"Black Holes in Higher Dimensions"

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∴ • Reference: "Higher Dimensional Black Holes"
Prog. Theor. Phys. Supple. 189 (2011)
Ed. Maeda, Shiromizu, Tanaka.

◎ Why Higher Dimensions?

▶ 4D ... Astrophysical / Cosmological importance.

▶ $D>4$ ...
- Theoretical Laboratory for Quantum Gravity.
- Phenomenological models.
  e.g. Brane-world / Large-extra. dimensions.
- Help better understand gravitation.

◎ Aim : ... give pedagogic account of basic properties of higher dimentional black holes in GR.

⊙ Plan.

§ Introduction — ⊙ Why? ✓.

⊙ Aim!

§ 4D Black Holes. … Basics

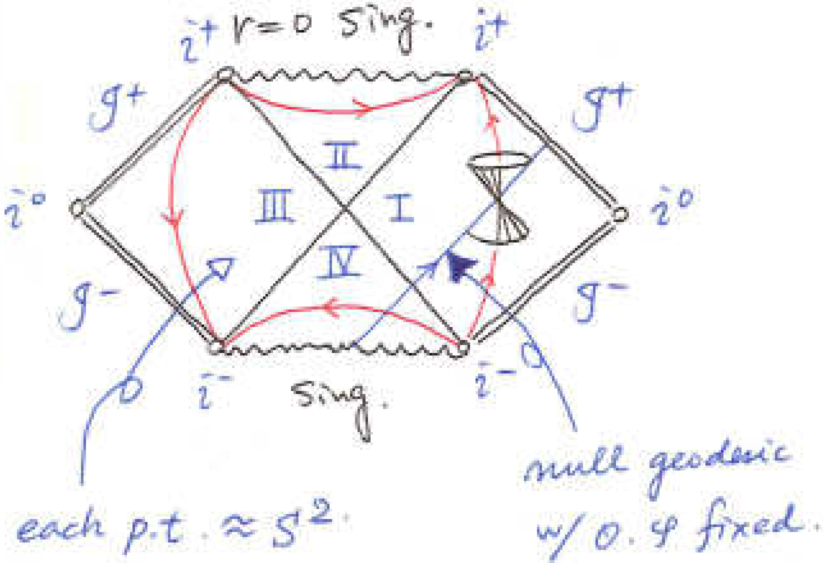
§ 5D & $D \geq 6$ BHs.

§ 4D BH Basics $\quad G = c = 1$.

◎ Schwarzschild BH

▶ metric $\quad ds^2_{(4)} = -f(r)\,dt^2 + \dfrac{dr^2}{f(r)} + r^2 d\Omega^2 \quad \left(\begin{array}{l} f = 1 - \dfrac{2M}{r} \\ d\Omega^2 : \text{2-sphere}. \end{array} \right.$

▶ Conformal Diagram



$\mathscr{I}^\pm$: Future (Past) null infinity

$$\begin{cases} t \to \pm\infty \\ r \to +\infty \\ |r \mp t| < \infty \quad (\text{finite}) \end{cases}$$

$i^\pm$: Future (Past) infinity

$$\begin{cases} t \to \pm\infty \\ \forall r < \infty \quad (\text{finite}) \end{cases}$$

i^0 : spatial infinity

$$\begin{cases} r \to +\infty \\ \forall t < \infty \quad (\text{finite}) \end{cases}$$

▶ Coordinate Systems

- Schwarzschild (t, r) : I (or II, III, or IV).
- Eddington-Finkelstein $(v_\pm, r)$:

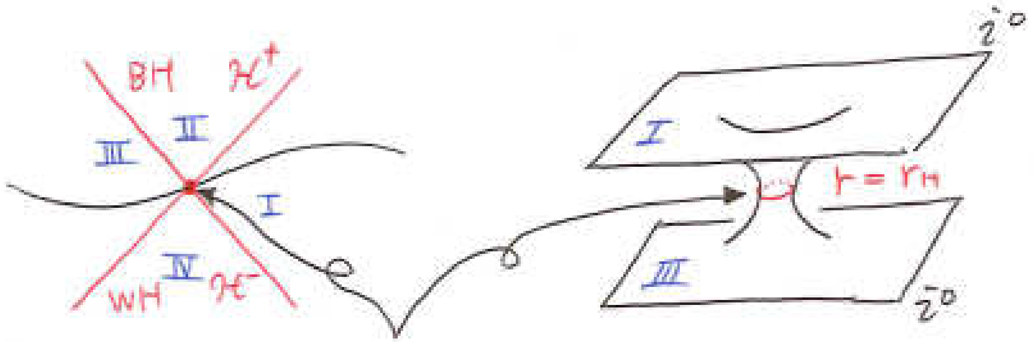
$$ds^2 = -f\,dv_\pm^2 \pm 2\,dr\,dv_\pm + r^2 d\Omega^2 \qquad \left(\begin{array}{ccc} \text{I \& II} & \text{or} & \text{III \& IV} \\ \text{I \& IV} & \text{or} & \text{II \& III} \end{array} \right.$$

- Kurstcal-Sezekeres (U, V) : The whole mfd.

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M}\, dU\,dV + r^2 d\Omega^2 \qquad \left\{ \begin{array}{l} UV = f\,e^{r/2M} \\ \dfrac{U}{V} = e^{-t/2M} \end{array} \right.$$

▶ Event Horizon

root of $f(r) = 0$. $r = r_H \equiv 2M$.



bifurcate surface.

