A general proof of the equivalence between the δN and covariant formalisms

Yukawa Institute for Theoretical Physics
Atsushi Naruko

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Introduction

- Inflation is one of the most promising candidates as the generation mechanism of primordial fluctuations.

- Inflation can be derived by a scalar field.

- We have hundreds or thousands of inflation models.
  → we have to discriminate those models.

- CMB : scale invariant spectrum, Gaussian statistics

- Non-Gaussianity may have the key of this puzzle.

Non-Gaussianity in CMB

- The deviations of CMB from the Gaussian statistics is parameterised by the non-linear parameter "f_{NL}" .

$$\frac{\Delta T}{T}(x) = \left.\frac{\Delta T}{T}\right|_{\text{Gaussian}} + f_{NL}\left.\left(\frac{\Delta T}{T}\right)^2\right|_{\text{Gaussian}} + \cdots$$

$$\left\langle\left(\frac{\Delta T}{T}\right)^3\right\rangle \sim f_{NL}\left\langle\left(\frac{\Delta T}{T}\right)^2\right\rangle^2$$

amplitude of 2nd order perturbation

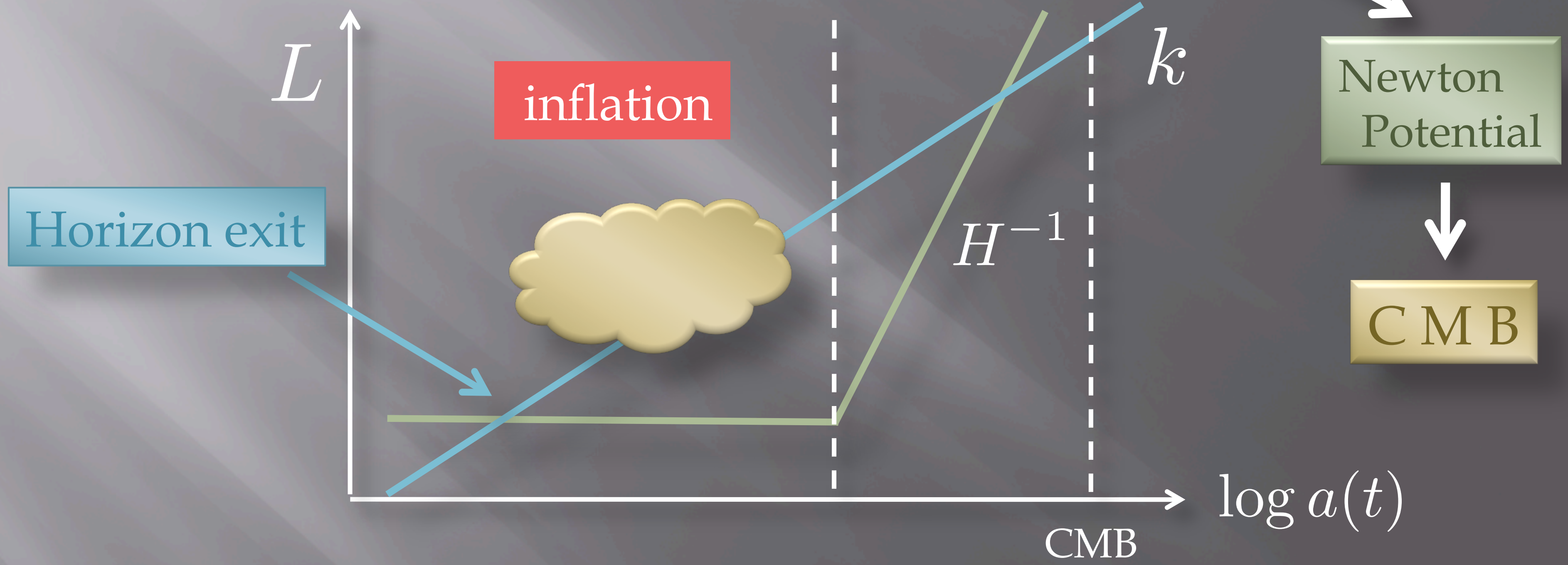
$$\text{WMAP 7}: \quad -10 < f_{NL}^{\text{local}} < 74 \qquad \text{PLANCK}: \left|\Delta f_{NL}^{\text{local}}\right| < 5$$

→ There exists a possibility to constrain inflation models by f_{NL} !!

