Post-Newtonian analysis of bobbing effects in spinning binary systems

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Outline

1. What is bobbing? How is it observed in spinning binary systems?

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   (Newtonian; circular orbits)

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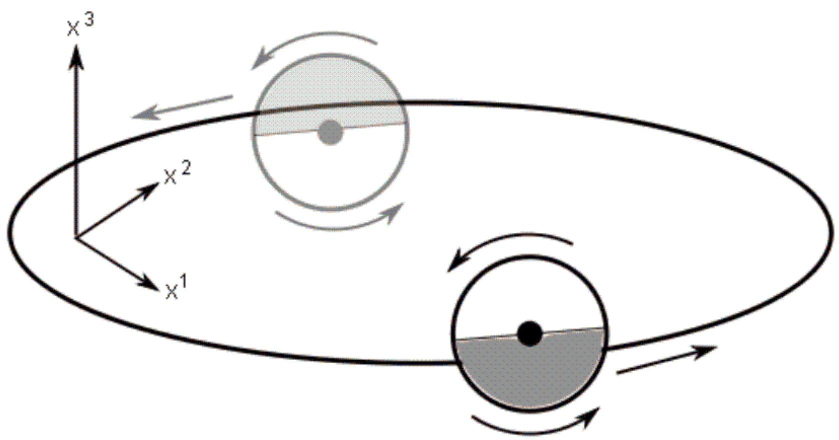
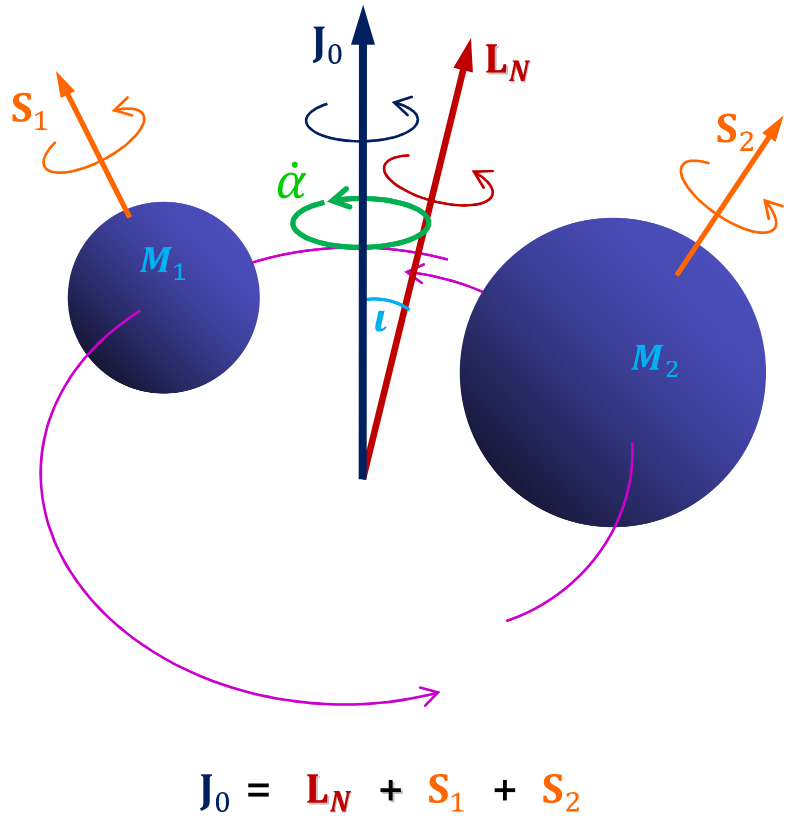
1. What is bobbing? How is it observed in spinning binary systems?



<Excerpted from ***Gralla et. al*** [Phys. Rev. D 81, 104012 (2010)]>

The lab frame centroid of a body (solid line) is confined to the x^1-x^2 plane. However, since the body has a (constant) spin vector lying in the plane, its center of mass (solid dot) is displaced in the x^3 direction by a velocity dependent factor, giving rise to bobbing:

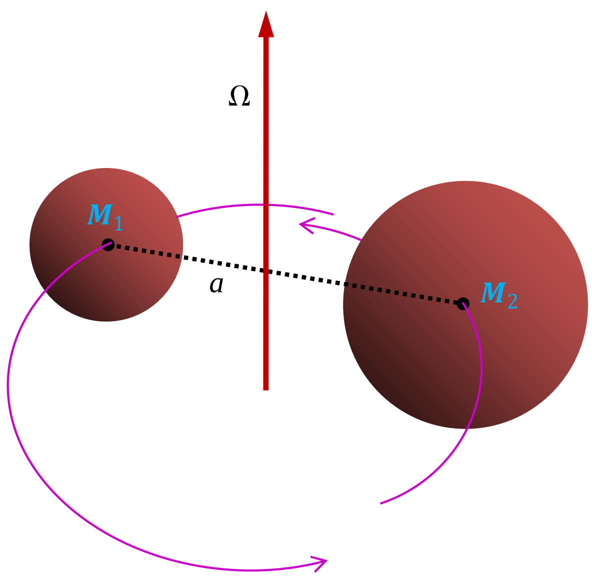
$$m\vec{z}_{\mathrm{CM}} = m\vec{z}_{\mathrm{Lab}} + \vec{S} \times \frac{\mathrm{d}\vec{z}_{\mathrm{L}}}{\mathrm{d}t} + \mathcal{O}\left(\left|\frac{\mathrm{d}\vec{z}_{\mathrm{L}}}{\mathrm{d}t}\right|^2\right)$$



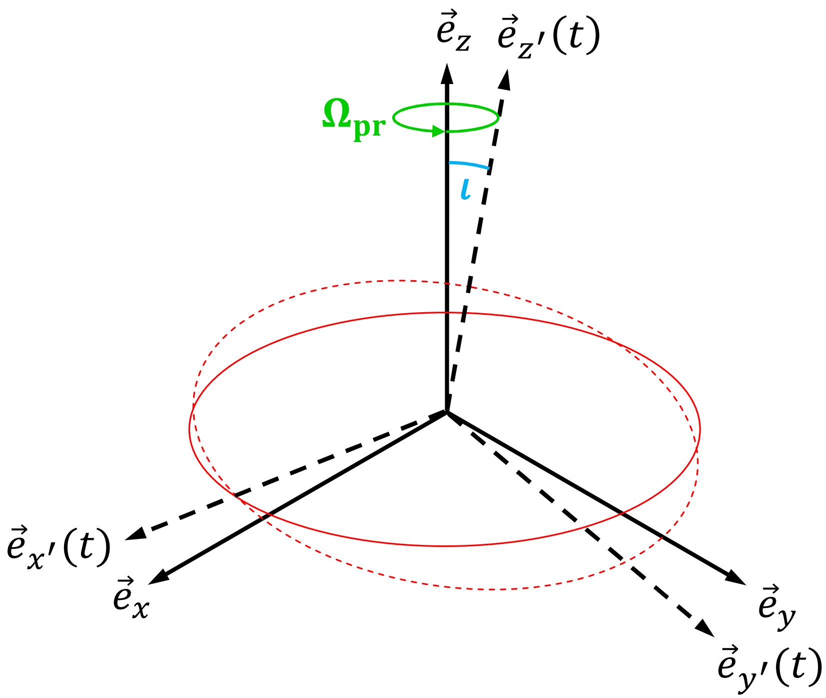
Spinning binary inspiral system

In spinning binary systems, this kinematics can be represented by the interactions between the orbital angular momentum ($\mathbf{L}_N$) and spins of two bodies ($\mathbf{S}_1$, $\mathbf{S}_2$), i.e. spin-orbit interactions. The net bobbing can be observed from the inclination angle (ι).