Properties

- single parameter M : Mass
- Topology (of Horizon X-section) $\approx$ spherical 
- Symmetry (Isometry)
  - spherical $SO(3)$
  - static $\mathbb{R}$ by $t^a = (\partial/\partial t)^a$ $\quad \mathcal{L}_t g_{ab} \overset{!}{=} 0$

    timelike in I & III

    hypersurface orthogonal $t_{[a}\nabla_b t_{c]} = 0$

    $\longleftrightarrow$ ($t \to -t$: invariant).
- Surface gravity $\kappa = \frac{1}{4M}$ (redshift $\times$ acceleration on $\mathcal{H}^+$)

  $t^c \nabla_c t^a = \kappa t^a$
- <u>Stable</u> $\quad g_{ab} \longrightarrow g_{ab} + h_{ab} \quad$ perturbations decay

▷ plus Maxwell field (Reissner-Nordström)

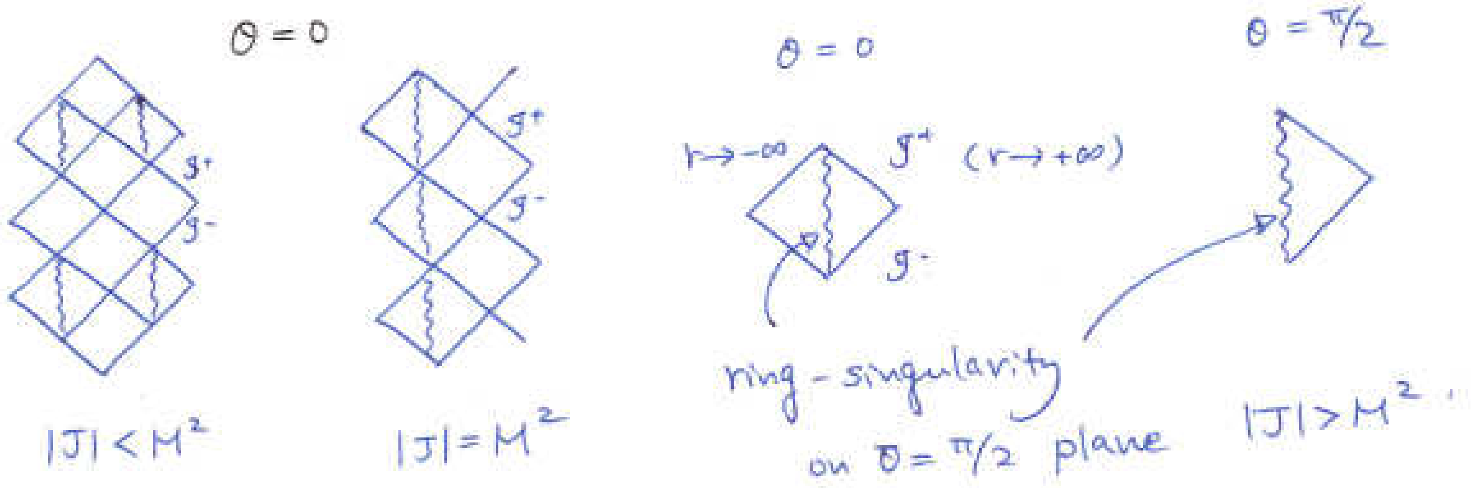
- $f \rightarrow f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$ $\qquad$ Q: charge.

  Maxwell potential $A = A_a dx^a = \frac{Q}{r} dt$.

- 2 parameters : (M, Q)

ⓔ Kerr BH

- metric $ds^2 = -dt^2 + \sin^2\theta\,(r^2+a^2)\,d\varphi^2 + \frac{2Mr}{\rho^2}\left(dt - a\sin^2\theta\, d\varphi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$
  $\begin{cases} \Delta = r^2 - 2Mr + a^2 \\ \rho^2 = r^2 + a^2\cos^2\theta \end{cases}$ $\quad a = J/M$
- 2 parameters (M, J)
- Event Horizon $\quad r_+$ & r_- : inner horizon $\quad g^{rr} = 0 \rightarrow \Delta = 0$
  $r_\pm = M \pm \sqrt{M^2 - a^2}$
- Kerr-bound $\quad |J| \leqslant M^2$



$|J| < M^2$ $\quad |J| = M^2$ $\quad |J| > M^2$.

- Topology (of Horizon X-sect.) $\approx$ spherical.

- Symmetry stationary & axi-symmetry
  $\mathbb{R}$ $U(1)$

  — discrete isometry $\begin{pmatrix} t \to -t \\ \varphi \to -\varphi \end{pmatrix}$

  — t^a : spacelike on $\mathcal{H}$: $\exists$ ergoregion

  $r_e = M + \sqrt{M^2 - a^2 \cos^2\theta}$

  $K^a = t^a + \Omega_H \varphi^a$ null on $\mathcal{H}$

  $\Omega_H = -\frac{g_{t\varphi}}{g_{\varphi\varphi}}\Big|_H$ angular velocity of $\mathcal{H}$ wrt $\mathcal{J}$

- Surface gravity κ

  $\kappa = \frac{\sqrt{M^4 - J^2}}{2M(M^2 + \sqrt{M^4 - J^2})}$ $K^c \nabla_c K^a = \kappa K^a$ ($\leftarrow$ not for t^a)

- Stable wrt perturbations

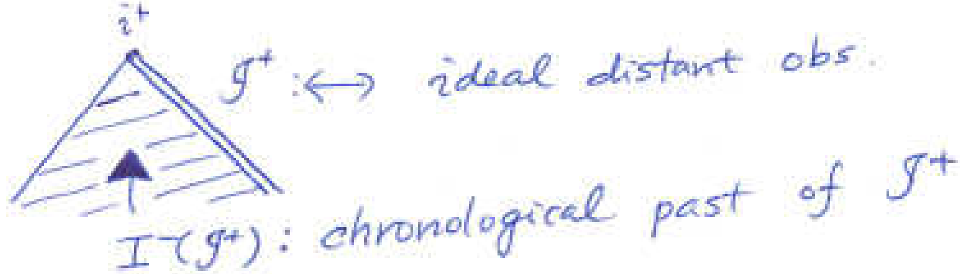
  … Teukolsky '72.

