h'_\mu{}^{\ \nu} \qquad\qquad h'_\mu{}^{\ \nu} \propto S$$

Generation of gravity waves by a fluctuating source:

Integrate $\Box_S h_\mu{}^\nu = -16\pi\, G\, S_\mu{}^\nu$

using a retarded Green's function

$$\Box_S G_R(x,x') = -\frac{\delta(x-x')}{\sqrt{-\gamma}}$$

and form the metric correlation function

$$K_\mu{}^\nu{}_\rho{}^\sigma(x,x') = \langle h_\mu{}^\nu(x)\; h_\rho{}^\sigma(x')\rangle - \langle h_\mu{}^\nu(x)\rangle\langle h_\rho{}^\sigma(x')\rangle$$

in terms of a stress tensor correlation function

$$C_\mu{}^\nu{}_\rho{}^\sigma(x,x') = \langle S_\mu{}^\nu(x)\; S_\rho{}^\sigma(x')\rangle - \langle S_\mu{}^\nu(x)\rangle\langle S_\rho{}^\sigma(x')\rangle$$

Conformally invariant fields:

$$C^{RW}_{\mu\nu\alpha\beta}(x,x') = a^{-4}(\eta)\, a^{-4}(\eta')\, C^{flat}_{\mu\nu\alpha\beta}(x,x')$$

(conformal anomaly cancels)

Take spatial Fourier transforms:

$$\hat{A}(\eta,\mathbf{k}) \equiv \frac{1}{(2\pi)^3}\int d^3x\, e^{i\mathbf{k}\cdot\mathbf{x}} A(\eta,\mathbf{x})$$

and take $\mathbf{k}$ to be in the z-direction.

Need only x & y components of $\hat{C}_{\mu}{}^{\nu}{}_{\rho}{}^{\sigma}(\eta_1,\eta_2,k)$

Power spectrum as a Fourier transform of a correlation function:

$$P(k) = \frac{1}{(2\pi)^3} \int d^3u \, \mathrm{e}^{i\,\mathbf{k}\cdot\mathbf{u}} \, C(\eta = \eta', \mathbf{u})$$

equal time
correlation function

Note: "power spectrum" in cosmology usually refers to

$$\mathcal{P}(k) = 4\pi k^3 \, P(k)$$

Here $P(k)$ is a spatial component of $\hat{K}_{\mu}{}^{\nu}{}_{\rho}{}^{\sigma}(\eta, \eta, k)$

First model: assume that the gravity wave fluctuations vanish at some initial time (the beginning of inflation) and then integrate forward in time to the end of inflation at

Power spectrum:

$$P_s(k) = 64(2\pi)^8 \int_{\eta_0}^{\eta_r} d\eta_1 \int_{\eta_0}^{\eta_r} d\eta_2\, \hat{G}(\eta, \eta_1, k)\, \hat{G}(\eta, \eta_2, k)\, \hat{C}_{flat}(\eta_1 - \eta_2, k)$$

Result:

$$P_s(k) = -\frac{H^2\, S^2}{3\pi^2\, k}(1 + k^2\, H^{-2})$$

S = expansion factor during inflation

Three remarkable features:

1) Negative power

2) Grows as the duration of inflation increases

$$\propto S^2$$

3) Highly blue tilted

$$P(k) \propto k$$

Second model: assume that the coupling to the fluctuating stress tensor is switch on gradually with a switching function $e^{p\eta}$

Now 1/p is the approximate conformal time at which the interaction begins.

Result for the power spectrum:

$$P_e(k) = -\frac{H^3(1+k^2/H^2)\,S}{8\pi^2\,k^2}$$

Same three remarkable features:

1) Negative power

2) Grows as the duration of inflation increases

$$\propto S$$

3) Highly blue tilted

$$P(k) \propto k^0$$

An aside on negative power spectra:

Usually, the Wiener-Khinchine theorem requires a non-negative spectrum.

However, for quadratic quantum operators, such as a stress tensor, the positive definite quantity in this theorem does not exist.

This allows allow negative spectra.

[J.T. Hsiang, C.H. Wu &LF, Phys. Lett. A 375, 2296 (2011)]

Example: the flat space EM energy density

$$\mathcal{E}(\Delta\eta, k) = -\frac{k^4 \sin(k\Delta\eta)}{960\pi^5\,\Delta\eta} \qquad \mathcal{E}(0, k) = -\frac{k^5}{960\pi^5}$$

Interpret a negative power spectrum as having the opposite correlation/anticorrelation behavior as a positive spectrum.

$$P(k) \to -P(k) \Rightarrow C(r) \to -C(r)$$

Normally, $C(0) > 0$ but this quantity is not defined when the power spectrum is negative.

Explore the exponentially switched model:

$$P_e(k) = -\frac{H^3(1+k^2/H^2)\,S}{8\pi^2\,k^2}$$

Corresponding spatial correlation function:

$$C_e(r) = -\frac{H^3\,S}{4r}$$

(A delta function term has been dropped.)

Assume efficient reheating at the end of inflation to an energy of E_R

Consider modes of the order of the horizon size today:

If $h > 10^{-5}$ $(C_e > 10^{-10})$

the tensor perturbations would have been detected.

This leads to the constraint:

$$S < 10^{40} \left(\frac{10^{16} GeV}{E_R} \right)^7$$

Allows enough inflation to solve the horizon and flatness problems

Can we take the power spectrum seriously for much smaller wavelengths?

At scales of the order of 100km, LIGO has set limits of $h < 10^{-24}$

This implies

$$S < 10^{25} \left(\frac{10^{13} GeV}{E_R} \right)^7$$

and

$$E_R < 10^{13} GeV$$

Dominant contribution comes from modes wavelengths far less than the Planck length at the beginning of inflation.

Analogous to the use of transplanckian modes in Hawking's derivation of black hole evaporation.

Alternative is to give up local Lorentz symmetry, e.g., postulate nonlinear dispersion relations (Jacobson, Unruh)

Ideally, the same physics should govern both cosmology and black holes.

This effect might offer a probe of transplanckian physics in the form of a non-Gaussian, non-scale invariant spectrum of gravity waves.

Summary

1) Density perturbations and gravity waves can be generated by quantum stress tensor fluctuations during inflation.

2) The resulting spectra are non-Gaussian and non-scale invariant.

3) These spectra seem to imply a limit on the duration of inflation.

4) These effects may offer a probe of transplanckian physics.