→ We need to go beyond the linear perturbation theory.

Outside Horizon

- The evolution of the curvature perturbation $R^{(3)}$



- To give a precise theoretical prediction,
  we need to solve the evolution of $R^{(3)}$ after horizon exit.

→ We focus on superhorizon dynamics of non-linear perturbations.

Non-linear Cosmological Pert's

- There are several approaches for non-linear pert's.
  - 1, higher order perturbation : most general, lengthy
  - 2, gradient expansion : superhorizon only, $\sim$ BG Eqs
  - 3, covariant formalism : coordinate-free, geometrical

- What is the relation between No.2 and No.3 ?
  - ✓ Equivalence at linear, 2^{nd} and 3^{rd} order

    Langlois et al., Enqvist et al,. Lehners et al,.
  - ✓ non-linear equivalence in the Einstein gravity

    Suyama et al.

Gradient expansion approach

- On large scales, spatial gradient expansion will be valid.

$$L \gg H^{-1} \Longrightarrow \left|\partial_i Q\right| (\sim L^{-1} Q) \ll \left|\partial_t Q\right| (\sim HQ)$$

$\longrightarrow$ We expand equations in powers of spatial gradients.

$\longrightarrow$ Full non-linear effects are taken into account.

- We express the metric in the ADM form

$$ds^2 = -\alpha^2 dt^2 + \hat{\gamma}_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

- The spatial metric is further decomposed

$$\hat{\gamma}_{ij} = a^2(t) e^{2\psi} \gamma_{ij}$$

ψ : curvature perturbation

$\det|\gamma_{ij}| = 1$

delta-N formalism

B.G. e-folding number

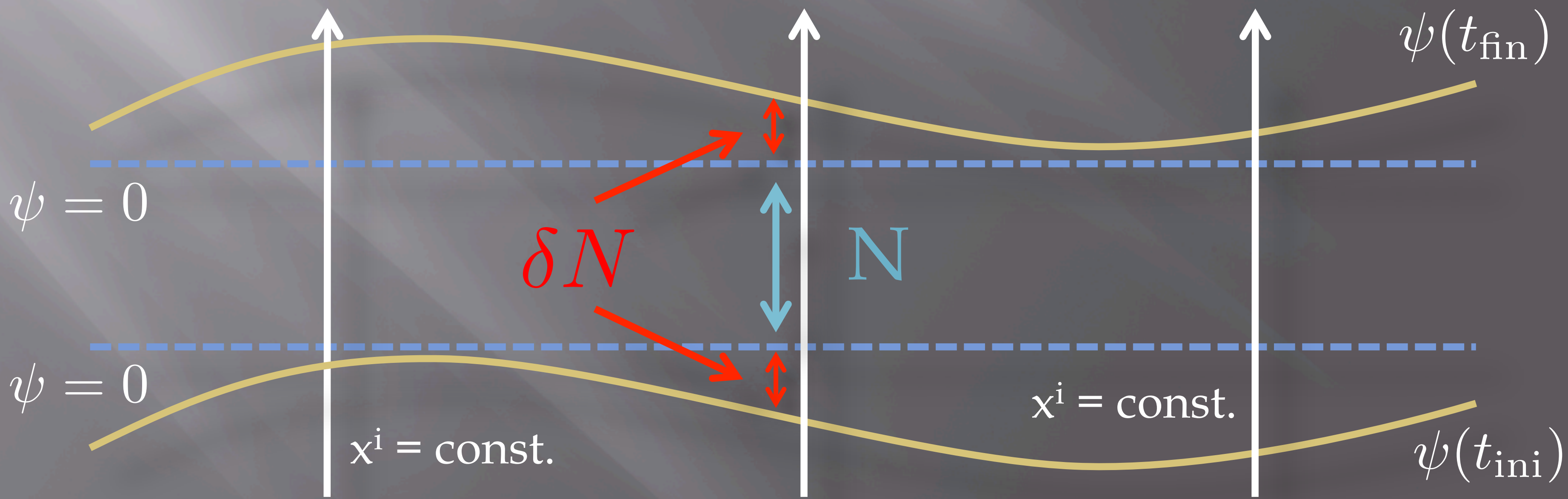
cf. $N \equiv \int H dt$

- We define the non-linear e-folding number.