▷ plus Maxwell field (Kerr - Newmann et al).

- $\Delta \to \Delta = r^2 - 2Mr + a^2 + Q^2$
- 3 parameters $(M. J. Q)$.

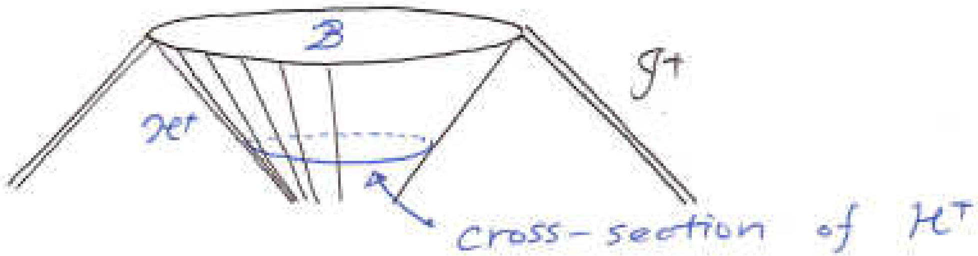
Black Holes in General Relativity

$$\mathcal{B} \equiv \underset{\text{(whole spacetime manifold)}}{\mathcal{M}} - I^-(\mathscr{I}^+)$$

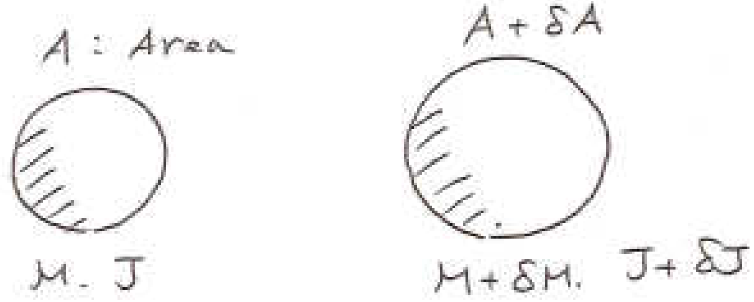


$\mathcal{H}^+ \equiv$ boundary of $\mathcal{B}$.

$\Longrightarrow$ $\mathcal{H}^+$: null surface by def.



▶ Stationary BH Mechanics (Bardeen, Carter, Hawking '73).



$\kappa = \text{const.}$ $\qquad \delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J \; (+ \Phi_H \delta Q)$

0th-law $\qquad$ 1st. law

$T = \text{const.}$ $\qquad \delta E = T\delta S - \underbrace{P\delta V}_{\text{work term}}$

Equilibrium thermodynamic laws.

--- Further more.

▶ $\boxed{\delta A \geq 0 \quad \text{Horizon Area never decreases}}$ $\longleftarrow$ 2nd - law $\longrightarrow$ $\boxed{\delta S \geq 0 \quad \text{Entropy}}$

Bekenstein '73.

▶ Quantum radiation. Hawking '74.

$$\boxed{T = \frac{\kappa}{2\pi} \qquad S = \frac{1}{4} A}$$

Lesson!

4D **Stationary** BH

- characteraized **uniquely** by (M. J. Q)

--- 《 BH Uniqueness Theorem 》.

$\Longleftrightarrow$

Equilibrium thermodynamic system.

- characteraized by a small # of parameters

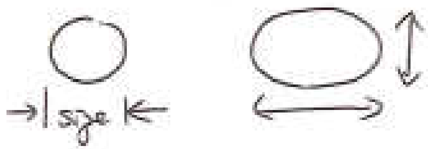
c.f. Stars

total Mass M. 

total rot. J 

--- not enough.

 diff. rotation.

shape 

oblateness ---

--- multipoles.

§ D > 4 Black Holes.

◎ Exact solutions. examples. (Focus on $R_{ab} = 0$. vacuum.).

▸ 4D Schwarzschild (static) '16

$$\begin{cases} f(r) = 1 - \frac{2M}{r} \\ d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2 \end{cases}$$

$\Longrightarrow$

▸ $D \geq 4$ + Tangherlini '63

$$\begin{cases} f(r) = 1 - \frac{2M}{r^{D-3}} \\ d\Omega^2_{(D-2)} = d\theta^2 + \sin^2\theta\, d\Omega^2_{(D-3)} \end{cases}$$

