Quasi-local energy for the Kerr space

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Hamiltonian and quasi-local quantities

- Action $S = \int_M \mathcal{L}$, (M, g) is a spacetime manifold with metric g.
- First order Lagrangian 4-form for a k-form field φ is $\mathcal{L} = \mathrm{d}\varphi \wedge p - \Lambda(\varphi, p)$
- The variation of $\mathcal{L}$

$$\delta\mathcal{L} = \mathrm{d}(\delta\varphi \wedge p) + \delta\varphi \wedge \frac{\delta\mathcal{L}}{\delta\varphi} + \frac{\delta\mathcal{L}}{\delta p} \wedge \delta p. \tag{1}$$

- Define the Euler-Lagrange equations by Hamilton's principle

$$(E.L._p)\ \frac{\delta\mathcal{L}}{\delta p} := \mathrm{d}\varphi - \partial_p\Lambda = 0, \tag{2}$$

$$(E.L._\varphi)\ \frac{\delta\mathcal{L}}{\delta\varphi} := -\varsigma\mathrm{d}p - \partial_\varphi\Lambda = 0,\ \varsigma := (-1)^k. \tag{3}$$

Hamiltonian and quasi-local quantities

- By the diffeomorphism invariant requirement (implies $L_{\mathbf{N}} \longrightarrow \delta$)

$$\mathrm{d}\iota_{\mathbf{N}}\mathcal{L} = L_{\mathbf{N}}\mathcal{L} = \mathrm{d}(L_{\mathbf{N}}\varphi \wedge p) + L_{\mathbf{N}}\varphi \wedge \frac{\delta\mathcal{L}}{\delta\varphi} + \frac{\delta\mathcal{L}}{\delta p} \wedge L_{\mathbf{N}}p,$$

$$L_{\mathbf{N}}\varphi \wedge \frac{\delta\mathcal{L}}{\delta\varphi} + \frac{\delta\mathcal{L}}{\delta p} \wedge L_{\mathbf{N}}p + \mathrm{d}(\underbrace{L_{\mathbf{N}}\varphi \wedge p - \iota_{\mathbf{N}}\mathcal{L}}_{\mathcal{H}}) \equiv 0. \qquad (4)$$

(Apply Cartan formula: $L_{\mathbf{N}} = \mathrm{d}\iota_{\mathbf{N}} + \iota_{\mathbf{N}}\mathrm{d}$)

- Hamiltonian is defined on the spatial hypersurface by

$$H(\mathbf{N}) = \int_{\Sigma} \mathcal{H} = \int_{\Sigma} (N^{\mu}\mathcal{H}_{\mu} + \mathrm{d}\mathcal{B}), \qquad (5)$$

where

$$N^{\mu}\mathcal{H}_{\mu} = \iota_{\mathbf{N}}\varphi \wedge (E.L._{\varphi}) + (E.L._{p}) \wedge \iota_{\mathbf{N}} p,$$
$$\mathcal{B}(\mathbf{N}) = \iota_{\mathbf{N}}\varphi \wedge p$$

We obtain $N^{\mu}\mathcal{H}_{\mu} = \iota_{\mathbf{N}}\varphi \wedge (E.L._{\varphi}) + (E.L._{p}) \wedge \iota_{\mathbf{N}} p$, which vanishes on shell. Consequently,

$$H(\mathbf{N}) = \oint_{\partial\Sigma} \mathcal{B}(\mathbf{N}). \qquad (6)$$

- Conserved quasilocal quantities and the corresponding symmetries
    1. Quasilocal quantity H: the Hamiltonian boundary term $\mathcal{B}$ integrated over a closed space-like 2−surface.
    2. Conservation and symmetries

conserved quantity H	$\leftrightarrow$	invariant under **N**
energy		time-like
momentum		space-like
angular mumentum		rotation
center of mass		boost

