Conformal thin sandwich puncture initial data with spectral method

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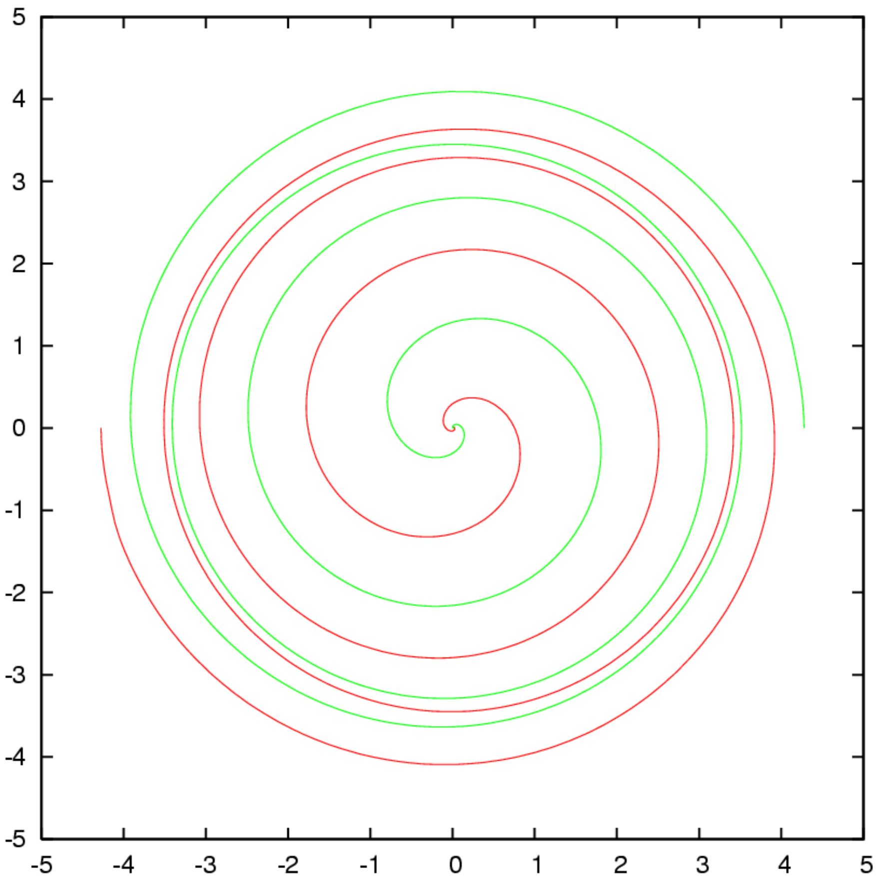
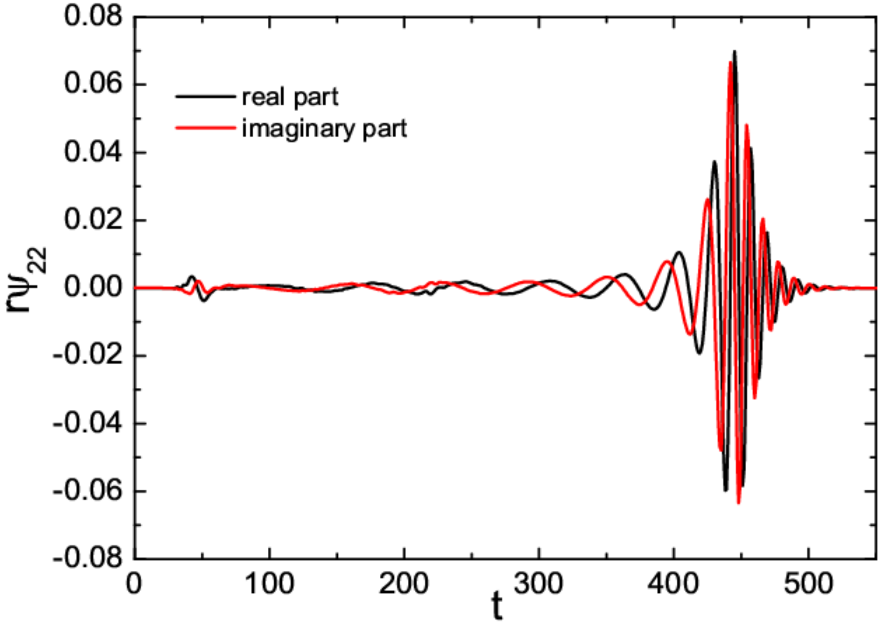
- Goal : construct BH initial data for GR simulation