2. Precessing binary inspirals and GWs (Newtonian; circular orbits)



Binary inspiral system



$$\vec{e}_{x'}(t) = -\sin(\Omega_{\mathrm{pr}}t)\,\vec{e}_x + \cos(\Omega_{\mathrm{pr}}t)\,\vec{e}_y$$

$$\vec{e}_{y'}(t) = -\cos\iota\cos(\Omega_{\mathrm{pr}}t)\,\vec{e}_x - \cos\iota\sin(\Omega_{\mathrm{pr}}t)\,\vec{e}_y + \sin\iota\;\vec{e}_z$$

$$\vec{e}_{z'}(t) = \sin\iota\cos(\Omega_{\mathrm{pr}}t)\,\vec{e}_x + \sin\iota\sin(\Omega_{\mathrm{pr}}t)\,\vec{e}_y + \cos\iota\;\vec{e}_z$$

Time-varying frame for precessing binary:

$$\begin{bmatrix} \vec{e}_{x'}(t) \\ \vec{e}_{y'}(t) \\ \vec{e}_{z'}(t) \end{bmatrix} = \begin{bmatrix} -\sin(\Omega_{\rm pr}t) & \cos(\Omega_{\rm pr}t) & 0 \\ -\cos\iota\cos(\Omega_{\rm pr}t) & -\cos\iota\sin(\Omega_{\rm pr}t) & \sin\iota \\ \sin\iota\cos(\Omega_{\rm pr}t) & \sin\iota\sin(\Omega_{\rm pr}t) & \cos\iota \end{bmatrix} \begin{bmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{bmatrix}$$

Thus,

$$\begin{bmatrix} \vec{e}_{\hat{r}}(t) \\ \vec{e}_{\hat{\theta}}(t) \\ \vec{e}_{\hat{\phi}}(t) \end{bmatrix} = \begin{bmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{bmatrix} \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} -\sin(\Omega_{\rm pr}t) & \cos(\Omega_{\rm pr}t) & 0 \\ -\cos\iota\cos(\Omega_{\rm pr}t) & -\cos\iota\sin(\Omega_{\rm pr}t) & \sin\iota \\ \sin\iota\cos(\Omega_{\rm pr}t) & \sin\iota\sin(\Omega_{\rm pr}t) & \cos\iota \end{bmatrix}^{-1} \begin{bmatrix} \vec{e}_{x'}(t) \\ \vec{e}_{y'}(t) \\ \vec{e}_{z'}(t) \end{bmatrix}$$

Now, from

$$I_{x'(t)x'(t)} = \mu a^2 \cos^2(\Omega t)\,,\; I_{y'(t)y'(t)} = \mu a^2 \sin^2(\Omega t)\,,\; I_{x'(t)y'(t)} = I_{y'(t)x'(t)} = \mu a^2 \cos(\Omega t)\sin(\Omega t)$$

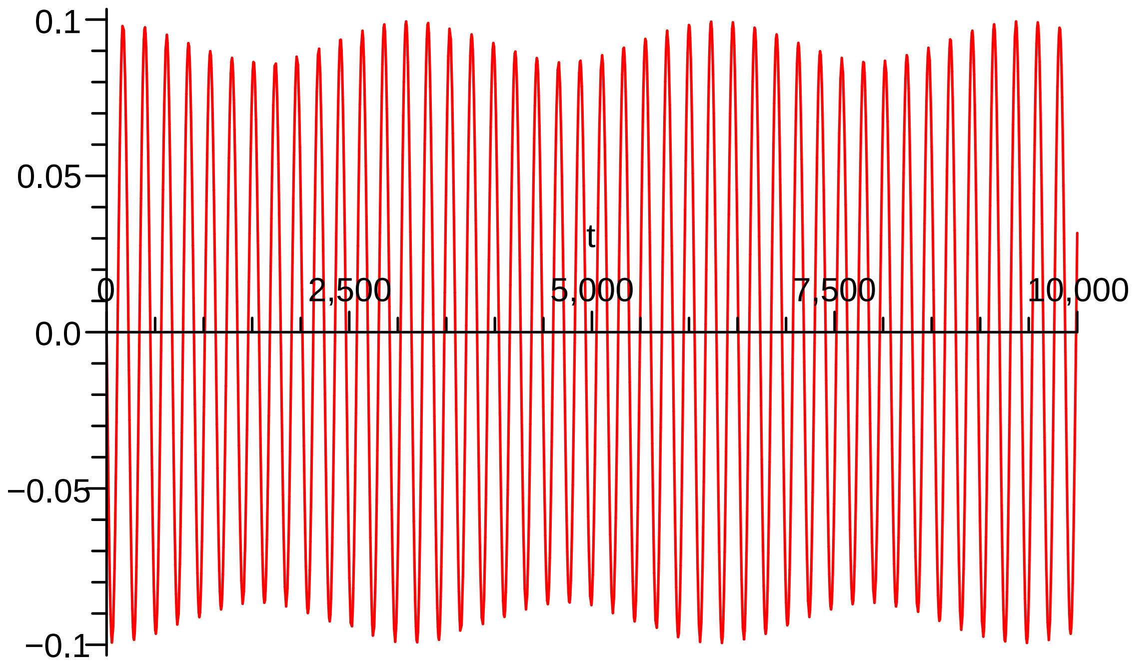
gravitational waves are computed (with polarization angle $\psi = 0$):

$$\mathbf{h}^{\rm TT} = h_+\mathbf{e}^+(t) + h_\times\mathbf{e}^\times(t)\;;$$ **(TT: Transverse and Trace-free)**

$$\mathbf{e}^+(t) = \left(\vec{e}_{\hat{\theta}}(t)\otimes\vec{e}_{\hat{\theta}}(t) - \vec{e}_{\hat{\phi}}(t)\otimes\vec{e}_{\hat{\phi}}(t)\right),\; \mathbf{e}^\times = \left(\vec{e}_{\hat{\theta}}(t)\otimes\vec{e}_{\hat{\phi}}(t) + \vec{e}_{\hat{\phi}}(t)\otimes\vec{e}_{\hat{\theta}}(t)\right)$$

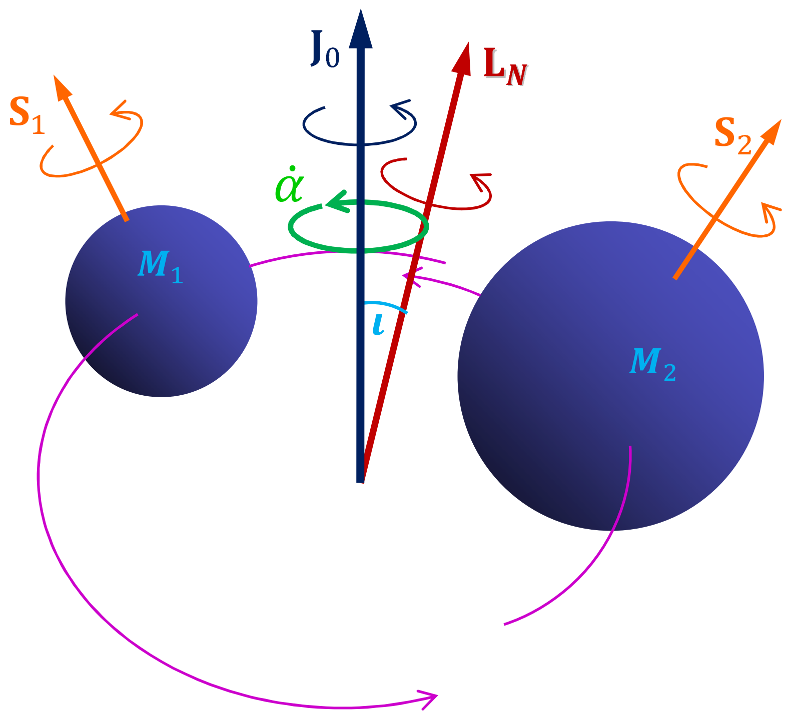
$$
\begin{aligned}
\leftarrow \quad h_+ &= h^{TT}_{\hat{\theta(t)}\hat{\theta(t)}} = \frac{2}{r}\left[\ddot{I}_{\hat{\theta(t)}\hat{\theta(t)}}(t-r)\right]^{TT} \\
&= -\frac{\mu(M\Omega)^{2/3}}{r}\left[\frac{(1+\cos\iota)^2(1+\cos^2\theta)}{2}\cos\left((2\Omega+2\Omega_{\rm pr})\,t-2\Omega r-2\phi\right)\right. \\
&\quad +\frac{(1-\cos\iota)^2(1+\cos^2\theta)}{2}\cos\left((2\Omega-2\Omega_{\rm pr})\,t-2\Omega r+2\phi\right) \\
&\quad +\sin\iota(1+\cos\iota)\sin 2\theta\cos\left((2\Omega+\Omega_{\rm pr})\,t-2\Omega r-\phi\right) \\
&\quad -\sin\iota(1-\cos\iota)\sin 2\theta\cos\left((2\Omega-\Omega_{\rm pr})\,t-2\Omega r+\phi\right) \\
&\quad \left.+3\sin^2\iota\sin^2\theta\cos\left(2\Omega t\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
h_\times &= h^{TT}_{\hat{\theta(t)}\hat{\phi(t)}} = \frac{2}{r}\left[\ddot{I}_{\hat{\theta(t)}\hat{\phi(t)}}(t-r)\right]^{TT} \\
&= -\frac{\mu(M\Omega)^{2/3}}{r}\left[(1+\cos\iota)^2\cos\theta\sin\left((2\Omega+2\Omega_{\rm pr})\,t-2\Omega r-2\phi\right)\right. \\
&\quad -(1-\cos\iota)^2\cos\theta\sin\left((2\Omega-2\Omega_{\rm pr})\,t-2\Omega r+2\phi\right) \\
&\quad +2\sin\iota(1+\cos\iota)\sin\theta\sin\left((2\Omega+\Omega_{\rm pr})\,t-2\Omega r-\phi\right) \\
&\quad \left.+2\sin\iota(1-\cos\iota)\sin\theta\sin\left((2\Omega-\Omega_{\rm pr})\,t-2\Omega r+\phi\right)\right]
\end{aligned}
$$



