A reference for the covariant Hamiltonian boundary term

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Outline

- Review quasi-local energy-momentum
- Review the covariant Hamiltonian formalism results
  - ▷ the Hamiltonian boundary term
- How to choose the reference: isometric matching and optimization
- results for spherically symmetric metrics

Quasi-local energy-momentum

Energy-momentum is the source of gravity.
Gravitating bodies can exchange energy-momentum with gravity—locally
yet there is no well defined energy-momentum density for gravity itself.
(a consequence of the equivalence principle)

Traditional approach: non-covariant, reference frame dependent,
energy-momentum complexes, i.e., pseudotensors
Ambiguity **1.**: no unique expression
(Einstein, Papapetrou, Landau-Lifshitz, Bergmann-Thompson, Møller,
Goldberg, Weinberg, . . .)
Ambiguity **2.**: which reference frame?

The modern idea is quasi-local (associated with a closed 2-surface)
[see Szabados, Living Reviews of Relativity, 2009]

One approach is via the *Hamiltonian* (the generator of time evolution).
This includes all the classical pseudotensors as special cases, while
taming their ambiguities, providing clear physical/geometric meaning.

covariant Hamiltonian formulation results

For geometric gravity theories the Hamiltonian 3-form is a conserved Noether current as well as the generator of the evolution of a spatial region along a space-time displacement vector field, it has the form

$$\mathcal{H}(N) = N^\mu \mathcal{H}_\mu + d\mathcal{B}(N), \qquad d\mathcal{H}(N) \propto \text{field eqns} \simeq 0$$

where $N^\mu \mathcal{H}_\mu$, which generates the evolution equations, is proportional to field equations (initial value constraints) and thus vanishes “on shell”. Hence the value is determined by the total differential (boundary) term,

$$E(N, \Sigma) := \int_\Sigma \mathcal{H}(N) = \oint_{\partial\Sigma} \mathcal{B}(N) \qquad \text{Thus it is } \textit{quasi-local}.$$

Note: $\mathcal{B}(N)$ can be modified—by hand—in any way without destroying the conservation property. One can arrange for almost any conserved value.

Fortunately the Hamiltonian’s role in generating evolution equations tames that freedom.

Boundary Variation Principle

[Lanczos (1949), Regge-Teitelboim (1974), Kijowski-Tulczjew (1979), . . .]
One must look to the boundary term in the variation of the Hamiltonian.
Requiring it to vanish yields the boundary conditions. The Hamiltonian is functionally differentiable on the phase space of fields satisfying these boundary conditions. Modifying the boundary term changes the boundary conditions.
[different pseudotensors correspond to different boundary conditions]

- The boundary term $\mathcal{B}(N)$ determines both the *quasi-local value* and the *boundary condition*.
- In order to accommodate suitable boundary conditions one must, in general, also introduce certain reference values which represent the ground state of the field—the “vacuum” (or background field) values.

For any quantity α, let $\Delta\alpha := \alpha - \bar{\alpha}$ where $\bar{\alpha}$ is the reference value.

Preferred Boundary Term for GR

Chen, N, Tung (1995) [also found by Katz, Bičák & Lynden-Bel]

$$\mathcal{B}(N) = \frac{1}{2\kappa}(\Delta\Gamma^\alpha{}_\beta \wedge i_N \eta_\alpha{}^\beta + \bar{D}_\beta N^\alpha \Delta\eta_\alpha{}^\beta) \qquad \eta^{\alpha\beta\dots} := {}^\star(\vartheta^\alpha \wedge \vartheta^\beta \wedge \cdots)$$

fix the orthonormal coframe ϑ^μ ($\sim$ metric) on the boundary:

$$\delta\mathcal{H}(N) \sim d i_N(\Delta\Gamma^\alpha{}_\beta \wedge \delta\eta_\alpha{}^\beta)$$

Like other choices, at spatial infinity it gives the ADM, MTW (1973), Regge-Teitelboim (1974), Beig-Ó Murchadha (1987), Szabados (2003) energy, momentum, angular-momentum, center-of-mass

Its special virtues:
(i) at null infinity: the Bondi-Trautman energy & the Bondi energy flux
(ii) it is "covariant"
(iii) it has a positive energy property
(iv) for small spheres, a positive multiple of the Bel-Robinson tensor
(v) first law of thermodynamics for black holes
(vi) in certain cases it reduces to Brown-York, hence for spherical solutions it has the hoop property

the reference and the quasi-local quantities

- Note: For all other fields it is appropriate to choose vanishing reference values as the reference ground state—the vacuum.
- But for geometric gravity the standard ground state is the non-vanishing Minkowski metric. A non-trivial reference is essential.
- With standard Minkowski coordinates y^i, a Killing field of the reference has the form $N^k = N_0^k + \lambda_0^k{}_l y^l$, where $\lambda_0^{kl} = \lambda_0^{[kl]}$, with N_0^k and λ_0^{kl} being constants. The 2-surface integral of the Hamiltonian boundary term then gives the value

$$\oint_S \mathcal{B}(N) = -N_0^k p_k(S) + \frac{1}{2}\lambda_0^{kl} J_{kl}(S),$$

  i.e., not only a quasi-local energy-momentum but also a quasi-local angular momentum/center-of-mass. The integrals $p_k(S),\ J_{kl}(S)$ in the spatial asymptotic limit agree with accepted expressions for these quantities.

the reference

- For energy-momentum take N^μ to be a translational Killing field of the Minkowski reference. Then the second quasi-local term vanishes.
- Remark: Holonomically (with vanishing reference) the first term is Freud's 1939 superpotential. Thus we are in effect making a proposal for best choice of coordinates for the Einstein pseudotensor.