- Boundary Variation principle
  From the variation of the Hamiltonian:

$$\delta H = \int_\Sigma \delta\mathcal{H} = \int_\Sigma (\cdots) + \oint_{\partial\Sigma} \mathcal{C}.$$

$$\begin{aligned}
\delta\mathcal{H} &= -\delta\varphi\wedge L_{\mathbf{N}}p + L_{\mathbf{N}}\varphi\wedge\delta p - \iota_{\mathbf{N}}[\delta\varphi\wedge(E.L._{\varphi}) + (E.L._{p})\wedge\delta p] \\
&\quad + \mathrm{d}[\iota_{\mathbf{N}}(\delta\varphi\wedge p)] \\
&= -\delta\varphi\wedge L_{\mathbf{N}}p + L_{\mathbf{N}}\varphi\wedge\delta p + \mathrm{d}[\iota_{\mathbf{N}}(\delta\varphi\wedge p)] \quad \text{“on shell”}
\end{aligned}$$

If $\oint_{\partial\Sigma}\mathcal{C} = \oint_{\partial\Sigma}\iota_{\mathbf{N}}(\delta\varphi\wedge p)$ vanishes, then the Hamiltonian is functional differentiable such that the Hamilton equations can be written

$$L_{\mathbf{N}}\varphi = \frac{\delta\mathcal{H}}{\delta p}, \quad L_{\mathbf{N}}p = -\frac{\delta\mathcal{H}}{\delta\varphi}$$

- Boundary condition comes from

$$\oint_{\partial\Sigma} \mathcal{C} = \oint_{\partial\Sigma} \iota_{\mathbf{N}}(\delta\varphi \wedge p) = 0,$$

which means $\mathcal{C} = \iota_{\mathbf{N}}(\delta\varphi \wedge p)$ vanishes on the closed 2−surface $\partial\Sigma$.

Note that if the 3−region Σ is compact without boundary, then $\oint_{\partial\Sigma} \mathcal{B}$ of the Hamiltonian is automatically vanishing, which implies the Hamiltonian is certainly well-defined (i.e. functionally differentiabe). But we are usually interested in the region which is asymptotically flat ($\mathbb{R}^3$ is non compact), so we need the boundary conditions.

- C.M. Chen's improved boundary terms
  With $\Delta\alpha := \alpha - \bar{\alpha}$, replace the natural boundary term $\iota_{\mathbf{N}}\varphi \wedge p$ by

$$\mathcal{B}(\mathbf{N}) = \iota_{\mathbf{N}} \left\{ \begin{array}{c} \varphi \\ \bar{\varphi} \end{array} \right\} \wedge \Delta p - \varsigma \Delta\varphi \wedge \iota_{\mathbf{N}} \left\{ \begin{array}{c} p \\ \bar{p} \end{array} \right\} \qquad (7)$$

the associated Hamiltonian variation boundary term has a symplectic form

$$\delta\mathcal{H}(\mathbf{N}) \sim d \left[\left\{ \begin{array}{c} \iota_{\mathbf{N}}\delta\varphi \wedge \Delta p \\ -\iota_{\mathbf{N}}\Delta\varphi \wedge \delta p \end{array} \right\} + \varsigma \left\{ \begin{array}{c} -\Delta\varphi \wedge \iota_{\mathbf{N}}\delta p \\ \delta\varphi \wedge \iota_{\mathbf{N}}\Delta p \end{array} \right\} \right]. \qquad (8)$$

[Chen, Nester, Tung, PRD **72**, 104020, (21)-(24)]

Regge-Teitelboim like asymptotic fall off and parity conditions:

$$\Delta\varphi \sim \mathcal{O}^{+}(r^{-1}) + \mathcal{O}^{-}(r^{-2}), \tag{9}$$
$$\Delta p \sim \mathcal{O}^{-}(r^{-2}) + \mathcal{O}^{+}(r^{-3}), \tag{10}$$

with $N^\mu = N_0^\mu + \lambda_0^\mu{}_\nu x^\nu$, where N_0^μ, $\lambda_0^{\mu\nu} = \lambda_0^{[\mu\nu]}$ are constant up to $\mathcal{O}^{+}(r^{-1})$, being asymptotically Killing, the quasi-local quantities have finite values, and the boundary term in the Hamiltonian vanishes asymptotically.