▶ 4D Kerr (stationary) '63

$$ds^2_{(4)} = -dt^2 + \sin^2\theta\,(r^2+a^2)\,d\varphi^2 + \boxed{\frac{2Mr}{\rho^2}}\,(dt - a\sin^2\theta\, d\varphi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \boxed{}$$

$$\Delta = r^2 - \boxed{2Mr} + a^2$$

$\Longrightarrow$

▶ $D \geq 5$ Myers - Perry '82

(w/ 1-spin)

$$ds^2_{(D)} = -dt^2 + \sin^2\theta\,(r^2+a^2)\,d\varphi^2 + \boxed{\frac{2M}{\rho^2 r^{D-5}}}\,(dt - a\sin^2\theta\, d\varphi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \boxed{+ r^2\cos^2\theta\, d\Omega^2_{(D-4)}}$$

$$\Delta^2 = r^2 - \boxed{\frac{2M}{r^{D-5}}} + a^2$$

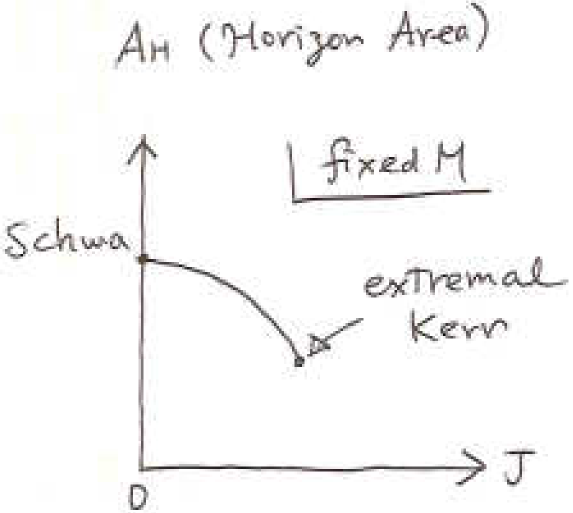
◎ Event Horizon

$r = r_H$: root of $\quad 0 = g^{rr} = \dfrac{\Delta}{\rho^2}$.

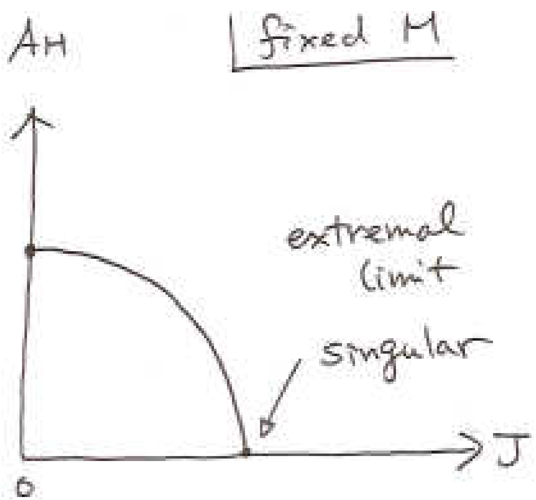
$$\boxed{0 = 1 + \frac{(J/M)^2}{r^2} - \frac{M}{r^{D-3}}}$$

— always has a solution for $D \geq 6$, $\forall M$, $\forall J$.

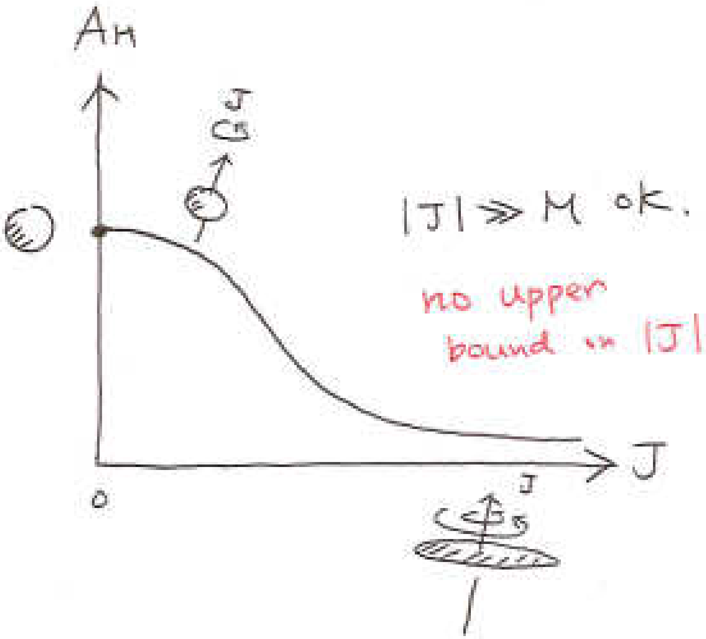
4D Kerr



5D MP w/ 1-spin



D≥6 MP

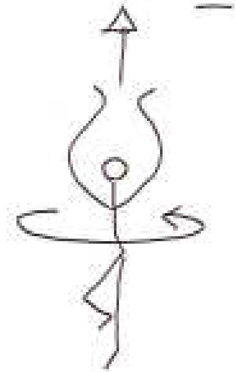


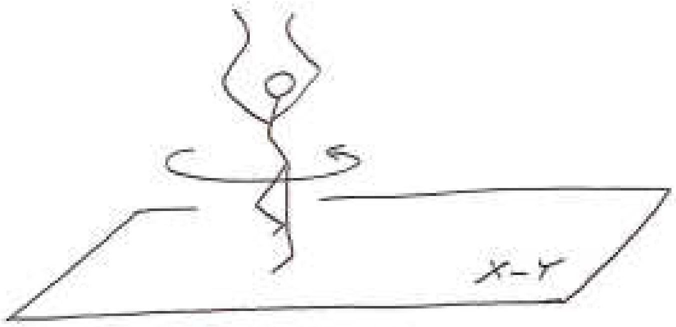
▶ How many independent rotations ?

--- How to specify indep. rotation ---

Z : spin "Axis" $\Longleftrightarrow$ (dual) spin on 2-plane (X-Y)

3D-space (X, Y, Z).





- 4D-spacetime

$$-dt^2 + \underbrace{dx^2 + dy^2}_{} + dz^2$$

1-spin.