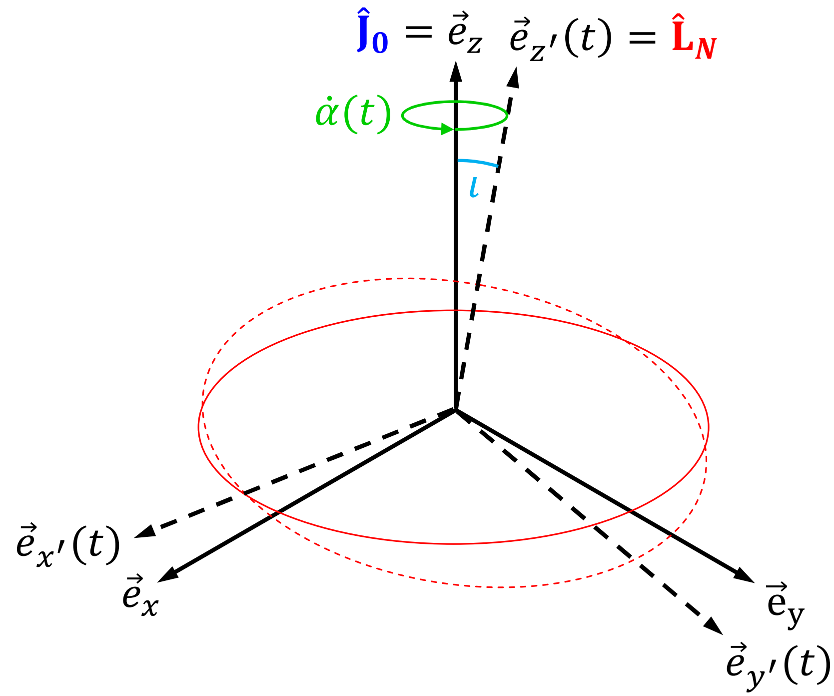
Plot of $h_\times$ **with** $\Omega = 0.012$, $\Omega_{\rm pr} = 0.002$, $\iota = \pi/15$, $\theta = \pi/10$, $\phi = 0$

3. Spinning binary inspirals and GWs (Post-Newtonian; circular orbits)



$\mathbf{J}_0 = \mathbf{L}_N + \mathbf{S}_1 + \mathbf{S}_2$

Spinning binary inspiral system



$$\vec{e}_{x'}(t) = -\sin\alpha\,\vec{e}_x + \cos\alpha\,\vec{e}_y$$

$$\vec{e}_{y'}(t) = -\cos\iota\cos\alpha\,\vec{e}_x - \cos\iota\sin\alpha\,\vec{e}_y + \sin\iota\,\vec{e}_z$$

$$\vec{e}_{z'}(t) = \sin\iota\cos\alpha\,\vec{e}_x + \sin\iota\sin\alpha\,\vec{e}_y + \cos\iota\,\vec{e}_z$$

Formally, following ***Kidder*** [Phys. Rev. D 52, 821 (1995)], ***Will et. al*** [Phys. Rev. D 54, 4813 (1996)] and ***Arun et. al*** [Phys. Rev. D 79, 104023 (2009)], one can perform the multipole expansion

$$h^{ij}(t,\mathbf{x}) = \frac{2}{D_L}\frac{d^2}{dt^2}\sum_{m=0}^{\infty}\hat{N}_{k_1}\cdots\hat{N}_{k_m}I_{\rm EW}^{ijk_1\cdots k_m}(t-r),$$

where $\hat{\mathbf{N}} \equiv \mathbf{x}/D_L$, $I_{\rm EW}^{ijk_1\cdots k_m}$: **Epstein-Wagoner (EW) moments**. Through quite involved analysis, one can determine

$$\begin{aligned} h_{+,\times} \ = \ & \frac{2M\nu v^2}{D_L}\Big[H_{+,\times}^{(0)} + H_{+,\times}^{(1/2)} + H_{+,\times}^{(1/2,\rm SO)} + H_{+,\times}^{(1)} + H_{+,\times}^{(1,\rm SO)} \\ & + H_{+,\times}^{(3/2)} + H_{+,\times}^{(3/2,\rm SO)}\Big], \end{aligned}$$

where $\nu = \frac{M_1 M_2}{M^2}$, and $H_{+,\times}^{(0)}$, $H_{+,\times}^{(1/2)}$, $H_{+,\times}^{(1/2,\rm SO)}$, $H_{+,\times}^{(1)}$, $H_{+,\times}^{(1,\rm SO)}$, $H_{+,\times}^{(3/2)}$, $H_{+,\times}^{(3/2,\rm SO)}$ are given as follows (**ι-expanded polarizations** with the assumption $S \ll L$):

<Excerpted from ***Arun et. al*** [Phys. Rev. D 79, 104023 (2009)]>

$$H_+^{(0)} = -(c_\theta{}^2 + 1)\cos 2(\alpha + \Psi), \tag{3.16a}$$

$$H_+^{(1/2)} = \upsilon\,\delta s_\theta\left[\left(\frac{c_\theta{}^2}{8} + \frac{5}{8}\right)\cos(\alpha + \Psi) - \frac{9}{8}(c_\theta{}^2 + 1)\cos 3(\alpha + \Psi)\right], \tag{3.16b}$$

$$H_+^{(1)} = \upsilon^2\left[\left(-\frac{c_\theta{}^4}{3} + \frac{3c_\theta{}^2}{2} + \frac{19}{6} + \left(c_\theta{}^4 + \frac{11c_\theta{}^2}{6} - \frac{19}{6}\right)\nu\right)\cos 2(\alpha + \Psi) + \frac{4}{3}(1 - c_\theta{}^4)(3\nu - 1)\cos 4(\alpha + \Psi)\right], \tag{3.16c}$$

$$\begin{aligned} H_+^{(3/2)} = \upsilon^3\Bigg[&\delta s_\theta\left(\frac{c_\theta{}^4}{192} - \frac{5c_\theta{}^2}{16} - \frac{19}{64} + \left(-\frac{c_\theta{}^4}{96} - \frac{c_\theta{}^2}{8} + \frac{49}{96}\right)\nu\right)\cos(\alpha + \Psi) - 2\pi(c_\theta{}^2 + 1)\cos 2(\alpha + \Psi) \\ &+ \delta s_\theta\left(\left(-\frac{81c_\theta{}^4}{128} + \frac{45c_\theta{}^2}{16} + \frac{657}{128}\right) + \left(\frac{81c_\theta{}^4}{64} + \frac{9c_\theta{}^2}{8} - \frac{225}{64}\right)\nu\right)\cos 3(\alpha + \Psi) + \delta s_\theta\frac{625}{384}(1 - c_\theta{}^4)(2\nu - 1) \\ &\times\cos 5(\alpha + \Psi)\Bigg], \end{aligned} \tag{3.16d}$$