- Each distinct choice of Hamiltonian boundary quasi-local expression is associated with a physically distinct boundary condition.
- In order to accommodate suitable boundary conditions one must, in general, introduce certain reference values $\bar{p}$, $\bar{\varphi}$, which represent the ground state of the field—the "vacuum" (or background field) values.

Application to General Relativity

- Lagrangian density is

$$\mathcal{L}[\mathbf{g}, \partial\mathbf{g}] = R\eta,$$

where η is the 4-D volume element $\sqrt{-g}\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$. In the differential form language and using the orthonormal frame basis rather then the coordinate basis

$$\mathcal{L}[\vartheta^\mu, \Gamma^\mu{}_\nu] = R^\alpha{}_\beta \wedge \eta_\alpha{}^\beta, \tag{11}$$

where $R^\alpha{}_\beta = \mathrm{d}\Gamma^\alpha{}_\beta + \Gamma^\alpha{}_\lambda \wedge \Gamma^\lambda{}_\beta$ is the curvature two-form, and $\eta_\alpha{}^\beta = \frac{1}{2}h^{\beta\lambda}\epsilon_{\alpha\lambda\mu\nu}\vartheta^\mu \wedge \vartheta^\nu$ is the dual basis two-form, $h^{\mu\nu}$ is the flat metric diag$(-1, +1, +1, +1)$.

Field variables are $\vartheta^\alpha \leftrightarrow \mathbf{g}$ and $\Gamma^\mu{}_\nu \leftrightarrow \partial\mathbf{g}$. Recall the first order Lagrangian $\mathcal{L} = \mathrm{d}\varphi \wedge p - \Lambda$ implies

$$\mathcal{L} = D\Gamma^\mu{}_\nu \wedge \rho_\mu{}^\nu + D\vartheta^\alpha \wedge \tau_\alpha - V^\mu{}_\nu \wedge (\rho_\mu{}^\nu - \frac{1}{2\kappa}\eta_\mu{}^\nu),$$

where τ_α, $\rho_\mu{}^\nu$ are conjugate momenta w.r.t ϑ^α and $\Gamma^\mu{}_\nu$; $V^\mu{}_\nu$ is the role multiplier.(Note that $\tau_\alpha = 0$ as the construction go back to the original Lagrangian.)

$$\begin{aligned}
&\delta\rho_\mu{}^\nu : \ D\Gamma^\mu{}_\nu = R^\alpha{}_\beta = V^\mu{}_\nu; \\
&\delta\tau_\alpha : \ D\vartheta^\alpha = 0 \text{ (torsion free)}; \\
&\delta V^\mu{}_\nu : \ \rho_\mu{}^\nu = \frac{1}{2\kappa}\eta_\mu{}^\nu, \\
&\delta\vartheta^\alpha : \ D\tau_\alpha = R^\alpha{}_\beta \wedge \eta_\alpha{}^\beta{}_\mu = G_\mu \text{ (Einstein three form)}; \\
&\delta\Gamma^\mu{}_\nu : \ D\rho_\mu{}^\nu = D\eta_\mu{}^\nu = 0 \text{ (followed by torsion free)}.
\end{aligned}$$

Preferred Boundary Term for GR

Chen, Nester, Tung, Phys Lett A **203**, 5 (1995)
[also found by Katz, Bičák & Lynden-Bel]

$$\mathcal{B}(N) = \frac{1}{2\kappa}(\Delta\Gamma^{\alpha}{}_{\beta} \wedge \iota_n \eta_{\alpha}{}^{\beta} + \bar{D}_{\beta} N^{\alpha} \Delta\eta_{\alpha}{}^{\beta})$$

It corresponds to holding the metric fixed on the boundary:

$$\delta\mathcal{H}(N) \sim d i_N (\Delta\Gamma^{\alpha}{}_{\beta} \wedge \delta\eta_{\alpha}{}^{\beta}) \qquad (12)$$

The choice of reference

Given a spacetime manifold $(M, \mathbf{g})$, and pick a local coordinate system $\{x^\mu\}$. The corresponding physical variables are the metric $g_{\mu\nu}$ and the connection (Christoffel symbol) $\Gamma^\mu{}_{\nu\lambda}$.