- 5D-spacetime

$$-dt^2 + \underbrace{dx^2 + dy^2}_{\text{① rot. plane}} + \underbrace{dz^2 + dw^2}_{\text{② rot. plane}}$$

2 independent spins.

- --- in "D"-spacetime dimension

$N = \lceil \frac{D-1}{2} \rceil$ independent spins ▶ $U(1)^N$ symmetry.

▶ Myers–Perry metric w/ multi spins

$$ds^2_{(0)} = -dt^2 + (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2) + p\, r^2 d\alpha^2$$
$$+ \frac{\Pi - \Delta}{\Pi F}(dt - a_i \mu_i^2 d\phi_i)^2 + \frac{\Pi F}{\Delta} dr^2 .$$

$\because$ $D =$ odd $\longrightarrow$ $p = 0$ $\quad$ (α, μ_i) : angular functions

even $\quad p = 1$ $\qquad$ $p\alpha^2 + \sum_{j=1}^{N} \mu_j^2 = 1.$

$$\begin{cases} \Delta \equiv \Pi - M r^{2-p} \\ F \equiv 1 - \dfrac{a_i^2 \mu_i^2}{r^2 + a_i^2} \\ \Pi \equiv \prod_{i=1}^{N} (r^2 + a_i^2) \end{cases}$$

▶ Topology $\approx S^{D-2}$

▶ Symmetry $\mathbb{R} \times U(1)^N$

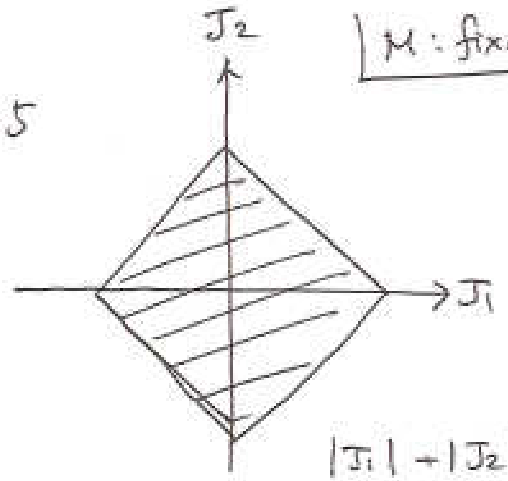
cf. $\left(\begin{array}{l} \cdot\ \text{1-spin} \to SO(D-3) \\ \cdot\ D = \text{odd} : J_1 = J_2 = \cdots = J_N \\ \qquad \to U(N). \end{array} \right.$

▶ Phase space of MP w/ 2-spin

$$\exists\, \mathcal{H}^{\pm} \iff \boxed{0 = \prod_{i=1}^{N} \left(1 + \frac{(J_i/M)^2}{r^2}\right) - \frac{GM}{r^{D-3}}}$$

M : fixed

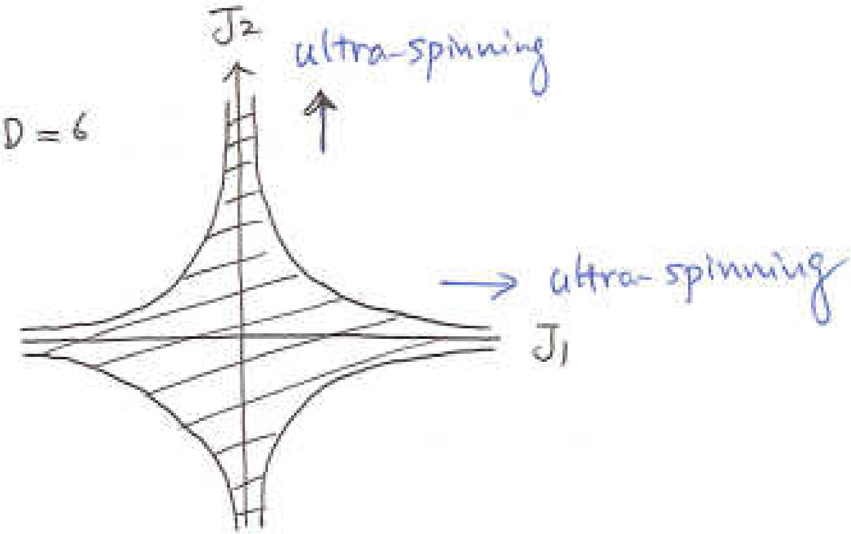
D = 5



$|J_1| + |J_2| \leq 1$

"=" ... extremal

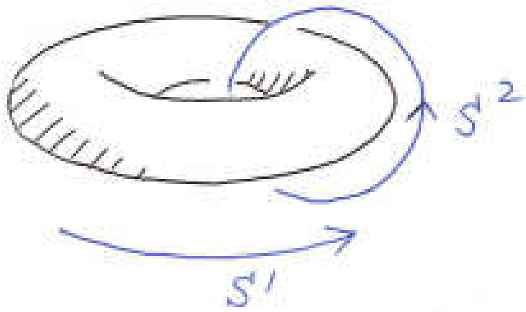
D = 6



▶ Black-Ring

- $D=5$ (so far)

- Topology $\quad S^1 \times S^2$



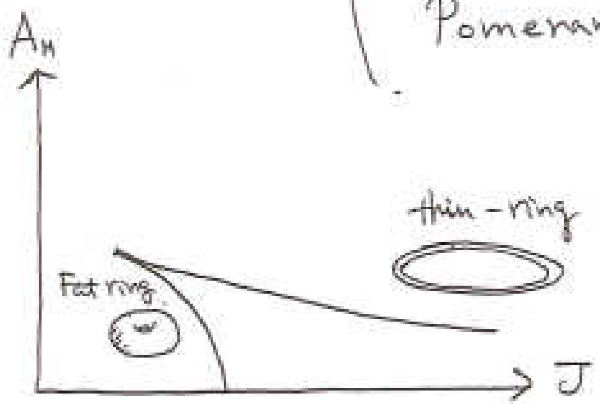
- Symmetry $\quad \mathbb{R} \times U(1)^2 \quad$ by $(t^a, \varphi_1^a, \varphi_2^a)$

