Global structure of cylindrically symmetric spacetimes

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03/March/2012@APS2012, Kyoto

1. Singularity theorems and cosmic censorship

Theorem 1 (Penrose) *Suppose the following conditions hold: (1) a Cauchy surface* Σ *is noncompact, (2) the null convergence condition, (3)* Σ *contains a closed trapped surface. Then the corresponding maxmal future development* $D^+(\Sigma)$ *is incomplete.*

Theorem 2 (Hawking) *Suppose the following conditions hold: (1) a Cauchy surface* Σ *is compact, (2) the timelike convergence condition, (3) the generic condition. Then the corresponding maxmal Cauchy development* $D(\Sigma)$ *is incomplete.*

These theorems say physically reasonable spacetimes have singularities in general.

However,

- the theorem does not say us nature of singularity.
- **predictability** is breakdown if singularity can be seen.

To solve these problems, one should analyze the Einstein equations by using PDE technique.

Conjecture 1 (Strong cosmic censorship (SCC)) *Generic Cauchy data sets have maximal Cauchy developments which are locally inextendible as Lorentzian manifolds.*

Remark 1 *This formulation is of Moncrief and Klainerman. The original is formulated by Penrose.*

To prove the SCC, one need to show

- global existence theorems in suitable coordinates,
- inextendibility.

However, it is too difficult to solve the Einstein equations without assumptions.

Therefore, we will use a **cylindrical symmetric spacetime**, which is one of the simplest inhomogeneous spacetime.

2. Field equations for cylindrically symmetric spacetimes

Cylindrical symmetric initial data

- noncompact Cauchy surfaces Σ (R^3-topology for spatial section),
- $U(1) \times \mathsf{R}^1$ isometry group with spacelike orbits (cylindrical symmetry),
- The metric h and the second fundamental form k of Σ satisfy

$$\mathcal{L}_{X^a} h_{\mu\nu} = \mathcal{L}_{X^a} k_{\mu\nu} = 0, \tag{1}$$

  where X^a, $a = 2, 3$ are two Killing vectors that generate the isometry group.

Metric

$$g = -e^{2(\eta - U)} dt^2 + e^{2(\eta - U)} dr^2 + e^{2U}(dx + A dy)^2 + e^{-2U} R^2 dy^2, \quad (2)$$

where R, η, U and A are functions of $t \in (0, \infty)$ and $r \in [0, \infty)$. Note that a metric with $A \equiv 0$ is given by Gowdy-Edmonds.

Constraint equations

$$\dot{U}^2 + U'^2 + \frac{e^{4U}}{4R^2}(\dot{A}^2 + A'^2) + \frac{R''}{R} - \frac{\dot{\eta}\dot{R}}{R} - \frac{\eta' R'}{R} = 0, \quad (3)$$

$$2\dot{U}U' + \frac{e^{4U}}{2R^2}\dot{A}A' + \frac{\dot{R}'}{R} - \frac{\dot{\eta}R'}{R} - \frac{\eta'\dot{R}}{R} = 0, \quad (4)$$

where dot and prime denote derivative with respect to time t and r, respectively.

Evolution equations

$$\ddot{R} - R'' = 0, \tag{5}$$

$$\ddot{\eta} - \eta'' = -\dot{U}^2 + U'^2 + \frac{e^{4U}}{4R^2}(\dot{A}^2 - A'^2), \tag{6}$$

$$\ddot{U} - U'' = -\frac{\dot{R}\dot{U}}{R} + \frac{R'U'}{R} + \frac{e^{4U}}{2R^2}(\dot{A}^2 - \alpha A'^2), \tag{7}$$

$$\ddot{A} - A'' = \frac{\dot{R}\dot{A}}{R} - \frac{R'A'}{R} - 4(\dot{A}\dot{U} - A'U'). \tag{8}$$

