A general maximum entropy principle for self-gravitating perfect fluid

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Ref. Sijie Gao, Phys.Rev.D 84, 104023 (2011);
Sijie Gao,Phys. Rev. D 85, 027503 (2012)

1. Introduction

General Relativity $\Rightarrow$ Black hole mechanics (Bekenstein, Bardeen,1973)

Hawking radiation (1974) $\Rightarrow$ Black hole thermodynamics

thermodynamics $\xrightarrow{?}$ General Relativity

Ted Jacobson (1995) assumed the first law $\delta Q = TdS$ holds for

local Rindler horizons. Then the Einstein equation can be derived.

The Einstein equation was also derived from thermodynamical laws of black hole horizons:

Y.Gong, A.Wang, Phys.Rev.Lett. **99**, 211301 (2007)

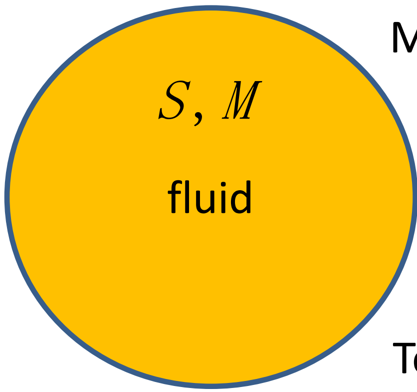
R.G. Cai and S.P. Kim, J.High Energy Phys. 02(2005)050.

C. Eling, R. Guedens, and T. Jacobson, Phys. Rev. Lett. **96**, 121301 (2006).

M.Akbar and R.G. Cai, Phys. Lett. B **635**, 7(2006).

R.G. Cai, and L.M.Cao, Phys. Rev. D **75**, 064008 (2007)

In 1965, W.J.Cocke (Ann. Inst. Henri Poincare, **2**, 283) proposed a maximum entropy principle for self-gravitating fluid.



S: total entropy of fluid
M: total mass of fluid

$$\boxed{\delta S = 0}$$

Tolman-Oppenheimer-Volkoff (TOV) equation:

$$\frac{d}{dr}(\rho/3) = -\frac{(\rho + \rho/3)[m(r) + 4\pi r^3(\rho/3)]}{r[r - 2m(r)]}$$

2. Maximum entropy principle for radiation

Sorkin, Wald, Zhang, Gen.Rel.Grav. 13, 1127 (1981)

In 1981, Sorkin, Wald, and Zhang (SWZ) derived the TOV equation of a self-gravitating radiation from the maximum entropy principle.

Consider a box of radiation (photon gas) confined within radius R . The stress-energy tensor is given by

$$T_{ab} = \rho u_a u_b + \frac{1}{3}\rho(g_{ab} + u_a u_b)$$

The radiation satisfies:

$$\rho = bT^4, \quad p = \frac{1}{3}\rho,$$

$$s = \alpha\rho^{3/4}$$

Assume the metric of the radiation takes the form

$$ds^2 = g_{tt}(r)dt^2 + \left[1 - \frac{2m(r)}{r}\right]^{-1} dr^2 + r^2 d\Omega^2$$

The constraint Einstein equation $G_{00} = 8\pi T_{00}$ yields

$$\rho = \frac{m'(r)}{4\pi r^2}$$

The total mass within R is $M = m(R)$.

The total entropy of the radiation is

$$
\begin{aligned}
S &= 4\pi \int_0^R s(r) \left[1 - \frac{2m(r)}{r}\right]^{-1/2} r^2 dr \\
&= 4\pi\alpha \int_0^R \rho^{3/4} \left[1 - \frac{2m(r)}{r}\right]^{-1/2} r^2 dr \\
&= (4\pi)^{1/4}\alpha \int_0^R \left[\frac{1}{r^2} m'(r)\right]^{3/4} \left[1 - \frac{2m(r)}{r}\right]^{-1/2} r^2 dr \, .
\end{aligned}
$$

$$
\text{Let } L = L(m, m') = \left[\frac{1}{r^2} m'(r)\right]^{3/4} \left[1 - \frac{2m(r)}{r}\right]^{-1/2} r^2
$$

Our purpose is to find the function $m(r)$ such that $\delta S = 0$ for fixed M.