$\uparrow$

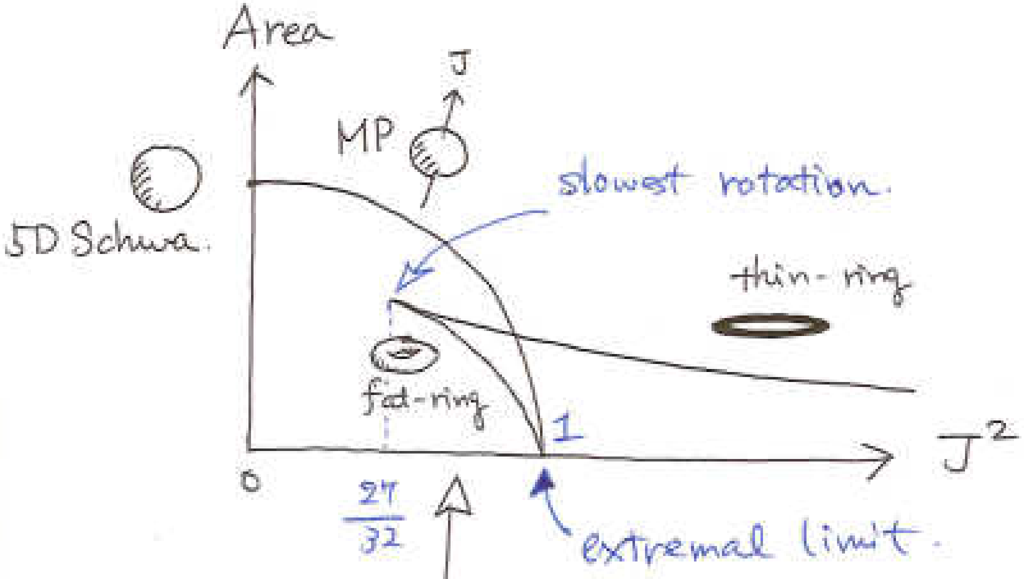
(M, J_1, J_2). conserved charges.

cf. Emparan - Reall '02 $\quad (M, J_1, J_2 = 0)$

Pomeransky - Senkov '06 $\quad (M, J_1, J_2)$.



▷ Phase space diagram of 5D BHs (connected Horizon w/ single spin ($J_2 = 0$)



Black "Hole"

Black ("Fat-Ring" / "Thin-Ring")

w/ the same (M, J_1, $J_2 = 0$) → cannot uniquely be determined by global charges!

※ $dS_{ab} \equiv * S^{D-2} = \overset{(2)}{\epsilon}_{ab}\, dS_{(D-2)}$

$$\begin{cases} M = -\dfrac{1}{16\pi G}\dfrac{(D-2)}{(D-3)}\displaystyle\oint_{S_\infty} dS_{ab}\nabla^a t^b \\[2ex] J = +\dfrac{1}{16\pi G}\displaystyle\oint_{S_\infty} dS_{ab}\nabla^a \varphi^b \end{cases}$$

— have seen basic properties of known exact solutions.

	4D vacuum GR	5D vacuum GR
• Exact Sol.	Kerr family	· Myers–Perry · Emparan–Reall (Rings ⋯).
• Stability	Stable → final state	un-stable
• Topology	spherical S^2	S^3, $S^1 \times S^2$
• Symmetry	$\mathbb{R} \times U(1)$	$\mathbb{R} \times U(1) \times U(1)$
• Uniqueness	Yes. by (M, J)	No: (M, J_1, J_2) NOT enough.
• Remark	Kerr–bound $\lvert J \rvert \leq M^2$	

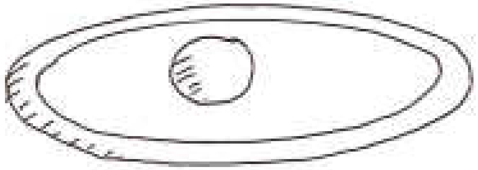
$D \geq 6$ NO Kerr-bound.

$D = 3$
No vacuum BH

--- so far ... "connected" horizons

Multi - BHs (Exact. Sol.)

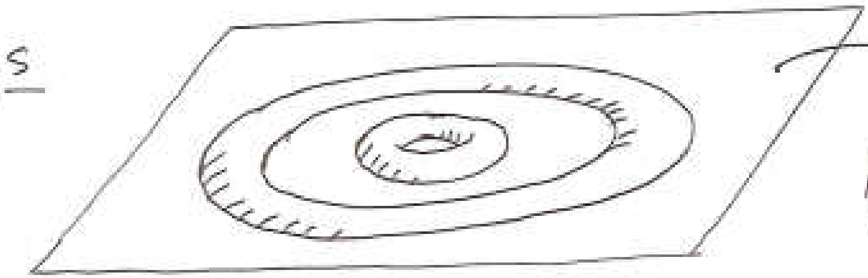
▶ Black Saturn



Elvang & Figueras '07

{ corotating
  counter-rotating

▶ di-rings

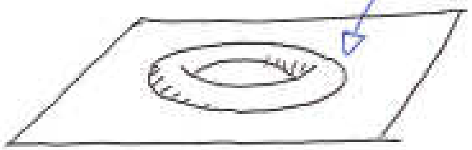


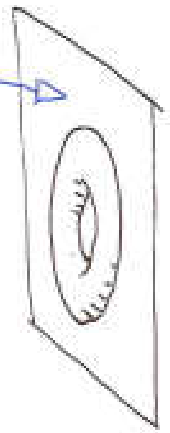
on the same rotation - plane.