Remark 2

- R *will be fixed by gauge condition:* $R = tr$.
- *Equation (6) can be derived from other equations.*
- *As the result, the evolution equations (7) and (8) are decoupled with the constraint equations (3) and (4).*

Remark 3 *If* $R = t$*, expanding universe is obtained. Also, if* $R = r$*, cylindrically symmetric gravitational waves in "asymptotically flat" spacetimes are given. Thus, our choice means that cylindrically symmetric gravitational waves in expanding universe is described.*

Remark 4 *In the both case* $R = t$ *and* $R = r$ *with or without matter fields, global existence theorems have been proved (Andreasson, Berger, Chruściel, Isenberg, Moncrief, Rendall, Ringström, MN).*

Lemma 1 *The cylindrically symmetric initial data do not contain trapped surfaces which are either compact or invariant under the isometry group.*

Thus, possible singularities would exist due to some other reason.

Geroch-Ernst potential

$$\dot{A} = -Re^{-4U}w', \quad A' = -Re^{-4U}\dot{w}.$$

From this and replacing U by $z/2$, we have

Constraint equations

$$\begin{aligned}
\dot{z}^2 + z'^2 + e^{-2z}\left(\dot{w}^2 + w'^2\right) + \frac{4R''}{R} - \frac{4\dot{R}\dot{\eta}}{R} - \frac{4R'\eta'}{R} &= 0 \qquad (9)\\
2\left(\dot{z}z' + e^{-2z}\dot{w}w'\right) + \frac{4\dot{R}'}{R} - \frac{4R'\dot{\eta}}{R} - \frac{4\dot{R}\eta'}{R} &= 0 \qquad (10)
\end{aligned}$$

Evolution equations

$$\ddot{z} + \frac{\dot{R}}{R}\dot{z} - z'' - \frac{R'}{R}z' + e^{-2z}\left(\dot{w}^2 - w'^2\right) = 0, \qquad (11)$$

$$\ddot{w} + \frac{\dot{R}}{R}\dot{w} - w'' - \frac{R'}{R}w' - 2\left(\dot{z}\dot{w} - z'w'\right) = 0 \qquad (12)$$

We have a **wave map** $\Psi : (\mathcal{M}^{2+1}, G) \mapsto (\mathcal{N}^2, h)$ as follows:

$$S_{\mathrm{WM}} = \int dtdr\sqrt{-G}G^{\alpha\beta}h_{AB}\partial_\alpha\Psi^A\partial_\beta\Psi^B, \tag{13}$$

where

$$G = -dt^2 + dr^2 + R^2 d\psi^2,$$

and

$$h = dz^2 + e^{-2z}dw^2.$$

Remark 5 *The target space is two-dimensional hyperbolic space. This is the same with Gowdy case.*

The energy-momentum tensor $T_{\alpha\beta}$ for this system is given of the form:

$$T_{\alpha\beta} = \tilde{h}_{AB}\left(\partial_\alpha\Psi^A\partial_\beta\Psi^B - \frac{1}{2}G_{\alpha\beta}\partial_\lambda\Psi^A\partial^\lambda\Psi^B\right). \tag{14}$$

3. Global existence

Theorem 3 *Let* (M, g) *be the maximal Cauchy development of* C_0^∞ *initial data for the cylindrically symmetric system. Then,* M *can be covered by Cauchy surfaces of constant time* t *with each value in the range* $(0, \infty)$*. Moreover, this maximal Cauchy development is timelike future geodesically complete, hence inextendible into the future direction.*

Remark 6 *To prove this theorem, our spacetime will be divided into two regions, one includes* $r = 0$ *and another is in* $r \geq \delta > 0$

3.1 Region in $r \geq \delta > 0$

Proposition 1 *Suppose $r \geq \delta > 0$. Then, there is a unique map Ψ satisfying the wave map equations (11) and (12).*

