Global structure of cylindrically symmetric spacetimes

Makoto Narita

Okinawa National College of Technology

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1. Singularity theorems and cosmic censorship

**Theorem 1 (Penrose)** Suppose the following conditions hold: (1) a Cauchy surface $\Sigma$ is noncompact, (2) the null convergence condition, (3) $\Sigma$ contains a closed trapped surface. Then the corresponding maximal future development $D^+(\Sigma)$ is incomplete.

**Theorem 2 (Hawking)** Suppose the following conditions hold: (1) a Cauchy surface $\Sigma$ is compact, (2) the timelike convergence condition, (3) the generic condition. Then the corresponding maximal Cauchy development $D(\Sigma)$ is incomplete.

These theorems say physically reasonable spacetimes have singularities in general.
However,

- the theorem does not say us nature of singularity.
- predictability is breakdown if singularity can be seen.

To solve these problems, one should analyze the Einstein equations by using PDE technique.

**Conjecture 1 (Strong cosmic censorship (SCC))** Generic Cauchy data sets have maximal Cauchy developments which are locally inextendible as Lorentzian manifolds.

**Remark 1** This formulation is of Moncrief and Klainerman. The original is formulated by Penrose.
To prove the SCC, one need to show

- global existence theorems in suitable coordinates,
- inextendibility.

However, it is too difficult to solve the Einstein equations without assumptions.

Therefore, we will use a **cylindrical symmetric spacetime**, which is one of the simplest inhomogeneous spacetime.
2. Field equations for cylindrically symmetric spacetimes

Cylindrical symmetric initial data

- noncompact Cauchy surfaces $\Sigma$ ($\mathbb{R}^3$-topology for spatial section),
- $U(1) \times \mathbb{R}^1$ isometry group with spacelike orbits (cylindrical symmetry),
- The metric $h$ and the second fundamental form $k$ of $\Sigma$ satisfy
  \[ \mathcal{L}_{X^a} h_{\mu\nu} = \mathcal{L}_{X^a} k_{\mu\nu} = 0, \]  
  (1)
  where $X^a$, $a = 2, 3$ are two Killing vectors that generate the isometry group.
Metric

\[ g = -e^2(\eta-U)dt^2 + e^2(\eta-U)dr^2 + e^{2U}(dx + Ady)^2 + e^{-2U}R^2dy^2, \]  

(2)

where \( R, \eta, U \) and \( A \) are functions of \( t \in (0, \infty) \) and \( r \in [0, \infty) \). Note that a metric with \( A \equiv 0 \) is given by Gowdy-Edmonds.

Constraint equations

\[ \ddot{U}^2 + U'^2 + \frac{e^{4U}}{4R^2}(\dot{A}^2 + A'^2) + \frac{R''}{R} - \frac{\eta\dot{R}}{R} - \frac{\eta'R'}{R} = 0, \]  

(3)

\[ 2\ddot{U}U' + \frac{e^{4U}}{2R^2}\dot{A}A' + \frac{\dot{R}'}{R} - \frac{\eta\dot{R}'}{R} - \frac{\eta'^{\prime}R}{R} = 0, \]  

(4)

where dot and prime denote derivative with respect to time \( t \) and \( r \), respectively.
Evolution equations

\[ \ddot{R} - R'' = 0, \]  

(5)

\[ \ddot{\eta} - \eta'' = -\dot{\eta}^2 + U'2 + \frac{e^{4U}}{4R^2}(\dot{A}^2 - A'^2), \]  

(6)

\[ \ddot{U} - U'' = -\frac{\dot{R}\dot{U}}{R} + \frac{R'U'}{R} + \frac{e^{4U}}{2R^2}(\dot{A}^2 - \alpha A'^2), \]  

(7)

\[ \ddot{A} - A'' = \frac{\dot{R}\dot{A}}{R} - \frac{R'A'}{R} - 4(\dot{A}\dot{U} - A'U'). \]  

(8)
Remark 2

- R will be fixed by gauge condition: $R = tr$.
- Equation (6) can be derived from other equations.
- As the result, the evolution equations (7) and (8) are decoupled with the constraint equations (3) and (4).