- by Solving constraint equations for initial 3-geometry
  - via conformal thin sandwich formulism
  - boundary value problem for elliptic PDEs
  - Finite difference vs Spectral method

Quasi-circular binary BH evolution





Merging time scale : 100M
~10ms for 20 $M_{\odot}$ BH

3+1 Einstein's eq

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} {}^{(4)}R = 8\pi G T_{\mu\nu}$$

(Hyperbolic-type) Evolution equation :

$$\begin{aligned} \partial_t \gamma_{ij} &= -2\alpha K_{ij} + \pounds_\beta \gamma_{ij} \\ \partial_t K_{ij} &= \alpha \left\{ R_{ij} - {}^{(4)}R_{ij} - 2K_{ik}K^k_j + KK_{ij} \right\} - D_i D_j \alpha + \pounds_\beta K_{ij} \end{aligned}$$

(Elliptic-type) Constraints :

$$\begin{aligned} R + K^2 - K_{ij}K^{ij} &= 16\pi E && \text{Hamiltonain constraint} \\ D_j(K^{ij} - \gamma^{ij}K) &= 8\pi p^i && \text{Momentum constraint} \end{aligned}$$

Conformal Thin Sandwich

Conformal factorization

$$\tilde{\gamma}_{ij} = \psi^{-4}\gamma_{ij}$$

$$\tilde{A}^{ij} \equiv \psi^4 A^{ij} = \frac{1}{2\alpha}\left(\tilde{\mathbb{L}}\beta^{ij} - \dot{\tilde{\gamma}}^{ij}_{TF}\right) \qquad \text{cf:} \hat{A}^{ij} \equiv \psi^{10}A^{ij} = \hat{A}^{ij}_{TT} + \tilde{\mathbb{L}}W^{ij}$$

Constraint equations [York 1999]

$$\tilde{D}^2\psi = -\frac{1}{8}\psi^5\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{12}\psi^5K^2 + \frac{1}{8}\psi\tilde{R} \tag{1}$$

$$\tilde{D}^2_j\left(\tilde{\mathbb{L}}\beta\right)^{ij} = 2\tilde{A}^{ij}(\tilde{D}_j\alpha - 6\alpha\tilde{D}_j\ln\psi) - \tilde{D}_j\dot{\tilde{\gamma}}^{ij}_{TF} + \frac{4}{3}\alpha\tilde{D}^iK \tag{2}$$

using $\dot{K}$ to determine the lapse [Pfeiffer&York 2002]

$$\tilde{D}^2(\alpha\psi) = \alpha\psi\left[\frac{7}{8}\psi^4\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{8}\tilde{R} + \frac{5}{12}\psi^4K^2\right] - \psi^5\left(\partial_t K - \beta^k\tilde{D}_kK\right) \tag{3}$$

Free $\{\tilde{\gamma}^{ij}, \dot{\tilde{\gamma}}^{ij}_{TF}, K, \dot{K}\}$ $\rightarrow$ Constrained $\{\alpha, \beta^i, \psi\}$