Method of the proof: **Light cone estimate**

3.2 Near $r = 0$

Theorem 4 (Christodoulou-Tahvildar-Zadeh) *Let Σ be a radial symmetric Cauchy surface in R^{2+1} and let Θ_0 and $\dot{\Theta}_0$ be any smooth radial symmetric Cauchy data for the wave map equations for a map $\Theta : (\mathrm{R}^{2+1}, \eta) \rightarrow (\mathrm{H}^2, h)$, where $\eta = -dt^2 + dr^2 + r^2 d\theta^2$. Then There is a unique smooth map $\Theta : (\mathrm{R}^{2+1}, \eta) \rightarrow (N, h)$ satisfying the wave map equations and assuming the Cauchy data Θ_0 and $\dot{\Theta}_0$.*

Lemma 2 *Let* $I = \{(t, r) \in (0, \infty) \times [0, \infty) : t > r\}$ *and let* $R = tr$. *Then there exists smooth coordinate* $\tau, \rho : I \leftrightarrow \mathcal{O} \subset \mathrm{R}^2$*, such that*

$$R|_I = \rho,$$

and

$$\left(-dt^2 + dr^2\right)|_I = \Omega^2 \left(-d\tau^2 + d\rho^2\right),$$

for some positive function $\Omega \in C^\infty(\mathcal{O})$.

From Theorem 4 and Lemma 2, a global existence theorem is obtained in $t > r$.

4. Nature of singularity (Construction of Kasner-like solutions)
We would like to know behavior of spacetimes near singularity.

Example: Kasner solution (spatially homogeneous and anisotropic):

$$g = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2,$$

where

$$\sum_{i=1}^{3} p_i = \sum_{i=1}^{3} p_i^2 = 1.$$

The Kretschmann invariant blows up at $t = 0$, thus Kasner spacetime is inextendible into the past direction except the case of $p_1 = 1$ and $p_2 = p_3 = 0$.

Our spacetime includes the Kasner in the sense that ours becomes the Kasner if metric functions are independent on r (spatially homogeneous).

One expects that the solutions should be Kasner-like ones near singularity (BKL conjecture).

Construction of Kasner-like solutions

Consider a system of PDEs on R^{n+1}, whose solutions are expected to have a singularity as $t \to 0$.

The Fuchsian technique:

- Decompose the unknown into a prescribed singular part and a regular part $\mathcal{U} = (u, Du, t\partial_x u)$.
- If the system can be written as a **first-order Fuchsian system** of the form

$$D\mathcal{U} + N(x)\mathcal{U} = t^{\alpha} f(t, x, \mathcal{U}, \partial_x \mathcal{U}), \quad \alpha > 0, \quad D = t\partial_t, \tag{15}$$

we can apply the following theorem.

Theorem 5 (Kichenassamy-Rendall) *Assume that N is an analytic matrix near $x = 0$ such that there is a constant C with $\| \sigma^N \| \leq C$ for $0 < \sigma < 1$, where σ^N is the matrix exponential of $N \ln \sigma$. Also, suppose that f is a locally Lipschitz function of $\mathcal{U}$ and $\partial_x \mathcal{U}$ which preserves analyticity in x and continuity in t. Then, the Fuchsian system (15) has a unique solution in a neighborhood of $x = 0$ and $t = 0$ which is analytic in x and continuous in t, and tends to zero as $t \to 0$.*

Remark 7 *The sufficient condition for N is non-negativity of eigenvalues of N.*

Formal solution(=leading-order+higher-order terms):

$$z(t,r) = z_*(r)\ln t + z_{**}(r) + t^{\epsilon} Z(t,r), \tag{16}$$

$$w(t,r) = w_*(r) + t^{2k(r)}\left(w_{**}(r) + W(t,r)\right), \tag{17}$$

where $\epsilon > 0$ is a small constant. We call z_*, z_{**}, w_*, w_{**} *asymptotic data*, while Z and W are *remainder* (higher order in t). We can get the Fuchsian system (15) with

$$N = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \epsilon^2 & 2\epsilon & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad 0 < \epsilon < \min\{2k, 2-2k\}. \tag{18}$$

Theorem 6 *Suppose that* z_*, z_{**}, w_* *and* w_{**} *are real analytic functions of* r *and* $0 < \epsilon < \min\{2k, 2-2k\}$. *Then, there is a unique solution of the Einstein equations of the form (16) and (17) in a neighborhood of* $t = 0$ *such that* Z *and* W *tend to zero as* $t \to 0$.