Iguchi - Mishima '07

▶ orthogonal

different rotation planes

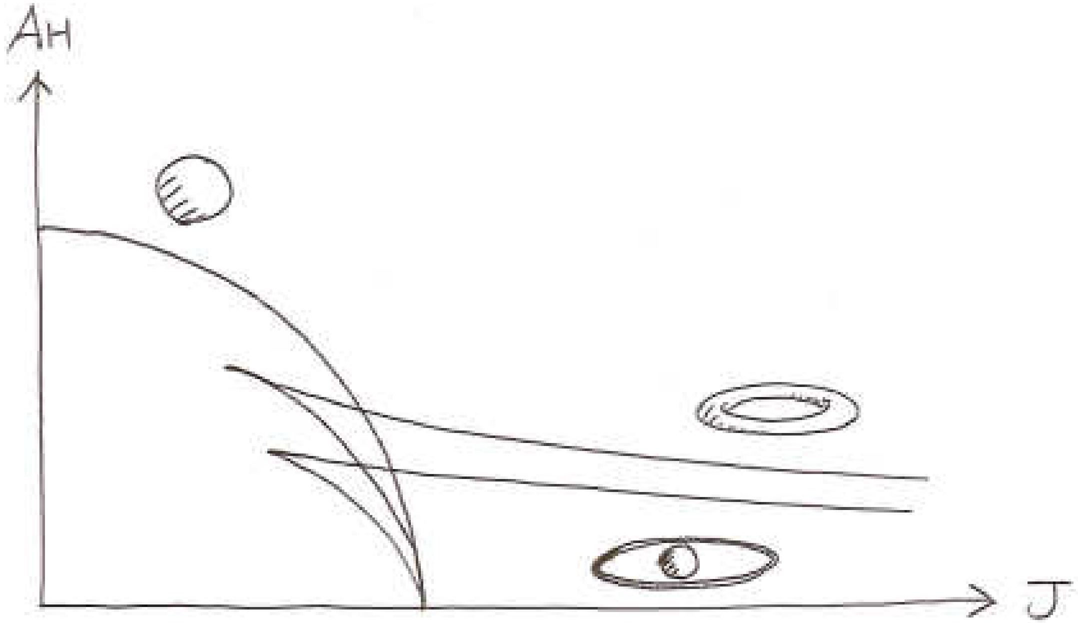




Izumi '07

Elvang - Rodriguez '07

- $D=5$



Qu : How to characterize HDBHs ?

Global charges $(M, J_1 \cdots J_N)$ ... NOT enough !

◎ Focus $\mathbb{R} \times U(1)^{D-3}$ case. $(t, \varphi_1 \cdots \varphi_{D-3})$.

$(D-3)$ - Killing vectors $\begin{cases} K_1^a = (\partial/\partial t)^a \\ K_j^a = (\partial/\partial \varphi_j)^a & j = 1 \cdots D-3. \end{cases}$

D-dimensional Einstein Equations

$\vdash$ reduce to $\longrightarrow$ Eqns on 2-dimensional space $\Sigma^{(2)}$

upper-half-plane.

spanned by (θ, r).

$$\boxed{\Sigma^{(2)} = {}^{(D)}M / \mathrm{Isom}(M).}$$

▶ Rod / Interval structure

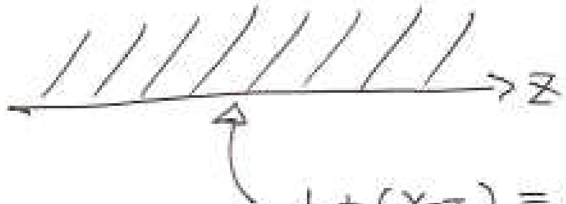
Harmark '04 . Hollands-Yazadjiev '08

$$K_I^a = \{ K_1^a = t^a , \; K_j^a = (\partial/\partial\varphi_j)^a \}.$$

Def. Gram matrix $\quad X_{IJ} \equiv g_{ab} X_I^a X_J^b$: $(D-2)\times(D-2)$ matrix .

$\longrightarrow$ specifies the structure of " orbit space $\Sigma^{(2)}$ "

$\Sigma^{(2)}$: simply connected mfd
w/ boundaries & corners.



$\det(X_{IJ}) = 0 \iff$

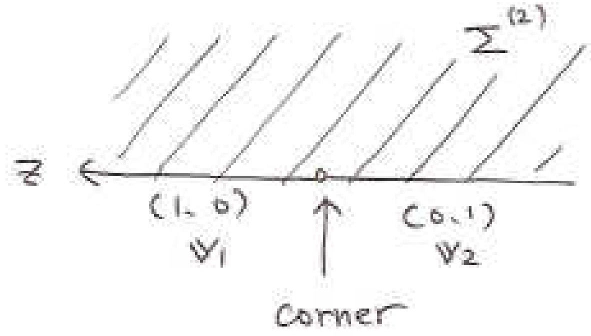
e.g. $D=5$. $U(1)\times U(1)$

$\exists\, v = (v_1 , v_2) \in \mathbb{Z}^2$

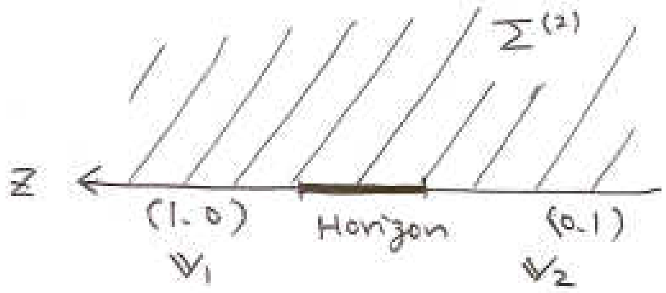
s.t. $v_1 \varphi_1^a + v_2 \varphi_2^a = 0$.