- Take a closed space-like two surface $\mathcal{S}$
- Define the reference variables
  1. The reference metric $\bar{\mathbf{g}}$, and
  2. the reference connection $\bar{\boldsymbol{\Gamma}}$ (note that it is not unique)
- **Then** the quasi-local expression is covariant.

The strategy of choosing reference

1. Directly defined from the physical variables: e.g. for Kerr case here, let $m = a = 0$

$$\bar{\mathbf{g}} := \mathbf{g}(m = a = 0); \quad \bar{\mathbf{\Gamma}} := \mathbf{\Gamma}(m = a = 0). \tag{13}$$

2. Determined by the local transformation only on $\mathcal{S}$.(For the spherical symmetric case, see [Phys.Lett.A **374** 3599 (arXiv:0909.2754), and PRD **84** 084047 (arXiv:1109.4738)])

$$\bar{\mathbf{g}} := \bar{g}_{ab} \mathrm{d}y^a \mathrm{d}y^b; \quad \bar{\mathbf{\Gamma}} = 0, \tag{14}$$

$$\bar{g}_{\mu\nu} = \bar{g}_{ab} \frac{\partial y^a}{\partial x^\mu} \frac{\partial y^b}{\partial x^\nu}; \quad \bar{\Gamma}^{\mu}{}_{\nu} = -\mathrm{d}\left(\frac{\partial x^\mu}{\partial y^a}\right) \frac{\partial y^a}{\partial x^\nu}.$$

Kerr space

$$
\begin{aligned}
\mathrm{d}s^2 \;=\; & -\frac{\Delta - a^2\sin^2\theta}{\rho^2}\mathrm{d}t^2 + \frac{4mar\sin^2\theta}{\rho^2}\mathrm{d}t\mathrm{d}\phi \\
& +\frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)^2 - a^2\Delta\sin^2\theta\right]\mathrm{d}\phi^2 \\
& +\frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2, \qquad (15)
\end{aligned}
$$

where $\Delta = r^2 + a^2 - 2mr$, $\rho^2 = r^2 + a^2\cos^2\theta$.

Quasi-local energy for $\mathbf{N} = \frac{\partial}{\partial_t}$

Using the strategy 1, for the constant t, r surface, the choice $\mathbf{N} = \partial_t$ for the quasi-local energy is

$$E = \frac{3a^4 + 3r^3(r-2m) + a^2 r(5r-6m)}{6r(a^2 + r(r-2m))} - \frac{[a^2 + r(r-2m)]^2 \arctan(\frac{a}{r})}{2a(a^2 + r(r-2m))}. \tag{16}$$

This is the exact value depending on m, a and r. For $r \to \infty$, it is the ADM energy

$$\lim_{r\to\infty} E(\partial_t) = m, \tag{17}$$

and for $a = 0$ one gets the result of Schwarzschild spacetime

$$E_{\mathrm{Sch}} = m. \tag{18}$$

Quasi-local energy for the extremal case $a = m$

Let $r = km$ for non-negative real constant k, then the quasi-local energy becomes

$$E = \frac{3 - 6k + 5k^2 - 6k^3 + 3k^4}{6k(k-1)^2} m - \frac{\tan^{-1}(1/k)}{2} m. \tag{19}$$

Note that it is linear in m.