$$H_+^{(1/2,\mathrm{SO})} = -2\iota c_\theta s_\theta\cos(\alpha + 2\Psi), \tag{3.16e}$$

$$\begin{aligned} H_+^{(1,\mathrm{SO})} = \upsilon^2&[(c_\theta(\chi_a^x + \delta\chi_s^x) - s_\theta(\chi_a^z + \delta\chi_s^z))\cos(\alpha + \Psi) - c_\theta(\chi_a^y + \delta\chi_s^y)\sin(\alpha + \Psi)] \\ &+ \upsilon\iota\delta c_\theta\left[\frac{1}{4}s_\theta{}^2\cos\Psi - \left(\frac{c_\theta{}^2}{8} + \frac{5}{8}\right)\cos(2\alpha + \Psi) + \left(-\frac{9}{8} + \frac{27c_\theta{}^2}{8}\right)\cos(2\alpha + 3\Psi)\right] \\ &+ \iota^2\left[-\frac{3}{2}s_\theta{}^2\cos 2\Psi + \frac{1}{2}(c_\theta{}^2 + 1)\cos 2(\alpha + \Psi)\right], \end{aligned} \tag{3.16f}$$

$$\begin{aligned} H_+^{(3/2,\mathrm{SO})} = \upsilon^3\Bigg[&s_\theta c_\theta(2\delta\chi_a^x + (2 - \nu)\chi_s^x) + \left(\frac{4}{3}(1 + c_\theta{}^2)\delta\chi_a^z + \frac{4}{3}((1 + c_\theta{}^2) + \nu(1 - 5c_\theta{}^2))\chi_s^z - s_\theta c_\theta(2\delta\chi_a^x + (2 + 7\nu)\chi_s^x)\right) \\ &\times\cos 2(\alpha + \Psi) - s_\theta c_\theta(2\delta\chi_a^y + (2 - \nu)\chi_s^y)\sin 2(\alpha + \Psi)\Bigg] + \upsilon^2\iota s_\theta\Bigg[c_\theta\left(-c_\theta{}^2 + 4 + \left(3c_\theta{}^2 + \frac{2}{3}\right)\nu\right)\cos(\alpha + 2\Psi) \\ &+ c_\theta\left(-\frac{c_\theta{}^2}{3} - 1 + (c_\theta{}^2 + 3)\nu\right)\cos(3\alpha + 2\Psi) + c_\theta^3\left(\frac{16}{3} - 16\nu\right)\cos(3\alpha + 4\Psi) - (\chi_a^y + \delta\chi_s^y)\sin\Psi\Bigg] \\ &+ \upsilon\iota^2 s_\theta\delta\Bigg[\left(-\frac{3c_\theta{}^2}{16} + \frac{9}{16}\right)\cos(\alpha - \Psi) - \left(\frac{11c_\theta{}^2}{32} + \frac{23}{32}\right)\cos(\alpha + \Psi) + \frac{27}{32}(c_\theta{}^2 + 1)\cos 3(\alpha + \Psi) - \frac{1}{32}(c_\theta{}^2 + 1) \\ &\times\cos(3\alpha + \Psi) + \left(-\frac{45}{32} + \frac{135c_\theta{}^2}{32}\right)\cos(\alpha + 3\Psi)\Bigg] + \iota^3 c_\theta s_\theta\left[\frac{1}{2}\cos(\alpha - 2\Psi) + \frac{5}{6}\cos(\alpha + 2\Psi)\right] \end{aligned} \tag{3.16g}$$

$$H_\times^{(0)} = -2c_\theta \sin 2(\alpha + \Psi), \tag{3.17a}$$

$$H_\times^{(1/2)} = \upsilon\delta c_\theta s_\theta\left[-\frac{9}{4}\sin 3(\alpha + \Psi) + \frac{3}{4}\sin(\alpha + \Psi)\right], \tag{3.17b}$$

$$H_\times^{(1)} = \upsilon^2 c_\theta\left[\left(-\frac{4c_\theta{}^2}{3} + \frac{17}{3} + \left(-\frac{13}{3} + 4c_\theta{}^2\right)\nu\right)\sin 2(\alpha + \Psi) + s_\theta{}^2\left(-\frac{8}{3} + 8\nu\right)\sin 4(\alpha + \Psi)\right], \tag{3.17c}$$

$$\begin{aligned} H_\times^{(3/2)} = \upsilon^3 c_\theta\Bigg[&\delta s_\theta\left(\left(-\frac{21}{32} + \frac{5c_\theta{}^2}{96}\right) + \left(-\frac{5c_\theta{}^2}{48} + \frac{23}{48}\right)\nu\right)\sin(\alpha + \Psi) - 4\pi \sin 2(\alpha + \Psi) \\ &+ \delta s_\theta\left(\left(-\frac{135c_\theta{}^2}{64} + \frac{603}{64}\right) + \left(-\frac{171}{32} + \frac{135c_\theta^2}{32}\right)\nu\right)\sin 3(\alpha + \Psi) + \delta s_\theta\left(\frac{625}{192}(2\nu - 1)s_\theta{}^2\right)\sin 5(\alpha + \Psi)\Bigg], \end{aligned} \tag{3.17d}$$

$$H_\times^{(1/2,\mathrm{SO})} = -2\iota s_\theta \sin(\alpha + 2\Psi), \tag{3.17e}$$

$$\begin{aligned} H_\times^{(1,\mathrm{SO})} = {}& \upsilon^2[(\chi_a^y + \delta\chi_s^y)\cos(\alpha + \Psi) + c_\theta(c_\theta(\chi_a^x + \delta\chi_s^x) - s_\theta(\chi_a^z + \delta\chi_s^z))\sin(\alpha + \Psi)] \\ &+ \iota\upsilon\delta\left[s_\theta{}^2 \sin\Psi - \left(\frac{c_\theta{}^2}{2} + \frac{1}{4}\right)\sin(2\alpha + \Psi) + \left(-\frac{9}{4} + \frac{9c_\theta{}^2}{2}\right)\sin(2\alpha + 3\Psi)\right] + \iota^2 c_\theta \sin 2(\alpha + \Psi), \end{aligned} \tag{3.17f}$$

$$\begin{aligned} H_\times^{(3/2,\mathrm{SO})} = {}& \upsilon^3\Bigg[s_\theta(2\delta\chi_a^y + (2 - \nu)\chi_s^y)(1 + \cos 2(\alpha + \Psi)) + \left(\frac{8}{3}c_\theta\delta\chi_a^z + c_\theta\left(\frac{8}{3} - \left(\frac{4}{3} + 4c_\theta^2\right)\nu\right)\chi_s^z\right. \\ &\left. - s_\theta(2\delta\chi_a^x + (2 + (3 + 4c_\theta{}^2)\nu)\chi_s^x)\right)\sin 2(\alpha + \Psi)\Bigg] + \iota\upsilon^2 s_\theta\Bigg[(c_\theta(\chi_a^x + \delta\chi_s^x) - s_\theta(\chi_a^z + \delta\chi_s^z))\sin\Psi \\ &+ \left((-3c_\theta{}^2 + 6) + \left(-\frac{16}{3} + 9c_\theta{}^2\right)\nu\right)\sin(\alpha + 2\Psi) + \left(-\left(c_\theta{}^2 + \frac{1}{3}\right) + (3c_\theta{}^2 + 1)\nu\right)\sin(3\alpha + 2\Psi) \\ &+ \left(\left(-\frac{8}{3} + 8c_\theta{}^2\right) + (-24c_\theta{}^2 + 8)\nu\right)\sin(3\alpha + 4\Psi)\Bigg] + \iota^2\upsilon\delta c_\theta s_\theta\Bigg[\frac{3}{8}\sin(\alpha - \Psi) - \frac{17}{16}\sin(\alpha + \Psi) \\ &+ \frac{27}{16}\sin 3(\alpha + \Psi) - \frac{1}{16}\sin(3\alpha + \Psi) + \frac{45}{16}\sin(\alpha + 3\Psi)\Bigg] + \iota^3\left[\frac{1}{2}s_\theta \sin(\alpha - 2\Psi) + \frac{5}{6}s_\theta \sin(\alpha + 2\Psi)\right], \end{aligned} \tag{3.17g}$$