$$\mathcal{N} \equiv \frac{1}{3}\int \Theta\, \alpha\, dt \sim \int (H + \partial_t \psi)\, dt \qquad \Theta \equiv \nabla_\mu n^\mu$$

- ψ is given by the difference of "N" → **δ N formalism**

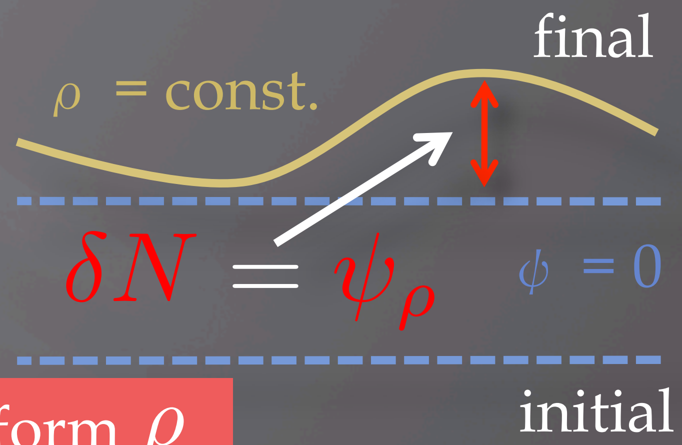
$$\delta N \equiv \mathcal{N} - N = \psi(t_{\rm fin}) - \psi(t_{\rm ini})$$



Evolution of curvature Pert.

- By choosing the slicings ;

  initial : flat & final : uniform ρ



δN gives the final ψ on the uniform ρ

- Let us consider a perfect fluid :

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}$$

- The energy cons. law gives the evolution eq. for ψ

$$u_\mu \nabla_\nu T^{\mu\nu} = 0 \quad \Rightarrow \quad H + \partial_t \psi = -\frac{1}{3}\frac{\partial_t \rho}{\rho + P}$$

Covariant formalism

- We define the curvature covector.

$$\zeta_\mu \equiv \partial_\mu \mathcal{N} - \frac{\dot{\mathcal{N}}}{\dot{\rho}} \partial_\mu \rho \qquad \dot{\mathcal{N}} \equiv \mathcal{L}_u \mathcal{N} = u^\mu \partial_\mu \mathcal{N}$$

- The energy cons. law gives the evolution eq. for ζ_μ ,

$$\dot{\zeta}_\mu \equiv \mathcal{L}_u \zeta_\mu = -\frac{\Theta}{3(\rho + P)} \left(\partial_\mu P - \frac{\dot{P}}{\dot{\rho}} \partial_\mu \rho \right)$$

- Notice !!

  1, the equation for ζ_μ is valid at all scales.

  $$\mathcal{N} \sim \int d\tau\, \Theta$$

  2, there is an ambiguity in the choice of the initial slice, since N is defined in terms of the integration.

Equivalence ①

- The relation between " ζ_μ " in the covariant formalism and " $\psi = \delta N$ " in the δN formalism is unclear.

- On the uniform energy density slicing : $\rho = \rho(t)$,

$$\zeta_i\Big|_E = \left(\partial_i \mathcal{N} - \cancel{\frac{\dot{\mathcal{N}}}{\dot{\rho}}\partial_i \rho}\right)_E = \partial_i\Big[\psi_E(t,x^j) - \psi(t_{\rm ini},x^j)\Big]$$

- We choose the initial flat slice as in the δN formalism,

$$\zeta_i\Big|_E = \partial_i\Big[\psi_E\Big] = \partial_i\Big[\delta N\Big]$$

➔ This shows that δN formalism = covariant formalism.