Since $\delta m(0) = \delta m(R) = 0$, the extrema of S is equivalent to the Euler-Lagrange equation:

$$
\frac{d}{dr}\left(\frac{\partial L}{\partial m'}\right) - \frac{\partial L}{\partial m} = 0
$$

Straightforward calculation gives

$$-\frac{3}{16}m''r^2 + \frac{3}{8}m''mr + \frac{3}{8}m'r - \frac{1}{4}m'^2 r - \frac{3}{2}m'm = 0\,.$$

Using $\rho = \frac{m'(r)}{4\pi r^2}$ to replace m' , m'' , we arrive at the TOV equation

$$\frac{d}{dr}\,(\rho/3) = -\frac{(\rho + \rho/3)[m(r) + 4\pi r^3(\rho/3)]}{r[r - 2m(r)]}\,.$$

3. Maximum entropy principle for a general fluid (Sijie Gao, arXiv:1109.2804)

- To generalize SWZ's treatment to a general fluid, we first need to find an expression for the entropy density s .
- The first law of the ordinary thermodynamics:

$$dS = \frac{1}{T}dE + \frac{p}{T}dV - \frac{\mu}{T}dN$$

Rewrite in terms of densities:

$$d(sV) = \frac{1}{T}d(\rho V) + \frac{p}{T}dV - \frac{\mu}{T}d(nV)$$

Expand: $$sdV + Vds = \frac{1}{T}\rho dV + Vd\rho + \frac{p}{T}dV - \frac{\mu}{T}ndV - \frac{\mu}{T}Vdn$$

The first law in a unit volume:

$$ds = \frac{1}{T}d\rho - \frac{\mu}{T}dn$$

Thus, we have the Gibbs-Duhem relation

$$s = \frac{1}{T}(\rho + p - \mu n) \qquad (20)$$

Choose (ρ, n) as independent thermodynamic variables. Assume

$$s = s(\rho, n), \;\; \mu = \mu(\rho, n), \quad p = p(\rho, n)$$

For example, the thermodynamic quantities for a monatomic ideal gas are

$$\begin{aligned}
\rho &= \frac{3}{2} nkT\,, \\
p &= nkT\,, \\
s &= \frac{3}{2} nk \ln T - nk \ln n + \frac{3}{2} nk \left[\frac{5}{3} + \ln\left(\frac{2\pi mk}{h^2}\right)\right].
\end{aligned}$$

Our task is to find functions $m(r)$ and $n(r)$ such that the total entropy

$$S = 4\pi \int_0^R s(r) \left[1 - \frac{2m(r)}{r}\right]^{-1/2} r^2 dr$$

is an extrema.

In addition to the constraint $\delta m(0) = \delta m(R) = 0$, it is also natural to require the total number of particles

$$N = 4\pi \int_0^R n(r) \left[1 - \frac{2m(r)}{r}\right]^{-1/2} r^2 dr$$

to be invariant, i.e., $\delta N = 0\,.$

Following the standard method of Lagrange multipliers, the equation of variation becomes

$$\delta S + \lambda \delta N = 0\,.$$

Define the "total Lagrangian" by

$$L(m, m', n) = s(\rho(m'), n)\left[1 - \frac{2m(r)}{r}\right]^{-1/2} r^2 + \lambda n(r)\left[1 - \frac{2m(r)}{r}\right]^{-1/2} r^2 .$$

Now the constrained Euler-Lagrange equation is given by

$$\frac{\partial L}{\partial n} = 0\,, \tag{30}$$

$$\frac{d}{dr}\frac{\partial L}{\partial m'} + \frac{\partial L}{\partial m} = 0\,. \tag{31}$$

Thus, Eq. (30) yields

$$\frac{\partial s}{\partial n} + \lambda = 0\,.$$

Using $ds = \frac{1}{T}d\rho - \frac{\mu}{T}dn$, we have

$$-\frac{\mu}{T} + \lambda = 0\,, \tag{33}$$

Note that $\frac{\partial L}{\partial m'} = \frac{\partial s}{\partial m'} r^2 \left(1 - \frac{2m}{r}\right)^{-1/2}, \quad \frac{\partial s}{\partial m'} = \frac{\partial s}{\partial \rho}\frac{\partial \rho}{\partial m'} = \frac{1}{T}\frac{1}{4\pi r^2}$

Thus, $\frac{d}{dr}\frac{\partial L}{\partial m'} = \frac{T(m'r - m) - r(r - 2m)T'}{4\pi T^2 (r - 2m)^{3/2} r^2}$

Using Eq. (34), we have

$$\begin{aligned}
\frac{\partial L}{\partial m} &= r\left(1-\frac{2m}{r}\right)^{-3/2}(n\lambda+s) \\
&= r\left(1-\frac{2m}{r}\right)^{-3/2}(n\frac{\mu}{T}+s) \\
&= r\left(1-\frac{2m}{r}\right)^{-3/2}(\frac{\rho+p}{T})
\end{aligned}$$