To complete the waveforms, one should determine the following quantities,

- **Invariant velocity:** $v(t)$
- **Total phase:** $\Psi(t)$
- **Precession phase:** $\alpha(t)$
- **Inclination angle:** $\iota(t)$
- **Spin vectors:** $\chi_i = \mathbf{S}_i / M_i^2$ $(i = 1, 2)$
- **Amplitude factors:** $F^{(n)} = M v^{n+2} / 2D_L$ $(n = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$: **PN-order**$)$,

by solving the following equations simultaneously,

- **Spin-precession eqns:** $\dot{\mathbf{S}}_1 = \mathbf{\Omega}_1 \times \mathbf{S}_1$, $\dot{\mathbf{S}}_2 = \mathbf{\Omega}_2 \times \mathbf{S}_2$;
  $\mathbf{\Omega}_{1,2} = M^{2/3} \omega_{\rm orb}^{5/3} \left(\frac{3}{4} + \frac{\nu}{2} \mp \frac{3}{4}\delta\right) \hat{\mathbf{L}}_N$, $\nu = \frac{M_1 M_2}{M^2}$, $\delta = \frac{M_1 - M_2}{M}$, $M = M_1 + M_2$
- **Spin/angular momentum evolution eqn:** $\dot{\hat{\mathbf{L}}}_N = -\frac{v}{\nu M^2} \left(\dot{\mathbf{S}}_1 + \dot{\mathbf{S}}_2\right)$; $v \equiv (M \omega_{\rm orb})^{1/3}$
- **Evolution eqn for orbital frequency:**

$$\frac{\dot{\omega}_{\rm orb}}{\omega_{\rm orb}^2} = \frac{96}{5} \nu v^5 \left\{1 - \left(\frac{743}{336} + \frac{11}{4}\nu\right) v^2 + \left[\left(\frac{19}{3}\nu - \frac{113}{12}\right) \chi_s \cdot \hat{\mathbf{L}}_N - \frac{113}{12} \delta \chi_a \cdot \hat{\mathbf{L}}_N\right] v^3 + 4\pi v^3\right\}; \;\; \chi_{s/a} = \frac{1}{2}(\chi_1 \pm \chi_2)$$

A. Equal-mass binaries

For binaries with $M_1 = M_2 = \frac{M}{2}$, $\delta = \frac{M_1 - M_2}{M} = 0$, $\nu = \frac{M_1 M_2}{M^2} = \frac{1}{4}$, we solve the following equations simultaneously:

- Spin-precession eqns: with $\mathbf{S} \equiv \mathbf{S}_1 + \mathbf{S}_2$ and $\bar{\mathbf{S}} \equiv \mathbf{S}_1 - \mathbf{S}_2$

$$\dot{\mathbf{S}} = \mathbf{\Omega} \times \mathbf{S}, \quad \dot{\bar{\mathbf{S}}} = \mathbf{\Omega} \times \bar{\mathbf{S}} \quad ; \quad \mathbf{\Omega} \equiv \tfrac{7}{8} M^{2/3} \omega_{\rm orb}^{5/3} \hat{\mathbf{L}}_N$$

- Spin/angular momentum evolution eqn: $\dot{\hat{\mathbf{L}}}_N = -\dfrac{4v}{M^2}\dot{\mathbf{S}}$; $v \equiv (M\omega_{\rm orb})^{1/3}$
- Evolution eqn for orbital frequency:

$$\frac{\dot{\omega}_{\rm orb}}{\omega_{\rm orb}^2} = \frac{24}{5} v^5 \left[1 - \frac{487}{168} v^2 + \left(4\pi - \frac{47}{6}\chi_s \cdot \hat{\mathbf{L}}_N\right) v^3\right]; \; \chi_s \equiv \frac{2\mathbf{S}}{M^2}$$

In the basis $\{\vec{e}_{x'}(t), \vec{e}_{y'}(t), \vec{e}_{z'}(t) = \hat{\mathbf{L}}_N\}$ we obtain trivial solutions:

$$\mathbf{S} = \mathbf{S}_{\rm o} = \text{const}, \; \bar{\mathbf{S}} = \bar{\mathbf{S}}_{\rm o} = \text{const} \quad \Rightarrow \quad \chi_s = \chi_{so} = 2\mathbf{S}_{\rm o}/M^2, \; \chi_a = \chi_{ao} = 2\bar{\mathbf{S}}_{\rm o}/M^2$$

Then project the solutions into $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$ basis via:

$$\begin{bmatrix} \vec{e}_{x'}(t) \\ \vec{e}_{y'}(t) \\ \vec{e}_{z'}(t) \end{bmatrix} = \begin{bmatrix} -\sin\alpha & \cos\alpha & 0 \\ -\cos\iota\cos\alpha & -\cos\iota\sin\alpha & \sin\iota \\ \sin\iota\cos\alpha & \sin\iota\sin\alpha & \cos\iota \end{bmatrix} \begin{bmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{bmatrix}$$

The following are determined:

- $v(t) = \frac{1}{2}\Theta^{-1/8}\left[1 + \frac{487}{4032}\Theta^{-1/4} + \left(-\frac{\pi}{10} + p\frac{47\chi_{so}\cos\beta_{\rm o}}{240}\right)\Theta^{-3/8} + \mathcal{O}\left(\Theta^{-1/2}\right)\right]$; $\Theta \equiv \frac{t_{\rm c}-t}{20M}$
  ($t_{\rm c}$: **coalescence time**)
- $\Psi(t) = -4\Theta^{5/8}\left[1 + \left(\frac{2435}{4032} - p\frac{35}{96}\right)\Theta^{-1/4} + \left(-\frac{3\pi}{4} + p\frac{59\chi_{so}\cos\beta_{\rm o}}{64}\right)\Theta^{-3/8} + \mathcal{O}\left(\Theta^{-1/2}\right)\right]$;
- $\alpha(t) = -p\frac{35}{24}\Theta^{3/8}\left[1 + \frac{3}{2}\chi_{so}\cos\beta_{\rm o}\Theta^{-1/8} + \mathcal{O}\left(\Theta^{-1/4}\right)\right]$
- $\iota \approx p\chi_s\sin\beta_{\rm o}\Theta^{-1/8}\left[1 + \frac{487}{4032}\Theta^{-1/4} + \left(-\frac{\pi}{10} + \frac{47}{240}\chi_{so}\cos\beta_{\rm o}\right)\Theta^{-3/8} + \mathcal{O}\left(\Theta^{-1/2}\right)\right]$
- $\begin{bmatrix}\chi_s^x \\ \chi_s^y \\ \chi_s^z\end{bmatrix} = \begin{bmatrix}-\chi_{so}\sin\beta_{\rm o}\cos\left(\alpha(t)\right) \\ -\chi_{so}\sin\beta_{\rm o}\sin\left(\alpha(t)\right) \\ \chi_{so}\cos\beta_{\rm o}\end{bmatrix}, \quad \begin{bmatrix}\chi_a^x \\ \chi_a^y \\ \chi_a^z\end{bmatrix} = \begin{bmatrix}-\chi_{ao}\sin\bar\beta_{\rm o}\cos\left(\alpha(t)\right) \\ -\chi_{ao}\sin\bar\beta_{\rm o}\sin\left(\alpha(t)\right) \\ \chi_{ao}\cos\bar\beta_{\rm o}\end{bmatrix}$
- $F^{(0)} = \frac{Mv^2}{2D_L} = \frac{M}{8D_L}\Theta^{-1/4}$, $F^{(1/2)} = \frac{Mv^3}{2D_L} = \frac{M}{16D_L}\Theta^{-3/8}$,
  $F^{(1)} = \frac{Mv^4}{2D_L} = \frac{M}{32D_L}\Theta^{-1/2}\left(1 + \frac{487}{1008}\Theta^{-1/4}\right)$,
  $F^{(3/2)} = \frac{Mv^5}{2D_L} = \frac{M}{64D_L}\Theta^{-5/8}\left[1 + \frac{2435}{4032}\Theta^{-1/4} + \left(-\frac{\pi}{2} + p\frac{47\chi_{so}\cos\beta_{\rm o}}{48}\right)\Theta^{-3/8}\right]$