Remark 8 *The Kretschmann invariant blows up as* $t \to 0$, *thus our spacetime is inextendible into the past direction if the solution (16) and (17) is generic.*

5. Nature of singularity (Cont.)

Definition 1 *A* **second-oder hyperbolic Fuchsian system** *is a set of partial differential equations of the form*

$$D^2\mathcal{V} + 2AD\mathcal{V} + B\mathcal{V} - t^2K^2\partial_x^2\mathcal{V} = f[\mathcal{V}], \tag{19}$$

in which the function $\mathcal{V} : (0, \delta] \times U \to \mathbf{R}^n$ *is the main unknown, while coefficients* $A(x)$*,* $B(x)$*,* $K(t, x)$ *are diagonal* $n \times n$ *matrix-valued maps and a smooth in* $x \in U$ *and* t *in the half-open interval* $(0, \delta]$*, and* $f = f[\mathcal{V}](t, x)$ *is an* n*-vector-valued map of the following form:*

$$f[\mathcal{V}](t, x) = f(t, x, \mathcal{V}(t, x), D\mathcal{V}(t, x), tK(t, x)\partial_x\mathcal{V}(t, x)).$$

We put

$$\lambda_1 = a + \sqrt{a^2 - b}, \qquad \lambda_2 = a - \sqrt{a^2 - b},$$

and

$$k(t,x) = t^{\beta(x)}\nu(t,x), \text{with} \beta : U \to (-1,\infty), \nu : [0,\delta] \times U \to (0,\infty),$$

where a, b and k are eigenvalues of A, B and K, respectively.

Theorem 7 (Beyer-LeFloch,MN) *For any asymptotic data in $H^3(U)$, there exists a unique solution of the singular initial value problem with remainder in $X_{\delta,\alpha,2}$ provided:*

- *we can choose $\delta, \alpha > 0$ so that the* **energy dissipation matrix**

$$\begin{bmatrix} \Re(\lambda_1-\lambda_2)+\alpha & ((\Im\lambda_1)^2/\eta-\eta)/2 & 0 & 0 \\ ((\Im\lambda_1)^2/\eta-\eta)/2 & \alpha & \Phi_x & \Phi_y \\ 0 & \Phi_x & \Psi_x & 0 \\ 0 & \Phi_y & 0 & \Psi_y \end{bmatrix} \tag{20}$$

  is positive semidefinite at each $(t,x) \in (0,\delta) \times U$ for a $\eta > 0$. Here, $\Phi_i = t\partial_i k - \partial_i \Re(\lambda_1 - \lambda_2 + \alpha)(tk_i \log t)$ *and* $\Psi_i = \Re(\lambda_1 - \lambda_2) + \alpha - 1 - Dk_i/k_i$.

- $f \in X_{\delta,\alpha+\epsilon,1}$ *for some $\epsilon > 0$.*

- $\alpha + \epsilon < 2(\beta+1) - \Re(\lambda_1 - \lambda_2)$.

Note that Theorem 7 can be formulated without difficulty for the C^∞-case.

Theorem 8 *Suppose that* z_*, z_{**}, w_* *and* w_{**} *are* **smooth** *functions of* r *and* $0 < \epsilon < \min\{2k, 2-2k\}$. *Then, there is a unique solution of the Einstein equations of the form (16) and (17) in a neighborhood of* $t = 0$ *such that* Z *and* W *tend to zero as* $t \to 0$.