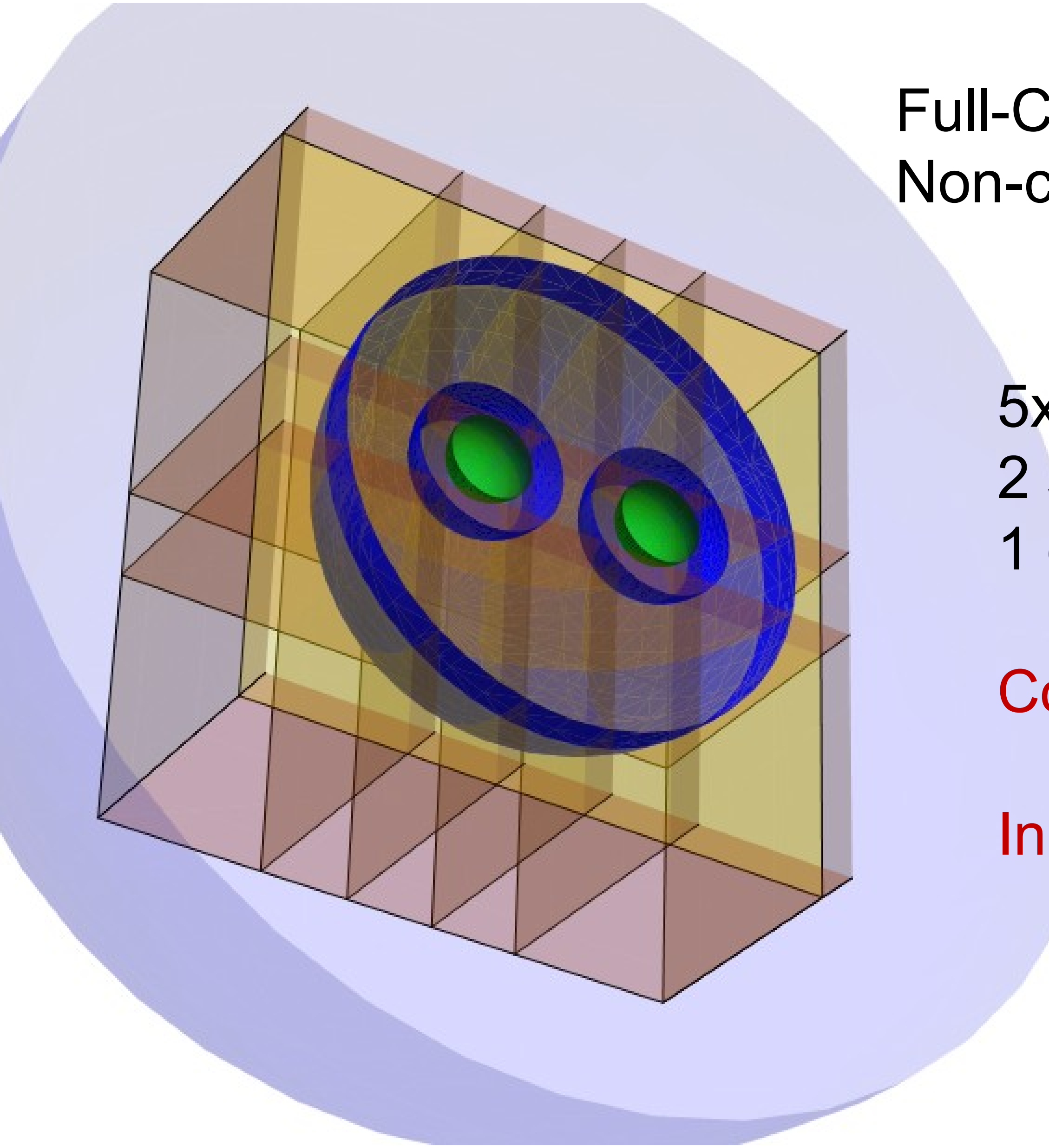
Various way to handle singularity…

Excision full CTS data

Puncture data
(conformal flatness
w/o specified initial gage)