Remark 3 If $R = t$, expanding universe is obtained. Also, if $R = r$, cylindrically symmetric gravitational waves in "asymptotically flat" spacetimes are given. Thus, our choice means that cylindrically symmetric gravitational waves in expanding universe is described.

Remark 4 In the both case $R = t$ and $R = r$ with or without matter fields, global existence theorems have been proved (Andreasson, Berger, Chruściel, Isenberg, Moncrief, Rendall, Ringström, MN).
Lemma 1 The cylindrically symmetric initial data do not contain trapped surfaces which are either compact or invariant under the isometry group.

Thus, possible singularities would exist due to some other reason.
Geroch-Ernst potential

\[ \dot{A} = -Re^{-4U}w', \quad A' = -Re^{-4U}\dot{w}. \]

From this and replacing \( U \) by \( z/2 \), we have

**Constraint equations**

\[ \dot{z}^2 + z'^2 + e^{-2z}(\dot{w}^2 + w'^2) + \frac{4R''}{R} - \frac{4\dot{R}\dot{\eta}}{R} - \frac{4R'\eta'}{R} = 0 \]  
\[ 2(\dot{z}z' + e^{-2z}\dot{w}w') + \frac{4\dot{R}'}{R} - \frac{4R'\dot{\eta}}{R} - \frac{4\dot{R}\eta'}{R} = 0 \]

**(9)**  
**(10)**

**Evolution equations**

\[ \ddot{z} + \frac{\dot{R}}{R} \dot{z} - z'' - \frac{R'}{R} z' + e^{-2z}(\dot{w}^2 - w'^2) = 0, \]
\[ \ddot{w} + \frac{\dot{R}}{R} \dot{w} - w'' - \frac{R'}{R} w' - 2(\dot{z}\dot{w} - z'w') = 0 \]

**(11)**  
**(12)**
We have a wave map $\psi : (\mathcal{M}^{2+1}, G) \mapsto (\mathcal{N}^2, h)$ as follows:

$$S_{WM} = \int dt dr \sqrt{-G} G^{\alpha\beta} h_{AB} \partial_\alpha \psi^A \partial_\beta \psi^B,$$

(13)

where

$$G = -dt^2 + dr^2 + R^2 d\psi^2,$$

and

$$h = dz^2 + e^{-2z} dw^2.$$

**Remark 5** The target space is two-dimensional hyperbolic space. This is the same with Gowdy case.

The energy-momentum tensor $T_{\alpha\beta}$ for this system is given of the form:

$$T_{\alpha\beta} = \tilde{h}_{AB} \left( \partial_\alpha \psi^A \partial_\beta \psi^B - \frac{1}{2} G_{\alpha\beta} \partial_\lambda \psi^A \partial_\lambda \psi^B \right).$$

(14)
3. Global existence

**Theorem 3** Let \((M, g)\) be the maximal Cauchy development of \(C^\infty_0\) initial data for the cylindrically symmetric system. Then, \(M\) can be covered by Cauchy surfaces of constant time \(t\) with each value in the range \((0, \infty)\). Moreover, this maximal Cauchy development is timelike future geodesically complete, hence inextendible into the future direction.

**Remark 6** To prove this theorem, our spacetime will be divided into two regions, one includes \(r = 0\) and another is in \(r \geq \delta > 0\).
3.1 Region in $r \geq \delta > 0$

**Proposition 1** Suppose $r \geq \delta > 0$. Then, there is a unique map $\Psi$ satisfying the wave map equations (11) and (12).