$p = 0$ **for non-spinning case,** $p = 1$ **for spinning case,** $\chi_{so} \equiv 2S_{\rm o}/M^2 =$ **const**, $\chi_{ao} \equiv 2\bar{S}_{\rm o}/M^2 =$ **const** ; $\mathbf{S} \equiv \mathbf{S}_1 + \mathbf{S}_2 = \mathbf{S}_{\rm o} =$ **const**, $\bar{\mathbf{S}} \equiv \mathbf{S}_1 - \mathbf{S}_2 = \bar{\mathbf{S}}_{\rm o} =$ **const**, $\beta_{\rm o} \equiv \cos^{-1}\left(\frac{\mathbf{L}_N\cdot\mathbf{S}}{L_N S}\right) =$ **const**, $\bar\beta_{\rm o} \equiv \cos^{-1}\left(\frac{\mathbf{L}_N\cdot\bar{\mathbf{S}}}{L_N \bar{S}}\right) =$ **const** $\Rightarrow$ **all determined by initial conditions**

B. Unequal-mass binaries

For binaries with $M_1 \neq M_2$, $\delta = \frac{M_1 - M_2}{M}$, $\nu = \frac{M_1 M_2}{M^2} = \frac{1-\delta^2}{4}$, we solve the following equations simultaneously:

- Spin-precession eqns: with $\mathbf{S} \equiv \mathbf{S}_1 + \mathbf{S}_2$ and $\bar{\mathbf{S}} \equiv \mathbf{S}_1 - \mathbf{S}_2$

$$\dot{\mathbf{S}} = \left(1 - \tfrac{\delta^2}{7}\right) \boldsymbol{\Omega} \times \mathbf{S} - \tfrac{6\delta}{7} \boldsymbol{\Omega} \times \bar{\mathbf{S}}, \quad \dot{\bar{\mathbf{S}}} = \left(1 - \tfrac{\delta^2}{7}\right) \boldsymbol{\Omega} \times \bar{\mathbf{S}} - \tfrac{6\delta}{7} \boldsymbol{\Omega} \times \mathbf{S} \quad ; \quad \boldsymbol{\Omega} \equiv \tfrac{7}{8} M^{2/3} \omega_{\rm orb}^{5/3} \hat{\mathbf{L}}_N$$

- Spin/angular momentum evolution eqn: $\dot{\hat{\mathbf{L}}}_N = -\dfrac{v}{\nu M^2} \dot{\mathbf{S}}$; $v \equiv (M\omega_{\rm orb})^{1/3}$
- Evolution eqn for orbital frequency:

$$\frac{\dot{\omega}_{\rm orb}}{\omega_{\rm orb}^2} = \frac{96}{5} \nu v^5 \left\{ 1 - \left(\frac{743}{336} + \frac{11}{4} \nu \right) v^2 + \left[\left(\frac{19}{3} \nu - \frac{113}{12} \right) \chi_s \cdot \hat{\mathbf{L}}_N - \frac{113}{12} \delta \chi_a \cdot \hat{\mathbf{L}}_N \right] v^3 + 4\pi v^3 \right\}; \quad \chi_s \equiv \frac{2\mathbf{S}}{M^2}, \, \chi_a \equiv \frac{2\bar{\mathbf{S}}}{M^2}$$

Solve the equations in the basis $\{\vec{e}_{x'}(t), \vec{e}_{y'}(t), \vec{e}_{z'}(t) = \hat{\mathbf{L}}_N\}$ first and project the solutions into $\{\vec{e}_x, \vec{e}_y, \vec{e}_z\}$ basis via:

$$\begin{bmatrix} \vec{e}_{x'}(t) \\ \vec{e}_{y'}(t) \\ \vec{e}_{z'}(t) \end{bmatrix} = \begin{bmatrix} -\sin\alpha & \cos\alpha & 0 \\ -\cos\iota\cos\alpha & -\cos\iota\sin\alpha & \sin\iota \\ \sin\iota\cos\alpha & \sin\iota\sin\alpha & \cos\iota \end{bmatrix} \begin{bmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{bmatrix}$$

In the basis $\{\vec{e}_{x'}(t), \vec{e}_{y'}(t), \vec{e}_{z'}(t) = \hat{\mathbf{L}}_N\}$ **the equations reduce to**

$$S^{x'} = 0, \quad S^{z'} = \mathbf{S}\cdot\hat{\mathbf{L}}_N = \mathbf{const} = S_\mathrm{o}\cos\beta_\mathrm{o}, \quad \bar{S}^{z'} = \bar{\mathbf{S}}\cdot\hat{\mathbf{L}}_N = \mathbf{const} = \bar{S}_\mathrm{o}\cos\bar{\beta}_\mathrm{o},$$
$$\dot{S}^{y'} = -\tfrac{6\delta}{7}\Omega\bar{S}^{x'}, \quad S^{y'}\bar{S}^{y'} = \mathbf{const} = S_\mathrm{o}^{y'}\bar{S}_\mathrm{o}^{y'}, \quad \dot{\bar{S}}^{x'} = -\tfrac{6\delta}{7}\Omega\left(S_\mathrm{o}^{y'}\bar{S}_\mathrm{o}^{y'}\right)^2\left(S^{y'}\right)^{-3} + \tfrac{6\delta}{7}\Omega S^{y'},$$
$$\dot{\alpha} = \left(1 - \tfrac{\delta^2}{7}\right)\Omega - \tfrac{6\delta}{7}\Omega\left(\bar{S}^{y'}/S^{y'}\right)\ ; \quad \Omega \equiv \tfrac{7}{8}M^{2/3}\omega_\mathrm{orb}^{5/3} = \tfrac{7}{8}M^{-1}v^5$$

Solving for $S^{y'}$**,**

$$\ddot{S}^{y'} - \frac{32\nu}{M}v^8\dot{S}^{y'} - \left(\frac{6\delta}{7}\Omega\right)^2\left(S_\mathrm{o}^{y'}\bar{S}_\mathrm{o}^{y'}\right)^2\left(S^{y'}\right)^{-3} + \left(\frac{6\delta}{7}\Omega\right)^2 S^{y'} = 0 \ \Rightarrow$$ **anharmonic** **oscillator**

Ignoring the damping term, and splitting $S^{y'} = S_\mathrm{o}^{y'} + \Delta^{y'}$**, and then taking** $\left(S^{y'}\right)^{-3} = \left(S_\mathrm{o}^{y'}\right)^{-3}\left[1 - 3\left(S_\mathrm{o}^{y'}\right)^{-1}\Delta^{y'} + \mathcal{O}\left(\Delta^2\right)\right]$ **(treating** $\mathcal{O}\left(\Delta^2\right)$ **as higher order perturbation pieces),**

$$\ddot{\Delta}^{y'} + \left(\frac{6\delta}{7}\sqrt{1 + 3\left(\frac{\bar{S}_\mathrm{o}^{y'}}{S_\mathrm{o}^{y'}}\right)^2}\,\Omega\right)^2\left(\Delta^{y'} + \frac{S_\mathrm{o}^{y'}\left[\left(S_\mathrm{o}^{y'}\right)^2 - \left(\bar{S}_\mathrm{o}^{y'}\right)^2\right]}{\left(S_\mathrm{o}^{y'}\right)^2 + 3\left(\bar{S}_\mathrm{o}^{y'}\right)^2}\right) = 0 \ \Rightarrow$$ **harmonic** **oscillator**