To construct a reference choose, in a neighborhood of the desired spacelike boundary 2-surface S, 4 smooth functions $y^i, \ i = 0, 1, 2, 3$ with $dy^0 \wedge dy^1 \wedge dy^2 \wedge dy^3 \neq 0$ and then define a Minkowski reference by $\bar{g} = -(dy^0)^2 + (dy^1)^2 + (dy^2)^2 + (dy^3)^2$.

equivalent to finding a diffeomorphism for a neighborhood of the 2-surface into Minkowski space. The reference connection is obtained from the pullback of the flat Minkowski connection.
Then with constant N^k our quasi-local expression takes the form

$$\mathcal{B}(N) = N^k x^\mu{}_k (\Gamma^\alpha{}_\beta - x^\alpha{}_j \, dy^j{}_\beta) \wedge \eta_{\mu\alpha}{}^\beta .$$

Isometric matching of the 2-surface

The reference metric on the dynamical space has the components

$$\bar{g}_{\mu\nu} = \bar{g}_{ij} y^i{}_\mu y^j{}_\nu. \tag{1}$$

Consider the usual embedding restriction: isometric matching of the 2-surface S. This can be expressed quite simply in terms of quasi-spherical foliation adapted coordinates t, r, θ, ϕ as

$$g_{AB} = \bar{g}_{AB} = \bar{g}_{ij} y^i_A y^j_B = -y^0_A y^0_B + \delta_{ij} y^i_A y^j_B \tag{2}$$

on S, where A, B range over $2, 3 = \theta, \phi$.

From a classic closed 2-surface into $\mathbb{R}^3$ embedding theorem, we expect that that—as long as one restricts S and $y^0(x^\mu)$ so that on S

$$g'_{AB} := g_{AB} + y^0_A y^0_B \tag{3}$$

is convex—one has a unique embedding.

Wang & Yau used this type of embedding in their recent quasi-local work.

Complete 4D isometric matching

• Our "new" proposal complete isometric matching on S:
[already suggested by Szabados in 2000]

$$\text{10 constraints :} \quad g_{\mu\nu}|_S = \bar{g}_{\mu\nu}|_S = \bar{g}_{ij} y^i{}_\mu y^j{}_\nu|_S.$$

on 12 embedding functions on the 2-surface of constant t, r:

$$y^i (\Longrightarrow y^i_\theta, y^i_\phi), \quad y^i_t, \quad y^i_r$$

In terms of the orthonormal coframe ϑ^α with 6 local Lorentz gauge d.o.f.
Lorentz transform to match the reference coframe dx^α on the 2-surface.
Integrability condition: the 2-forms $d\vartheta^\alpha$ should vanish when restricted to the 2-surface:

$$d\vartheta^\alpha|_S = 0, \quad \text{4 restrictions}$$

Determine the optimal "best matched" reference by energy extremization.

The best matched reference geometry

- 12 embedding variables subject to 10 isometric conditions
- equivalently, 6 local Lorentz gauge subject to 4 embedding conditions
- To fix the remaining 2, regard the quasi-local value as a measure of the difference between the dynamical and the reference boundary values.
- We propose taking the optimal embedding as the one which gives the extreme value to the associated invariant mass $m^2 = -p_i p_j \bar{g}^{ij}$. Reasonable, since quasi-local energy should be non-negative and vanish only for Minkowski space.
- minimize. There are 2 different situations.

I: Given a 2-surface S take the inf of m^2. This should determine the reference up to Poincaré transformations.
II: Given a 2-surface S and a vector field N, take the inf of $E(N, S)$. [Afterward one could extremize over the choice of N.]

Based on some physical and practical computational arguments it is reasonable to expect a unique solution.

Static, spherically symmetric spacetime

Reissner-Nordström–(Anti)-de Sitter metric:

$A = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2 \quad B = 1 - \frac{\Lambda}{3}r^2$

Program A: $N = \partial_T \quad E_A(N) = r(\sqrt{-g(N,N)B} - AN^t)$

$$N_{\text{static}} = \frac{1}{\sqrt{A}}\partial_t \;\Rightarrow\; E_A(N_{\text{static}}) = \frac{2m - Q^2/r}{\sqrt{B} + \sqrt{A}}$$

- non-negative except for the small region $r < Q^2/2m$ inside the inner horizon, where the gravitational "force" is repulsive
- For Schwarzschild ($Q = 0 = \Lambda$), the result is the standard one obtained by many people using different quasi-local energy expressions. (Brown-York, Liu-Yau, Wang-Yau, Chen-N-Tung, etc.)