Method of the proof: **Light cone estimate**

3.2 Near $r = 0$

**Theorem 4 (Christodoulou-Tahvildar-Zadeh)** Let $\Sigma$ be a radial symmetric Cauchy surface in $\mathbb{R}^{2+1}$ and let $\Theta_0$ and $\dot{\Theta}_0$ be any smooth radial symmetric Cauchy data for the wave map equations for a map $\Theta : (\mathbb{R}^{2+1}, \eta) \to (\mathbb{H}^2, h)$, where $\eta = -dt^2 + dr^2 + r^2d\theta^2$. Then there is a unique smooth map $\Theta : (\mathbb{R}^{2+1}, \eta) \to (\mathbb{N}, h)$ satisfying the wave map equations and assuming the Cauchy data $\Theta_0$ and $\dot{\Theta}_0$. 
Lemma 2  Let \( I = \{(t, r) \in (0, \infty) \times [0, \infty) : t > r\} \) and let \( R = tr \). Then there exists smooth coordinate \( \tau, \rho : I \leftrightarrow \mathcal{O} \subset \mathbb{R}^2 \), such that

\[
R|_I = \rho,
\]

and

\[
\left(-dt^2 + dr^2\right)|_I = \Omega^2 \left(-d\tau^2 + d\rho^2\right),
\]

for some positive function \( \Omega \in C^\infty(\mathcal{O}) \).

From Theorem 4 and Lemma 2, a global existence theorem is obtained in \( t > r \).
4. Nature of singularity (Construction of Kasner-like solutions)

We would like to know behavior of spacetimes near singularity.

*Example:* Kasner solution (spatially homogeneous and anisotropic):

$$g = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2,$$

where

$$\sum_{i=1}^{3} p_i = \sum_{i=1}^{3} p_i^2 = 1.$$

The Kretschmann invariant blows up at $t = 0$, thus Kasner spacetime is inextendible into the past direction except the case of $p_1 = 1$ and $p_2 = p_3 = 0.$
Our spacetime includes the Kasner in the sense that ours becomes the Kasner if metric functions are independent on $r$ (spatially homogeneous).

One expects that the solutions should be Kasner-like ones near singularity (BKL conjecture).
Construction of Kasner-like solutions

Consider a system of PDEs on $\mathbb{R}^{n+1}$, whose solutions are expected to have a singularity as $t \to 0$.

The Fuchsian technique:

• Decompose the unknown into a prescribed singular part and a regular part $\mathcal{U} = (u, Du, t\partial_x u)$.

• If the system can be written as a first-order Fuchsian system of the form

$$DU + N(x)\mathcal{U} = t^\alpha f(t, x, \mathcal{U}, \partial_x \mathcal{U}), \quad \alpha > 0, \quad D = t\partial_t,$$

we can apply the following theorem.
Theorem 5 (Kichenassamy-Rendall) Assume that $N$ is an analytic matrix near $x = 0$ such that there is a constant $C$ with $\| \sigma^N \| \leq C$ for $0 < \sigma < 1$, where $\sigma^N$ is the matrix exponential of $N \ln \sigma$. Also, suppose that $f$ is a locally Lipschitz function of $U$ and $\partial_x U$ which preserves analyticity in $x$ and continuity in $t$. Then, the Fuchsian system (15) has a unique solution in a neighborhood of $x = 0$ and $t = 0$ which is analytic in $x$ and continuous in $t$, and tends to zero as $t \to 0$.

Remark 7 The sufficient condition for $N$ is non-negativity of eigenvalues of $N$. 
Formal solution (=leading-order+higher-order terms):

\[ z(t, r) = z_*(r) \ln t + z_{**}(r) + t^\epsilon Z(t, r), \]
\[ w(t, r) = w_*(r) + t^{2k(r)} (w_{**}(r) + W(t, r)), \]

where \( \epsilon > 0 \) is a small constant. We call \( z_*, z_{**}, w_*, w_{**} \) asymptotic data, while \( Z \) and \( W \) are remainder (higher order in \( t \)). We can get the Fuchsian system (15) with

\[
N = \begin{bmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\epsilon^2 & 2\epsilon & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2k & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad 0 < \epsilon < \min\{2k, 2-2k\}. \]
Theorem 6 Suppose that $z_*, z_{**}, w_*$ and $w_{**}$ are real analytic functions of $r$ and $0 < \epsilon < \min\{2k, 2-2k\}$. Then, there is a unique solution of the Einstein equations of the form (16) and (17) in a neighborhood of $t = 0$ such that $Z$ and $W$ tend to zero as $t \to 0$.