Solutions:

$$
\begin{aligned}
S^{x'} &= 0,\\
S^{y'} &= S_{\mathrm{o}}^{y'} + \mathcal{A}\sin\left(f\left(v\right)+\varphi\right) + C,\\
S^{z'} &= S_{\mathrm{o}}\cos\beta_{\mathrm{o}},\\
\bar{S}^{x'} &= \sqrt{1+3\left(\bar{S}_{\mathrm{o}}^{y'}/S_{\mathrm{o}}^{y'}\right)^{2}}\mathcal{A}\cos\left(f\left(v\right)+\varphi\right),\\
\bar{S}^{y'} &= -\left(\bar{S}_{\mathrm{o}}^{y'}/S_{\mathrm{o}}^{y'}\right)\mathcal{A}\sin\left(f\left(v\right)+\varphi\right) + \bar{S}_{\mathrm{o}}^{y'} - C\left(\bar{S}_{\mathrm{o}}^{y'}/S_{\mathrm{o}}^{y'}\right),\\
\bar{S}^{z'} &= \bar{S}_{\mathrm{o}}\cos\bar{\beta}_{\mathrm{o}},
\end{aligned}
$$

where

$$
f\left(v\right) \equiv -\frac{5\delta}{32\left(1-\delta^{2}\right)}\sqrt{1+3\left(\frac{\bar{S}_{\mathrm{o}}^{y'}}{S_{\mathrm{o}}^{y'}}\right)^{2}}v^{-3}, \quad C \equiv \frac{S_{\mathrm{o}}^{y'}\left[\left(\bar{S}_{\mathrm{o}}^{y'}\right)^{2}-\left(S_{\mathrm{o}}^{y'}\right)^{2}\right]}{\left(S_{\mathrm{o}}^{y'}\right)^{2}+3\left(\bar{S}_{\mathrm{o}}^{y'}\right)^{2}}, \quad \mathcal{A} \equiv \sqrt{C^{2}+\frac{\left(\bar{S}_{\mathrm{o}}^{x'}\right)^{2}\left(S_{\mathrm{o}}^{y'}\right)^{2}}{\left(S_{\mathrm{o}}^{y'}\right)^{2}+3\left(\bar{S}_{\mathrm{o}}^{y'}\right)^{2}}},
$$

$$
S_{\mathrm{o}} = \sqrt{S_{1}^{2}+S_{2}^{2}+2S_{1}S_{2}\left(\cos\theta_{1}\cos\theta_{2}+\sin\theta_{1}\sin\theta_{2}\cos\left(\phi_{1}-\phi_{2}\right)\right)},
$$

$$
\bar{S}_{\mathrm{o}} = \sqrt{S_{1}^{2}+S_{2}^{2}-2S_{1}S_{2}\left(\cos\theta_{1}\cos\theta_{2}+\sin\theta_{1}\sin\theta_{2}\cos\left(\phi_{1}-\phi_{2}\right)\right)},
$$

$$
\beta_{\mathrm{o}} = \arccos\left(\frac{S_{1}\cos\theta_{1}+S_{2}\cos\theta_{2}}{S_{\mathrm{o}}}\right), \quad \bar{\beta}_{\mathrm{o}} = \arccos\left(\frac{S_{1}\cos\theta_{1}-S_{2}\cos\theta_{2}}{\bar{S}_{\mathrm{o}}}\right),
$$

$$
S_{\mathrm{o}}^{y'} = S_{\mathrm{o}}\sin\beta_{\mathrm{o}}, \quad \bar{S}_{\mathrm{o}}^{x'} = -\frac{2S_{1}S_{2}\csc\beta_{\mathrm{o}}\sin\theta_{1}\sin\theta_{2}\sin(\phi_{1}-\phi_{2})}{\bar{S}_{\mathrm{o}}}, \quad \bar{S}_{\mathrm{o}}^{y'} = \frac{\left(S_{1}^{2}-S_{2}^{2}\right)\csc\beta_{\mathrm{o}}}{\bar{S}_{\mathrm{o}}} - \bar{S}_{\mathrm{o}}\cot\beta_{\mathrm{o}}\cos\bar{\beta}_{\mathrm{o}}
$$

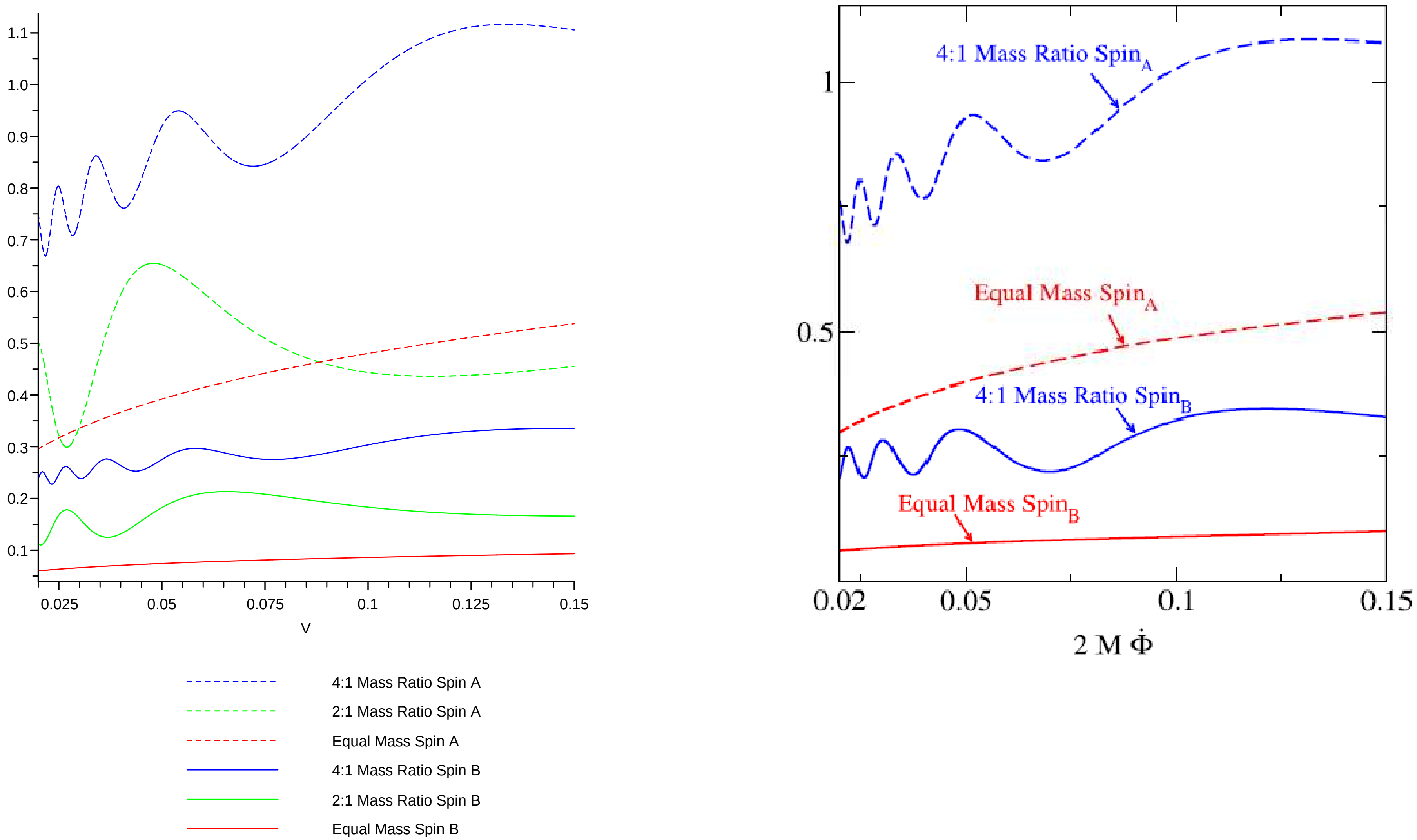
Determine the following quantities accordingly:

- Invariant velocity: v
- Total phase: Ψ
- Precession phase: α
- Inclination angle: ι
- Amplitude factors: $F^{(n)} = Mv^{n+2}/2D_L$ ($n = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$: **PN-order**),