So the Eular-Lagrange Eq. (31) yields

$$(4\pi p r^3+m)T+(r-2m)rT'=0\,. \tag{40}$$

The constraint Eq. (33) yields

$$\mu'=\lambda T'\,. \tag{41}$$

Rewrite Eq. (20) as

$$p=Ts+\mu n-\rho\,. \tag{42}$$

The differential of p is

$$dp = Tds + sdT + \mu dn + nd\mu - d\rho\,. \tag{43}$$

By substituting Eq. (19), we have

$$dp = sdT + nd\mu\,. \tag{44}$$

It follows immediately that

$$p'(r) = sT'(r) + n\mu'(r)\,. \tag{45}$$

Substituting Eqs. (33), (20) and (41) into Eq. (45), we have

$$T' = \frac{T}{p+\rho}p'(r)\,. \tag{46}$$

Substituting Eq. (46) into Eq. (40), we obtain the desired TOV equation

$$p' = -\frac{(p+\rho)(4\pi r^3 p + m)}{r(r-2m)}\,. \tag{47}$$

4. Maximum entropy principle for charged fluid

In coordinates (t, r, θ, ϕ), assume that a spherically symmetric charged fluid has the line element

$$ds^2 = g_{tt}(r)dt^2 + \left[1 - \frac{2m(r)}{r} + \frac{Q^2(r)}{r^2}\right]^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \,. \quad (53)$$

The stress tensor is $$T_{ab} = \tilde{T}_{ab} + T^{EM}_{ab} \,,$$

where $$\tilde{T}_{ab} = \rho u^a u^b + p(g_{ab} + u_a u_b) \,,$$

$$u^a = \frac{1}{\sqrt{-g_{tt}}} \left(\frac{\partial}{\partial t}\right)^a$$

$$T^{EM}_{ab} = \frac{1}{4\pi} \left(F_a{}^c F_{bc} - \frac{1}{4} g_{ab} F^{cd} F_{cd} \right) .$$

Maxwell's equations: $$\nabla_b F^{ab} = 4\pi j^a = 4\pi \rho_e u^a \,,$$
$$\nabla_{[a} F_{bc]} = 0 \,,$$

The solution is $$F^{tr} = \frac{1}{r^2\sqrt{-g_{tt}g_{rr}}}Q(r)$$

and $$Q(r) = \int_0^r 4\pi r'^2 \sqrt{g_{rr}}\rho_e dr' \,. \qquad (57)$$

Then the time-time component of the Einstein's equation gives

$$m'(r) = 4\pi r^2 \rho + \frac{QQ'}{r} \,. \qquad (59)$$

The total entropy of matter takes the form

$$S = \int_0^R s(r)\left[1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right]^{-1/2} r^2 dr \,.$$

For simplicity, we assume all the particles have the same charge q. Thus, the charge density is proportional to the particle number density n

$$\rho_e = qn \,.$$

Together with Eq. (62), we have

$$n = \frac{Q'}{4\pi r^2 q}\left[1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right]^{1/2} \,. \qquad (62)$$

So the Lagrangian is written as

$$L(m, m', Q, Q') = s\left[1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right]^{-1/2} r^2 \,.$$

The conservation of particle number N is equivalent to the conservation of charge with the radius R. Now the constraints are

$$m(0) = Q(0) = 0, \quad m(R) = constant, \quad Q(R) = constant \,.$$

With these constraints, the extrema of S leads to the following Euler-Lagrange equations

$$\frac{d}{dr}\frac{\partial L}{\partial Q'} + \frac{\partial L}{\partial Q} = 0 \qquad (65)$$

$$\frac{d}{dr}\frac{\partial L}{\partial m'} + \frac{\partial L}{\partial m} = 0 \qquad (66)$$

Note that $s = s(\rho, n) = s(\rho(m', Q, Q'), n(Q, m, Q'))\,.$

$$\begin{aligned}\frac{\partial s}{\partial Q'} &= \frac{\partial s}{\partial \rho}\frac{\partial \rho}{\partial Q'} + \frac{\partial s}{\partial n}\frac{\partial n}{\partial Q'} \\ &= -\frac{1}{T}\frac{Q}{4\pi r^3} - \frac{\mu}{T}\frac{1}{q}\frac{1}{4\pi r^2}\left[1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right]^{1/2} \,.\end{aligned}$$