Remark 8 The Kretschmann invariant blows up as $t \to 0$, thus our spacetime is inextendible into the past direction if the solution (16) and (17) is generic.
5. Nature of singularity (Cont.)

Definition 1 A second-order hyperbolic Fuchsian system is a set of partial differential equations of the form

\[ D^2\mathcal{V} + 2A D\mathcal{V} + B\mathcal{V} - t^2 K^2 \partial_x^2 \mathcal{V} = f[\mathcal{V}], \tag{19} \]

in which the function \( \mathcal{V} : (0, \delta] \times U \to \mathbb{R}^n \) is the main unknown, while coefficients \( A(x), B(x), K(t, x) \) are diagonal \( n \times n \) matrix-valued maps and a smooth in \( x \in U \) and \( t \) in the half-open interval \( (0, \delta] \), and \( f = f[\mathcal{V}](t, x) \) is an \( n \)-vector-valued map of the following form:

\[ f[\mathcal{V}](t, x) = f(t, x, \mathcal{V}(t, x), D\mathcal{V}(t, x), tK(t, x)\partial_x \mathcal{V}(t, x)). \]
We put

\[ \lambda_1 = a + \sqrt{a^2 - b}, \quad \lambda_2 = a - \sqrt{a^2 - b}, \]

and

\[ k(t, x) = t^{\beta(x)} \nu(t, x), \text{ with } \beta : U \to (-1, \infty), \nu : [0, \delta] \times U \to (0, \infty), \]

where \( a, b \) and \( k \) are eigenvalues of \( A, B \) and \( K \), respectively.
Theorem 7 (Beyer-LeFloch,MN) For any asymptotic data in $H^3(U)$, there exists a unique solution of the singular initial value problem with remainder in $X_{\delta,\alpha,2}$ provided:

- we can choose $\delta, \alpha > 0$ so that the energy dissipation matrix

$$
\begin{bmatrix}
\Re(\lambda_1 - \lambda_2) + \alpha & ((\Im \lambda_1)^2/\eta - \eta)/2 & 0 & 0 \\
((\Im \lambda_1)^2/\eta - \eta)/2 & \alpha & \Phi_x & \Phi_y \\
0 & \Phi_x & \Psi_x & 0 \\
0 & \Phi_y & 0 & \Psi_y
\end{bmatrix}
$$

is positive semidefinite at each $(t, x) \in (0, \delta) \times U$ for a $\eta > 0$. Here, $\Phi_i = t \partial_i k - \partial_i \Re(\lambda_1 - \lambda_2 + \alpha)(tk_i \log t)$ and $\Psi_i = \Re(\lambda_1 - \lambda_2) + \alpha - 1 - Dk_i/k_i$.

- $f \in X_{\delta,\alpha+\epsilon,1}$ for some $\epsilon > 0$.

- $\alpha + \epsilon < 2(\beta + 1) - \Re(\lambda_1 - \lambda_2)$. 
Note that Theorem 7 can be formulated without difficulty for the $C^\infty$-case.

**Theorem 8** Suppose that $z^*, z^{**}, w^*$ and $w^{**}$ are smooth functions of $r$ and $0 < \epsilon < \min\{2k, 2 - 2k\}$. Then, there is a unique solution of the Einstein equations of the form (16) and (17) in a neighborhood of $t = 0$ such that $Z$ and $W$ tend to zero as $t \to 0$. 