Plot $E = \frac{3-6k+5k^2-6k^3+3k^4}{6k(k-1)^2} - \frac{\tan^{-1}(1/k)}{2}$ for $m = 1$:

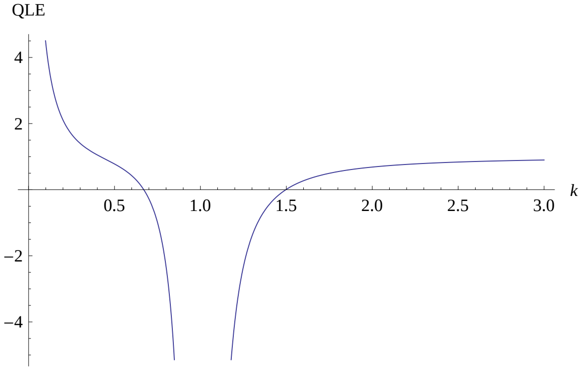

Figure: extremal Kerr quasi-local energy

The event horizon is at $k = 1$ and two roots appear at $k \approx 0.67$ and $k \approx 1.5$.

Quasi-local angular momentum for $\mathbf{N} = \frac{\partial}{\partial_\phi}$

$\mathbf{N} = \partial_\phi$ for the quasi-local angular momentum is

$$E = am. \tag{20}$$

Boost transformation in $t - r$ plane

Consider a new time coordinate τ such that $\mathrm{d}\tau = \mathrm{d}t + f(r)\mathrm{d}r$. Let the Kerr metric

$$\begin{aligned} \mathrm{d}s^2 &= F\mathrm{d}t^2 + 2G\mathrm{d}t\mathrm{d}\phi + H\mathrm{d}\phi^2 + R\mathrm{d}r^2 + \rho\mathrm{d}\phi^2 \\ &= F\mathrm{d}\tau^2 + 2Ff\mathrm{d}\tau\mathrm{d}r + 2G\mathrm{d}\tau\mathrm{d}\phi + 2Gf\mathrm{d}r\mathrm{d}\phi \\ &\quad +(Ff^2 + R)\mathrm{d}r^2 + \rho\mathrm{d}\theta^2 + H\mathrm{d}\phi^2. \end{aligned}$$

Boundary expression for $\mathbf{N} = \partial/\partial_\tau (= \partial_t)$

$$\begin{aligned} 2\kappa\mathcal{B} &= [\sqrt{-g}(g^{\beta 2}\Delta\Gamma^1{}_{\beta 2} + g^{\beta 3}\Delta\Gamma^1{}_{\beta 3} - g^{\beta 1}\Delta\Gamma^2{}_{\beta 2} - g^{\beta 1}\Delta\Gamma^3{}_{\beta 3}) \\ &\quad +\bar{\Gamma}^0{}_{\beta 0}\Delta(\sqrt{-g}g^{\beta 1}) - \bar{\Gamma}^1{}_{\beta 0}\Delta(\sqrt{-g}g^{\beta 0})]\mathrm{d}\theta \wedge \mathrm{d}\phi. \end{aligned}$$

$$\mathbf{g} = \begin{pmatrix} F & Ff & 0 & G \\ Ff & Ff^2 + R & 0 & Gf \\ 0 & 0 & \Sigma & 0 \\ G & Gf & 0 & H \end{pmatrix},$$

$$\mathbf{g}^{-1} = \begin{pmatrix} \frac{f^2}{R} + \frac{H}{K} & -\frac{f}{R} & 0 & -\frac{G}{K} \\ -\frac{f}{R} & \frac{1}{R} & 0 & 0 \\ 0 & 0 & \frac{1}{\Sigma} & 0 \\ -\frac{G}{K} & 0 & 0 & \frac{F}{K} \end{pmatrix}.$$

$$\Gamma^1{}_{03} = -\frac{G_r}{2R}, \quad \Gamma^1{}_{22} = -\frac{\Sigma_r}{2R}, \quad \Gamma^1{}_{33} = -\frac{H_r}{2R}, \quad \Gamma^2{}_{02} = 0,$$

$$\Gamma^2{}_{12} = \frac{\Sigma_r}{2\Sigma}, \quad \Gamma^3{}_{03} = 0, \quad \Gamma^3{}_{13} = \frac{FH_r - GG_r}{2(FH - G^2)},$$

$$\Gamma^0{}_{00} = \frac{fF_r}{2R}, \quad \Gamma^0{}_{10} = \frac{f^2 F_r(FH - G^2) + R(F_r H - GG_r)}{2R(FH - G^2)},$$

$$\Gamma^1{}_{00} = -\frac{F_r}{2R}, \quad \Gamma^1{}_{10} = -\frac{fF_r}{2R}, \quad \Gamma^1{}_{30} = -\frac{G_r}{2R}.$$