- CTS-Puncture data
  - retain conformal flatness
  - gauge determined by CTS

Multi-domain spectral solver for binary BH



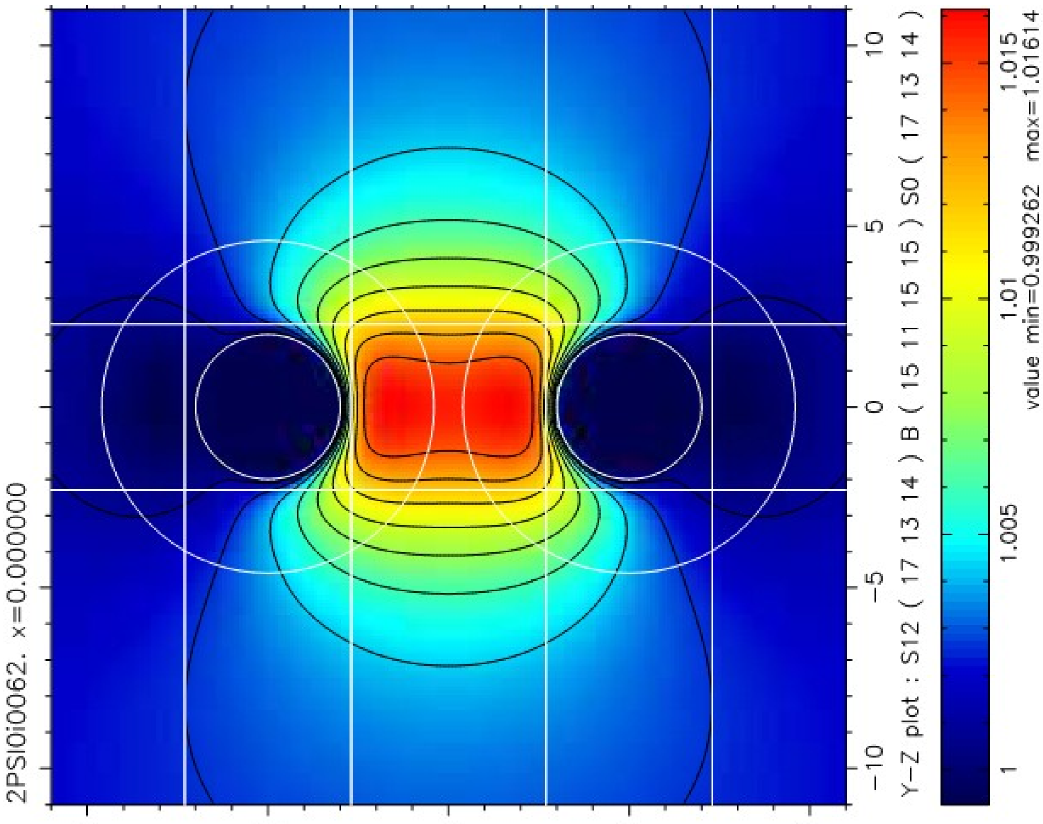
Full-CTS
Non-conformal flat background

5x3x3 block domains
2 sphere
1 common shell

Compactified boundary :
Asymptotic flat
Inner boundary :
AH, Kerr-Schild, etc.

[Pfeiffer+ 2006]

Superposed Kerr BH as conformal background



Conformal factor ~1 within ~ 2%

Puncture data

Conformal flatness & $K = 0 \Longrightarrow$

$$
\begin{aligned}
\tilde{D}_i \hat{A}^{ij} &= 0 \\
\tilde{D}^2 \psi &= -\frac{1}{8} \psi^{-7} \hat{A}_{ij} \hat{A}^{ij}
\end{aligned}
$$

$\Longrightarrow$Bowen-York curvature

$$\hat{A}^{ij} = \frac{f(P)}{r^2} + \frac{g(S)}{r^3}$$

$\Longrightarrow \psi = \psi_s + u$ as perturbation of isotropic BHs solution

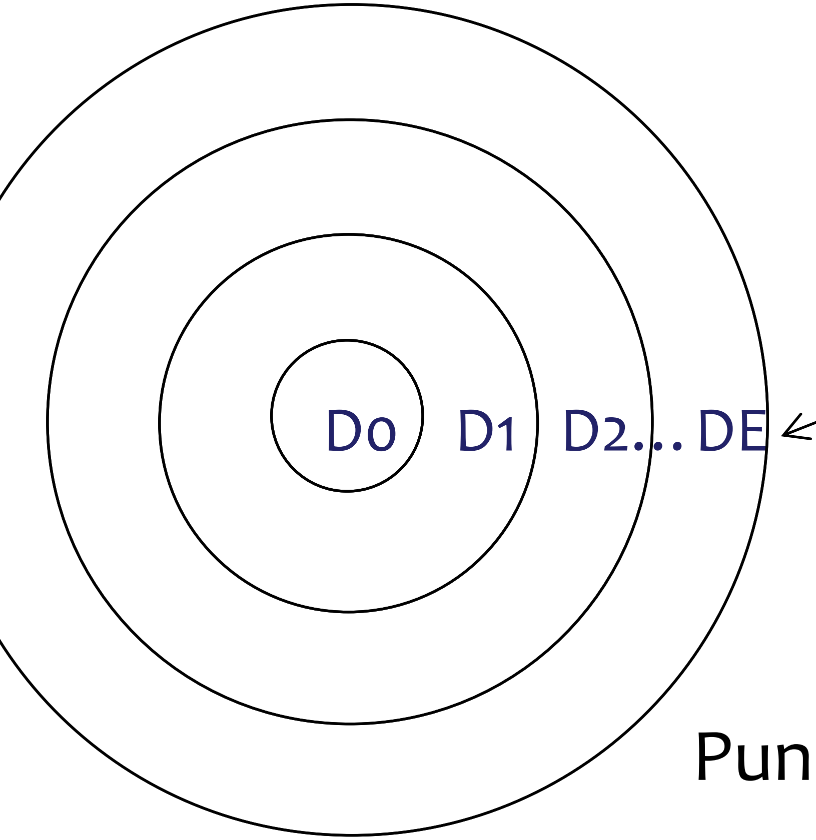
$$\psi_s = 1 + \sum_a \frac{m_a}{2\,|r - r_a|}$$