Program B : $\pounds_N(\text{area}) = \pounds_N(\overline{\text{area}}), \quad E_B = \dfrac{-g(N,N)(2m - Q^2/r)}{\sqrt{-g(N,N)B + (N^r)^2} + AN^t}$

$E_B(N_{\text{static}}) = E_A(N_{\text{static}})$

4D isometric matching

$$E_{\text{iso}} = r(BN^T - AN^t) = \frac{l^2(2m - Q^2/r) + r((N^R)^2 - (N^r)^2)}{\sqrt{l^2B + (N^R)^2} + \sqrt{l^2A + (N^r)^2}}$$

where $l^2 = -g(N, N)$.

$$N = \partial_T \quad \Longrightarrow \quad E_{\text{iso}} = E_A$$

$$\pounds_N(\text{area}) = \pounds_N(\overline{\text{area}}) \quad \Longrightarrow \quad E_{\text{iso}} = E_B = \frac{-g(N, N)(2m - Q^2/r)}{\sqrt{-g(N, N)B + (N^r)^2} + AN^t}$$

FLRW $A = a(t)/\sqrt{1-kr^2}$

Program A: $E_A = ar(\sqrt{-g(N,N)} - A^{-1}aN^t - A\dot{a}rN^r)$

▷ comoving observer: $E_A(\partial_t) = ar(1-\sqrt{1-kr^2}) = \frac{kar^3}{1+\sqrt{1-kr^2}}$

note: proportional to k, hence positive, negative or vanishing

$$E_A(N_{\mathrm{dmc}}) = \frac{ar^3(k+\dot{a}^2)}{1+\sqrt{1-kr^2-\dot{a}^2r^2}} = \frac{\frac{8\pi}{3}\rho(ar)^3}{1+\sqrt{1-\frac{8\pi}{3}\rho(ar)^2}} \geq 0$$

$$E_B = \frac{ar\left(-g(N,N)(k+\dot{a}^2)r^2\right)}{\sqrt{-g(N,N)+(\dot{a}rN^t+aN^r)^2}+\sqrt{1-kr^2}N^t+\frac{a\dot{a}r}{\sqrt{1-kr^2}}N^r} \geq 0$$

comoving $E_B(\partial_t) = \frac{ar^3(k+\dot{a}^2)}{\sqrt{1+\dot{a}^2r^2}+\sqrt{1-kr^2}}$, $\qquad E_B(N_{\mathrm{dmc}}) = E_A(N_{\mathrm{dmc}})$

$$E_{\mathrm{iso}} = ar\left(N^T - \sqrt{1-kr^2}N^t - \frac{a\dot{a}r}{\sqrt{1-kr^2}N^r}\right),$$

$E_{\mathrm{isoB}} = E_B, \qquad E_{\mathrm{iso}}(\partial_T) = E_A$

approach	energy for RN-AdS
iso	$E = r(BN^T - AN^t)$
isoA	$E = r(\sqrt{-g(N,N)B} - AN^t)$
isoB	$E = r(\sqrt{-g(N,N)B + (N^r)^2} - AN^t)$

where $A = 1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2, \;\; B = 1 - \frac{\Lambda}{3}r^2$.

approach	energy for FLRW
iso	$E = ar(N^T - \frac{aN^t}{A} - A\dot{a}rN^r)$
isoA	$E = ar(\sqrt{l^2} - \frac{aN^t}{A} - A\dot{a}rN^r)$
isoB	$E = ar(\sqrt{l^2 + (\dot{a}rN^t + aN^r)^2} - \frac{aN^t}{A} - A\dot{a}rN^r)$

where $A = \frac{a}{\sqrt{1-kr^2}}, \;\; l^2 = -g(N,N)$.

iso means matching the orthonormal frames.

isoA means iso with the restriction $N = \partial_T$.

isoB means iso with the restriction $\pounds_N(\text{area}) = \pounds_N(\overline{\text{area}})$.

References

- C.-M. Chen, J.M. Nester & R.S. Tung, Phys Lett A **203** (1995) 5
- C.-M. Chen & J.M. Nester, Class Quantum Grav **16** (1999) 1279
- C.-M. Chen & J.M. Nester, Gravitation Cosmol. **6** (2000) 257
- J.M. Nester, Class Quantum Grav **21** (2004) S261
- C.-M. Chen, J.M. Nester & R.S. Tung, Phys Rev D **72** (2005) 104020
- J.M. Nester, Prog Theor Phys Supp **172** (2008) 30
- M.F. Wu, C.-M. Chen, J.L. Liu & J.M. Nester, PLA **374** (2010) 3599
- J.L. Liu, C.M. Chen & J.M. Nester, CQG **28** (2011) 195019
- M.-F. Wu, C.-M. Chen, J.-L. Liu & J.M. Nester, PRD **84** (2011) 085047
- L.B. Szabados, Living Rev Relativity **12** (2009) 4
- L.B. Szabados, “Quasi-local energy-momentum and angular momentum in GR: the covariant Lagrangian approach”, unpublished draft (2005)