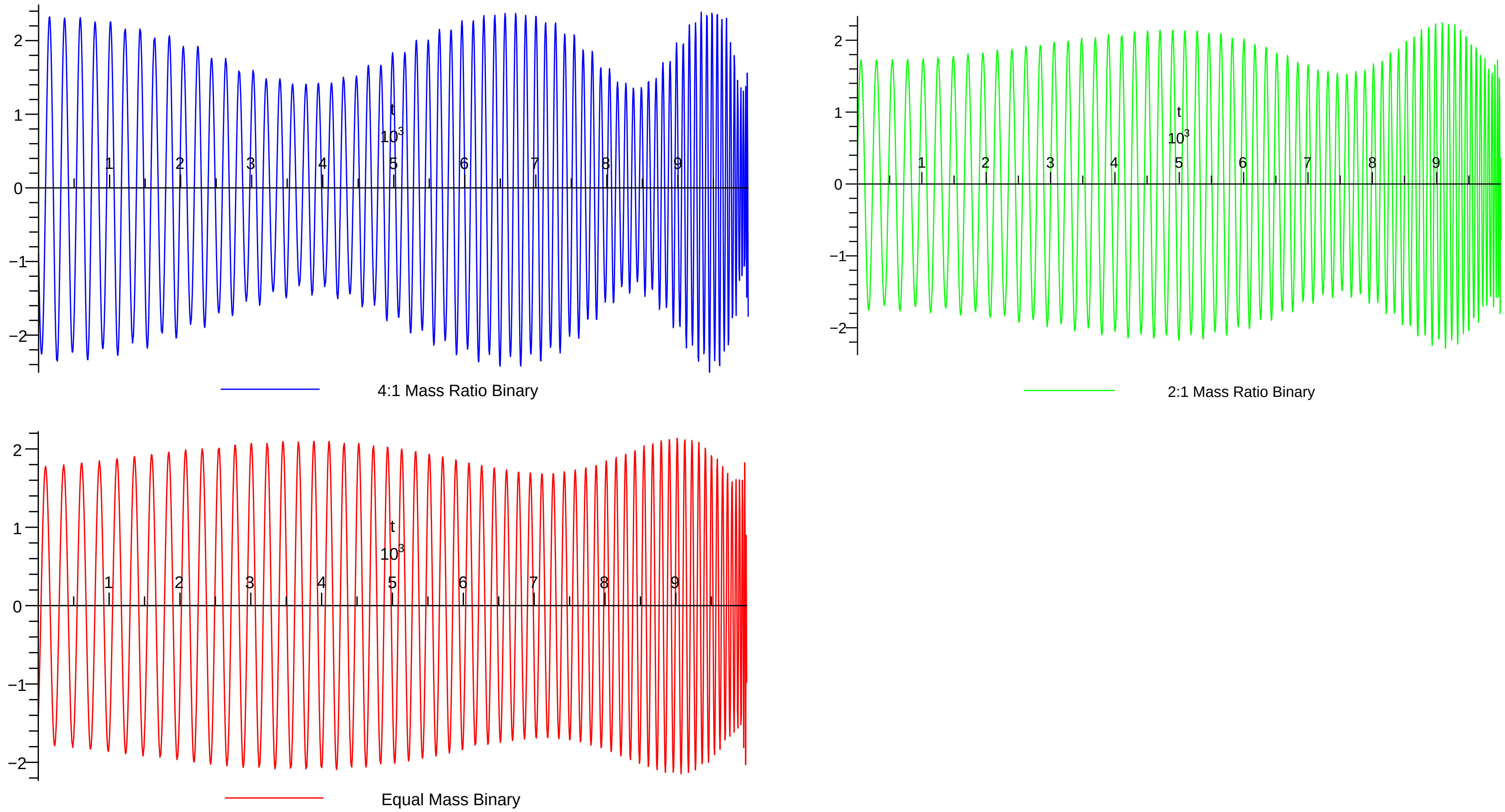
In particular, the inclination angle ι undergoes bobbing (oscillation) unlike the equal-mass case:

$$\begin{aligned} \iota &= \arcsin\left(\frac{S^{y'}}{J_0}\right) \\ &\approx \arcsin\left(\frac{S_{\rm o}^{y'} + \mathcal{A}\sin\left(f\left(v\right)+\varphi\right)+C}{\sqrt{L_N^2 + 2L_N S_{\rm o}^{z'} + \left(S_{\rm o}^{x'}\right)^2 + \left(S_{\rm o}^{y'}\right)^2 + \left(S_{\rm o}^{z'}\right)^2}}\right); \; L_N = \frac{\left(1-\delta^2\right)M^2}{4v} \end{aligned}$$

$f\left(v\right) \equiv -\frac{5\delta}{32\left(1-\delta^2\right)}\sqrt{1+3\left(\frac{\bar{S}_{\rm o}^{y'}}{S_{\rm o}^{y'}}\right)^2} v^{-3}$, $C \equiv \frac{S_{\rm o}^{y'}\left[\left(\bar{S}_{\rm o}^{y'}\right)^2 - \left(S_{\rm o}^{y'}\right)^2\right]}{\left(S_{\rm o}^{y'}\right)^2 + 3\left(\bar{S}_{\rm o}^{y'}\right)^2}$, $\mathcal{A} \equiv \sqrt{C^2 + \frac{\left(\bar{S}_{\rm o}^{x'}\right)^2\left(S_{\rm o}^{y'}\right)^2}{\left(S_{\rm o}^{y'}\right)^2 + 3\left(\bar{S}_{\rm o}^{y'}\right)^2}}$ will characterize the bobbing patterns: mass-ratios and initial spin configurations are main factors.



<LEFT> Plots of inclination angle ι vs. $V \equiv 2M\dot{\Phi} \approx 2v^3$ for 4:1 mass-ratio binary ($\delta = \frac{3}{5}$), 2:1 mass-ratio binary ($\delta = \frac{1}{3}$) and equal-mass binary ($\delta = 0$) constructed from analytic solutions (to 1st order perturbation).
<RIGHT> Numerical plots of ι vs. $V \equiv 2M\dot{\Phi} \approx 2v^3$ for 4:1 mass-ratio binary ($\delta = \frac{3}{5}$) and equal-mass binary ($\delta = 0$) from *Arun et. al*, PRD79-104023 (2009). Initial spin orientations relative to orbital angular momentum are chosen to be Spin$_\mathbf{A}$ $= \{\theta_1 = \pi/2, \phi_1 = 0, \theta_2 = \pi/2, \phi_2 = \pi/2\}$, Spin$_\mathbf{B}$ $= \{\theta_1 = \pi/6, \phi_1 = \pi/4, \theta_2 = \pi/6, \phi_2 = \pi\}$.



Plots of $h_+(t)$ for 4:1 mass-ratio binary ($\delta = \frac{3}{5}$), 2:1 mass-ratio binary ($\delta = \frac{1}{3}$) and equal-mass binary ($\delta = 0$) through 0.5 PN order: with initial spin orientation relative to orbital angular momentum,
$\{\theta_1 = \pi/2, \phi_1 = 0, \theta_2 = \pi/2, \phi_2 = \pi/2\}$.

4. Conclusions and discussions

- $h_+(t)$ and $h_\times(t)$ for spinning binaries with unequal masses can be determined analytically via perturbation - through 1.5 PN order (spin-orbit interactions) and the spins exhibit bobbing effects, which depend on the mass-ratios and initial spin orientations relative to the orbital angular momentum.

- The bobbing is caused by a purely kinematical effect of spin, and in fact ubiquitous in relativistic mechanics, occurring independently of the type of force holding two spinning bodies in orbit: an electromagnetic analog studied by ***Gralla et. al*** [Phys. Rev. D 81, 104012 (2010)].

- When spin-spin interactions (2 PN) and radiation reaction (2.5 PN) are involved, spinning binaries with general mass ratios assume much more complicated dynamic evolutions. The spin evolutions can be described by nonlinear oscillators and damping (***Racine*** [Phys. Rev. D 78, 044021 (2008)]), which makes the analysis of higher-order PN effects much more difficult.

- Recently, some new techniques (e.g. effective field theory approach) have provided more systematic computational schemes for higher-order PN terms (beyond 2.5 PN). Collaborations with Y. Chen (Caltech), B. L. Hu (Maryland) and C. Galley (JPL) are currently on-going.