Regular part solved by the puncture equation

$$D^2 u = -\frac{1}{8} (\psi_s + u)^{-7} \hat{A}^{ij} \hat{A}_{ij}$$

[80 Bowen, York; 97 Brandt, Brugmann]

Multi-domain spectral puncture solver



Compactified outer boundary
Asymptotic flat (u=0)

Puncture equation split for each hole

$$\tilde{D}^2 u_a = -\frac{1}{8}(\psi_s + u)^{-7} \hat{A}_a^{ij} \hat{A}_{ij}$$

Conformal Thin Sandwich Puncture

$$\psi = 1 + \frac{m_i}{2r_i} + u$$

$$\tilde{\alpha}\psi^7 = \alpha\psi = 1 - \frac{c_i m_i}{2r_i} + v$$

$$\boxed{\begin{aligned} \tilde{D}^2 u_a &= -\frac{1}{8}\frac{1}{2\tilde{\alpha}}\left(\tilde{\mathbb{L}}\beta_a^{ij}\right)\hat{A}_{ij}\psi^{-7} \\ \tilde{D}^2 v_a &= \frac{7}{8}\frac{1}{2}\left(\tilde{\mathbb{L}}\beta_a^{ij}\right)\hat{A}_{ij}\psi^{-1} \\ \tilde{D}_j\left(\tilde{\mathbb{L}}\beta_a\right)^{ij} &= \left(\tilde{\mathbb{L}}\beta_a^{ij}\right)D_j \ln\tilde{\alpha} \end{aligned}}$$

$$\hat{A}^{ij} = \frac{1}{2\tilde{\alpha}}\left(\tilde{\mathbb{L}}\beta^{ij}\right)$$

u,v solvable w/o specified inner boundary
& point-boundary for Shift

[03 Laguna, 05 Hannam & Cook]

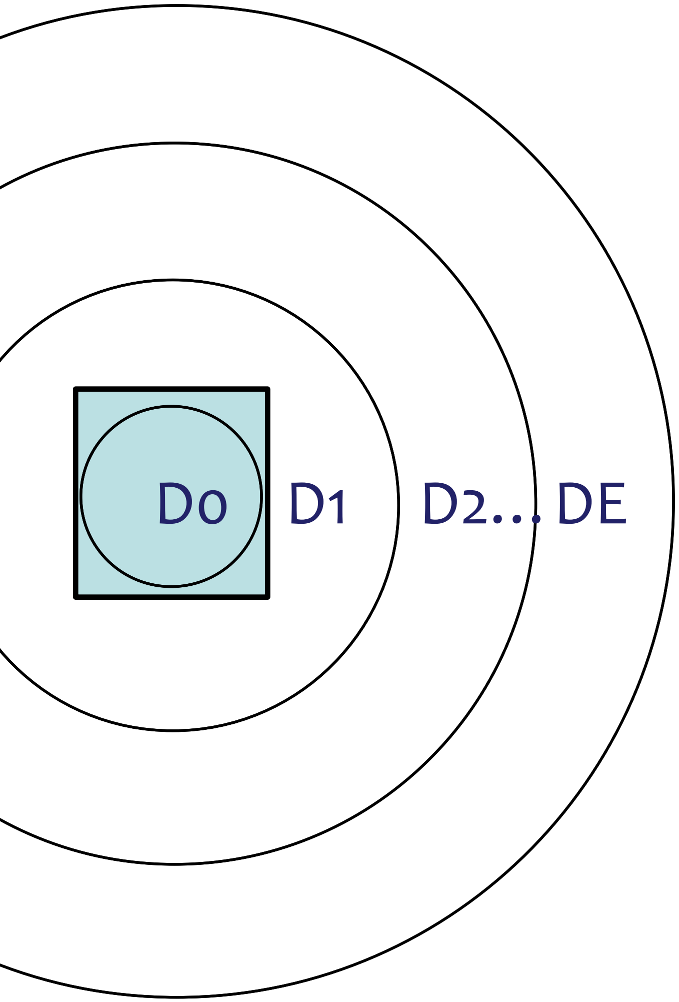
Has exact solution for single boosted BH (eg. in z-axis)

$$
\begin{aligned}
\beta^i &= \frac{-P}{5(m+2r)^6}\left\{xzf, yzf, \frac{1}{4}\left(4z^2 f + g\right)\right\} \\
f &\sim r^3 \\
g &\sim r^5
\end{aligned}
$$