$\Rightarrow E(\partial_\tau)$ is independent of $f(r)$.
The angular momentum is also invariant under the boost transformation $\mathrm{d}\tau = \mathrm{d}t + f(r)\mathrm{d}r$.

Remark

1. Note that under the boost transformation $d\tau = dt + f(r)dr$, the choice of displacement $\mathbf{N} = \partial_\tau$ is invariant. Let new coordinate is $\{\tau, R\}$ and old one is $\{t, r\}$. Under the transformation

$$d\tau = dt + f(r)dr, \quad dR = dr,$$

which implies $\boxed{\partial_\tau = \partial_t}$, $\partial_R = -f(r)\partial_t + \partial_r$.

2. This kind of boost transformation includes the Eddington-Finkelstein and Painlevé-Gullstrand coordinates.

Remark

3. The quasi-local angular momentum is a constant $E(\partial_\phi) = am$ for $\mathbf{N} = \partial_\phi$, which is independent of the boost transformation.

$$\begin{aligned}
2\kappa\mathcal{B} &= \Delta\Gamma^{\alpha}{}_{\beta 3}\sqrt{-g}g^{\beta\gamma}\epsilon_{\alpha\gamma 32}\mathrm{d}x^3\wedge\mathrm{d}x^2 \\
&\quad +\bar{\Gamma}^{\alpha}{}_{\beta 3}\Delta(\sqrt{-g}g^{\beta\gamma})\epsilon_{\alpha\gamma 23}\mathrm{d}\theta\wedge\mathrm{d}\phi \\
&= \sqrt{-g}(g^{11}\Delta\Gamma^{0}{}_{13}+g^{01}\Delta\Gamma^{0}{}_{03}-g^{00}\Delta\Gamma^{1}{}_{03} \\
&\quad -g^{10}\Delta\Gamma^{1}{}_{13}-g^{30}\Delta\Gamma^{1}{}_{33})\mathrm{d}\theta\wedge\mathrm{d}\phi \\
&\quad +[\bar{\Gamma}^{0}{}_{13}(\Delta\sqrt{-g}g^{11})-\bar{\Gamma}^{1}{}_{03}(\Delta\sqrt{-g}g^{00})-\bar{\Gamma}^{1}{}_{33}(\Delta\sqrt{-g}g^{30}) \\
&\quad +\bar{\Gamma}^{0}{}_{03}(\Delta\sqrt{-g}g^{01})-\bar{\Gamma}^{1}{}_{13}(\Delta\sqrt{-g}g^{10})]\mathrm{d}\theta\wedge\mathrm{d}\phi.
\end{aligned}$$

Here the only contributed connection terms are

$$\Gamma^{0}{}_{13}=\frac{f^2G_r}{2R}+\frac{HG_r-GH_r}{2K},\quad \Gamma^{1}{}_{03}=-\frac{G_r}{2R},\quad \Gamma^{1}{}_{33}=-\frac{H_r}{2R}$$

$$\Gamma^{0}{}_{03}=\frac{fG_r}{2R},\quad \Gamma^{1}{}_{13}=-\frac{fG_r}{2R},\quad \bar{\Gamma}^{1}{}_{33}=-r\sin^2\theta.$$

$$\begin{aligned}
\text{First term} &: \sqrt{-g}\left[(HG_r-GH_r)/KR+Gr\sin^2\theta/K\right]\mathrm{d}\theta\wedge\mathrm{d}\phi, \\
\text{Second term} &: -\sqrt{-g}Gr\sin^2\theta/K\mathrm{d}\theta\wedge\mathrm{d}\phi.
\end{aligned}$$

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