▷ D=5 example

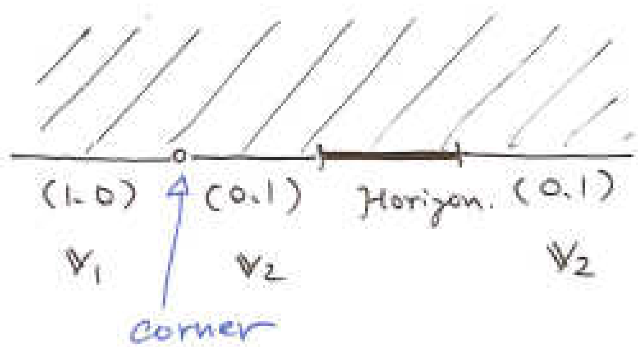
• Minkowski



• Myers-Perry BH

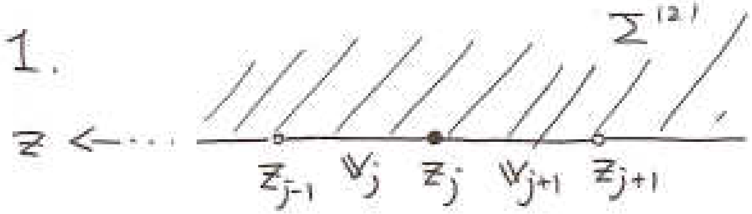


• Ring

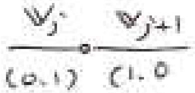


(i) If intervals (z_{j-1}, z_j) and (z_j, z_{j+1}) are NOT Horizon

$\Rightarrow$ then $\det(\mathbb{V}_j, \mathbb{V}_{j+1}) = \pm 1$.



e.g.



$\det(\mathbb{V}_j, \mathbb{V}_{j+1})$

$= \det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$ //.

(ii) If a finite interval (z_H, z_{H+1}) is Horizon.

$\Rightarrow$ then $\boxed{\det(\mathbb{V}_{H-1}, \mathbb{V}_{H+1}) = p}$

vectors on "adjacent" rods.

e.g. Black "Hole"

$\Rightarrow \det(\mathbb{V}_{H-1}, \mathbb{V}_{H+1})$

$= \det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$

Black "Ring"

$\Rightarrow \det(\mathbb{V}_{H-1}, \mathbb{V}_{H+1})$

$= \det\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = 0$.

$$\boxed{\underline{\text{invariants}} \quad \begin{cases} \text{moduli} : & l_k := z_k - z_{k+1} \\ \text{winding \#} : & \mathbb{v}_k := (v_k^1, v_k^2) \end{cases}}$$

Interval / Rod structure, $\{ l_k, \mathbb{v}_k \}$

determines the global structure of $\Sigma^{(2)}$

p	Horizon Topology
0	$S^1 \times S^2$
± 1	S^3
others	$L(p, q)$.

§ Open issues

▶ Exact solutions

- Black Holes w/ non-trivial topology in $D \geq 6$.

  (effective theory approach (blackfold) : done!).
  Emparan et al

- AdS-ring — not well understood.

- Brane-world localized black hole
  (numerical study Figueras. Lucietti. Wiseman).
  analytical ···

▶ Stability

"hole" — { analytic { linear perturbations / Myers-Perry w/ extra symmetries. — Kodama - AI, Kunduri et al, Murata - Soda ···

numerical { soft Dias. Figueras. Monteiro. Santos. Reall ··· et al / hard Shibata - Yoshino.

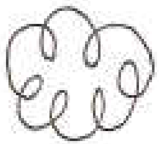
"ring" ··· not yet. c.f. Black string Lehner-Pretorius '10

► Topology ... of ⊕ve Yamabe type. Galloway - Schoen '05

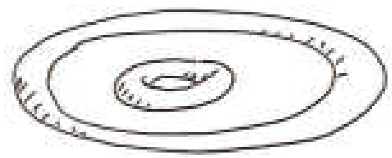
► Symmetry ... $\mathbb{R} \times U(1)$. $\exists$ single $U(1)$. Hollands - Al - Wald '07

↳ restriction on topology from symmetry.

less - symmetric solutions. (w/ single $U(1)$).

helical 

{ numerical — Dias et al
{ approx. method — Emparan et al

► Thermodynamics multi - black holes 

1st law $\delta M = \Sigma \left(\frac{\kappa_i}{8\pi} \delta A_i + \Omega_i \delta J_i \right)$

? ↗ stationary sol. w/ different temperatures $\kappa_1 \neq \kappa_2 \ldots$

↘ equilibrium ?

◎ 4D Black Holes

⇒ — under control by (M. J. Q) : Uniqueness!

◎ Higher Dimensional Black Holes

— out of control $(M.\ J_1.\ J_2 \cdots)$ not enough

⇒ or (e.g. D=9. 70 parameters Dias et al '10

— More surprises ?