Regular and Smooth at puncture, so is its derivative
→ Spectral method applicable

[03 Laguna]

Multiple CTS-Puncture

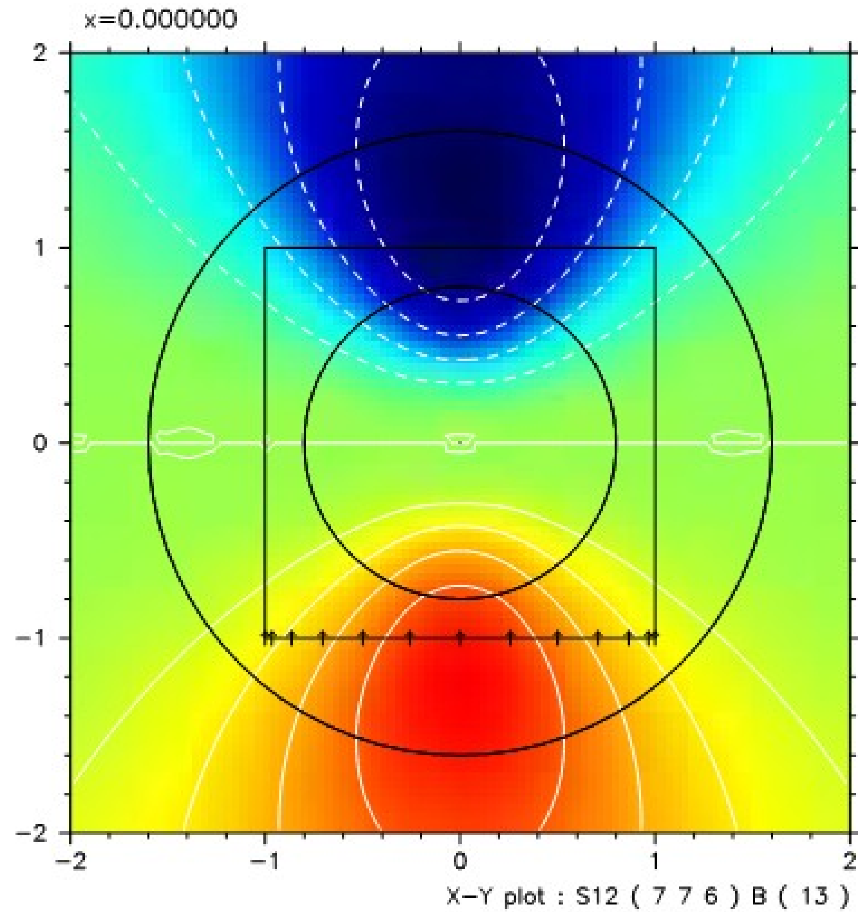


point boundary for shift at puncture to be the exact solution

→ Singular in terms of polar coordinate in shell domain

→ cover Cartesian block for central region

extrinsic curvature
for single black hole



Conclusion

Excision CTS data

Implement the multi-domain spectral solver for binary BH