Equivalence ②

- We can show the equivalence between two evolution eqs.
- The evolution eq. for ζ_μ on large scales

$$\dot{\zeta}_\mu = -\frac{\Theta}{3(\rho+P)}\left(\partial_\mu P - \frac{\dot{P}}{\dot{\rho}}\partial_\mu \rho\right)$$

$$\frac{1}{\alpha}\left[\partial_i \psi' + \frac{\partial_i \rho'}{3(\rho+P)} - \frac{\partial_i \rho(\rho'+P')}{3(\rho+P)^2}\right] \qquad \frac{1}{\alpha}\frac{\rho'}{3(\rho+P)^2}\left(\partial_i P - \frac{P'}{\rho'}\partial_i \rho\right)$$

$$\boxed{\partial_i \psi' = -\partial_i\left[\frac{\rho'}{3(\rho+P)}\right]}$$

→ This also shows that δ N formalism = covariant formalism.

Conclusion

- We have shown that the non-linear equivalence between the δN and covariant formalisms on superhorizon scales.

- In the proof, we have not assumed the gravity theory, which means the equivalence holds in any gravity theory.

Linear theory

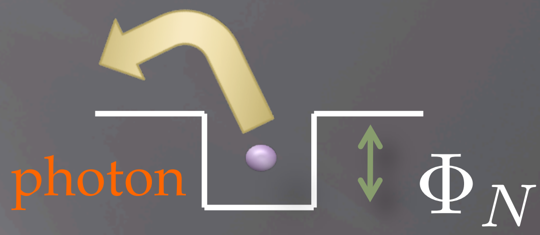
$$R^{(3)} \sim \Delta \mathcal{R}$$

- Let us consider perturbations around the FLRW universe.

$$ds^2 = a^2(\eta)\left[-(1+2A)d\eta^2 - 2\Delta^{-1/2}B_{,i}dx^i d\eta + \left\{(1+2\mathcal{R})\delta_{ij} - 2\Delta^{-1}C_{,ij}\right\}dx^i dx^j\right]$$

$$\mathcal{R} \sim \Phi_N \sim \Delta T/T$$

(gravitational redshift)



- The Einstein equations $\longrightarrow$ the master equation for R

$$\mathcal{R}_c'' + 2\frac{z'}{z}\mathcal{R}_c' - \Delta\mathcal{R}_c = 0$$

$\mathcal{R}_c : \mathcal{R}$ in $\delta\phi = 0$

$z \equiv a \times \left(\phi_0'/\mathcal{H}\right)$

- On superhorizon scales $\lambda \gg H^{-1}$,

Slow-roll

$\mathcal{R}_c =$ const. and $\mathcal{R}_c' \propto z^{-2} \sim a^{-2}$

Curvature covector

- We define the e-folding number, which is the integration of the expansion along an integral curve of u^μ,

$$\mathcal{N} \equiv \frac{1}{3}\int d\tau\Theta = \frac{1}{3}\int d\tau\nabla_\mu u^\mu$$

- We define the curvature covector, which is one of the most important quantities in the covariant formalism.

$$\zeta_\mu \equiv \partial_\mu\mathcal{N} - \frac{\dot{\mathcal{N}}}{\dot{\rho}}\partial_\mu\rho$$

where the dot denotes the Lie derivative with respect to u^μ,

$$\dot{\mathcal{N}} \equiv \mathcal{L}_u\mathcal{N} = u^\mu\partial_\mu\mathcal{N} \qquad \dot{\zeta}_\mu \equiv \mathcal{L}_u\zeta_\mu = u^\nu\partial_\nu\zeta_\mu + \zeta_\nu\partial_\mu u^\nu$$

Evolution equation for CC

- The energy cons. law gives the evolution eq. for CC,

$$\dot{\zeta}_\mu = -\frac{\Theta}{3(\rho + P)}\left(\partial_\mu P - \frac{\dot{P}}{\dot{\rho}}\partial_\mu \rho\right)$$

- When the adiabatic condition "P = P (ρ)" is satisfied, the RHS vanishes and CC is conserved.

- Notice !!
  1, the above equation is valid at all scales.
  2, there is an ambiguity in the choice of the initial slice, since N is defined in terms of the integration.

$$\mathcal{N} \sim \int d\tau\, \Theta$$