Thus,

$$\frac{\partial L}{\partial Q'} = -\frac{1}{T}\frac{Q}{4\pi r}\left[1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right]^{-1/2} - \frac{\mu}{T}\frac{1}{q}\frac{1}{4\pi}\,. \tag{70}$$

To calculate $\frac{\partial L}{\partial Q}$, first note that

$$\begin{aligned}\frac{\partial s}{\partial Q} &= \frac{\partial s}{\partial \rho}\frac{\partial \rho}{\partial Q} + \frac{\partial s}{\partial n}\frac{\partial n}{\partial Q} \\ &= -\frac{1}{T}\frac{Q'}{4\pi r^3} - \frac{\mu}{T}\frac{QQ'}{4\pi q r^4}\left[1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right]^{-1/2}\,. \end{aligned} \tag{71}$$

Then

$$\frac{\partial L}{\partial Q} = -\frac{4\pi r^2 qQsT + (fqr + \sqrt{f}Q\mu)Q'}{4\pi r^2 qTf^{3/2}}\,, \tag{72}$$

where $$f = 1 - \frac{2m}{r} + \frac{Q^2}{r^2}\,. \tag{73}$$

$$
\begin{aligned}
0 &= qQ^3T' + Q[-mqT + qrT - qrTm' + 4\pi qr^3 sT^2 + \sqrt{f}rT\mu Q' - 2mqrT' \\
&+ qr^2T'] + \sqrt{f}r^2(r-2m)(\mu T' - T\mu') + Q^2[qTQ' + \sqrt{f}r(\mu T' - T\mu')] \,. \qquad (74)
\end{aligned}
$$

Using (45)

$$
p'(r) = sT'(r) + n\mu'(r) \,. \qquad (45)
$$

to eliminate μ' in Eq.(79), we have

$$
\begin{aligned}
0 &= qQ^3T' + \frac{\sqrt{f}r^2(r-2m)(sTT' + n\mu T' - Tp')}{n} + \frac{Q^2\sqrt{f}r(sTT' + n\mu T' - Tp')}{n} \\
&+ qTQ^2Q' + Q[-mqT + qrT - qrTm' + 4\pi qr^3 sT^2 + \sqrt{f}rT\mu Q' - 2mqrT' + qr^2T'] \,. \qquad (75)
\end{aligned}
$$

Eliminating s, μ and n via Eqs. (20) and (62), we rewrite Eq. (75) as

$$
\begin{aligned}
0 &= 4\pi r^3(r^2 - 2mr + Q^2)(p+\rho)T' - 4\pi r^3(r^2 - 2mr + Q^2)Tp' + TQ^2Q'^2 \\
&+ QQ'(rT + 4\pi r^3(p+\rho)T + Q^2T' + r^2T' - mT - 2rmT' - rm'T) \qquad (76)
\end{aligned}
$$

Now we begin to calculate Eq. (66). Note that

$$\frac{\partial s}{\partial m'} = \frac{\partial s}{\partial \rho}\frac{\partial \rho}{\partial m'} = \frac{1}{4\pi r^2 T}\,. \tag{77}$$

Then

$$\frac{\partial L}{\partial m'} = \frac{1}{4\pi r^2 T}\left[1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right]^{-1/2} r^2\,. \tag{78}$$

$$\frac{\partial L}{\partial m} = r\left[1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right]^{-3/2}\frac{\rho + p}{T}\,.$$

Thus, Eq. (66) becomes

$$\begin{aligned}&Q^2T - 4\pi r^4 T(p+\rho) + m'Tr^2 - TrQQ' - rT'Q^2 - r^3T'\\&-mrT + 2mr^2T' = 0\,.\end{aligned} \tag{81}$$

Combining Eq. (76) and Eq. (81), one can eliminate T'. Then by substituting Eq. (59) for m', we finally find

$$p' = \frac{QQ'}{4\pi r^4} - (\rho + p)\left(4\pi r p + \frac{m}{r^2} - \frac{Q^2}{r^3}\right)\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1}\,. \tag{82}$$

This is exactly the generalized Oppenheimer-Volkoff equation for charged fluid (J.D. Bekenstein, Phys.Rev. D, 4, 2185 (1971))

5.Conclusions

- By applying the maximum entropy principle to a general self-gravitating system, we have derived the TOV equation of hydrostatic equilibrium, which was originally derived from the Einstein equation. We only used the constraint Einstein equation and the ordinary thermodynamic relations. This is a strong evidence for the fundamental relationship between gravitation and thermodynamics.

Thank you!