Puncture data

Develop spectral multi-BH solver.
Restrict to conformal flatness

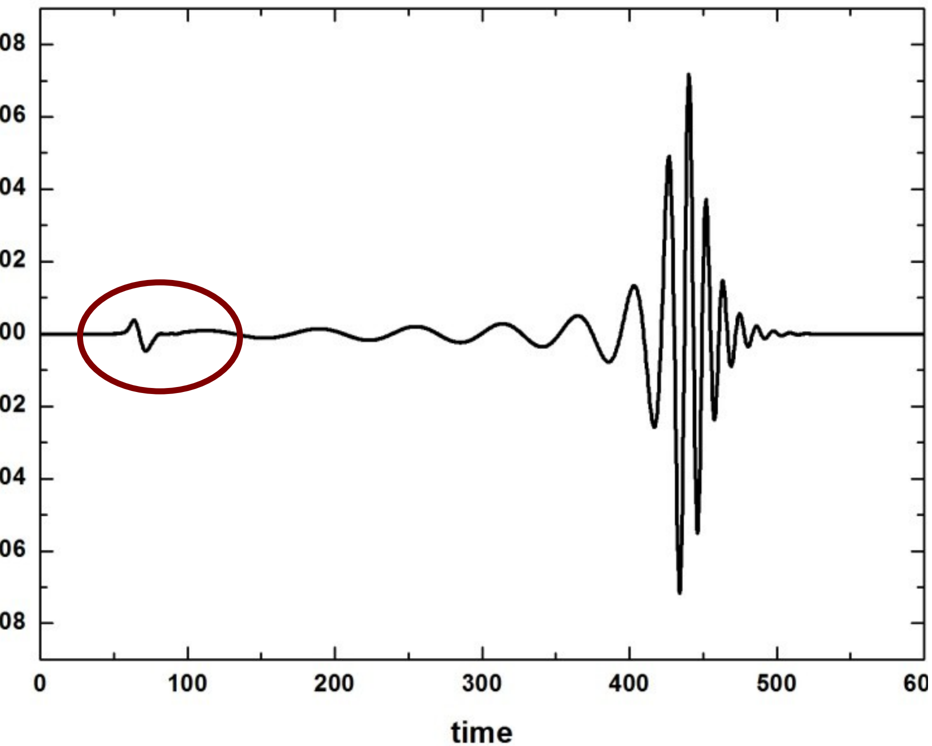
- **CTS-Puncture data**

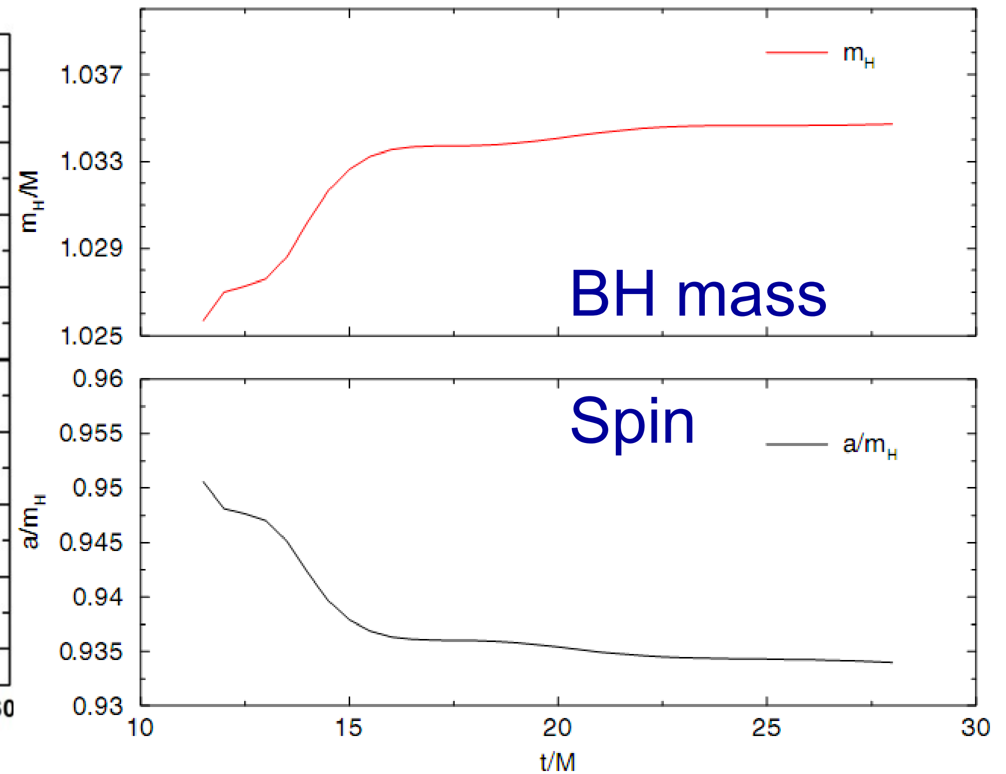
  Realize a multi-BH spectral solver.
  Be conformal flat, consistent with CTS

Initial Data with incorrect GW content

Spurious radiation

fail to evolve Extreme BH





Might due to

Conformal flatness

Bowen-York curvature