Some recipes for BSSN formulation

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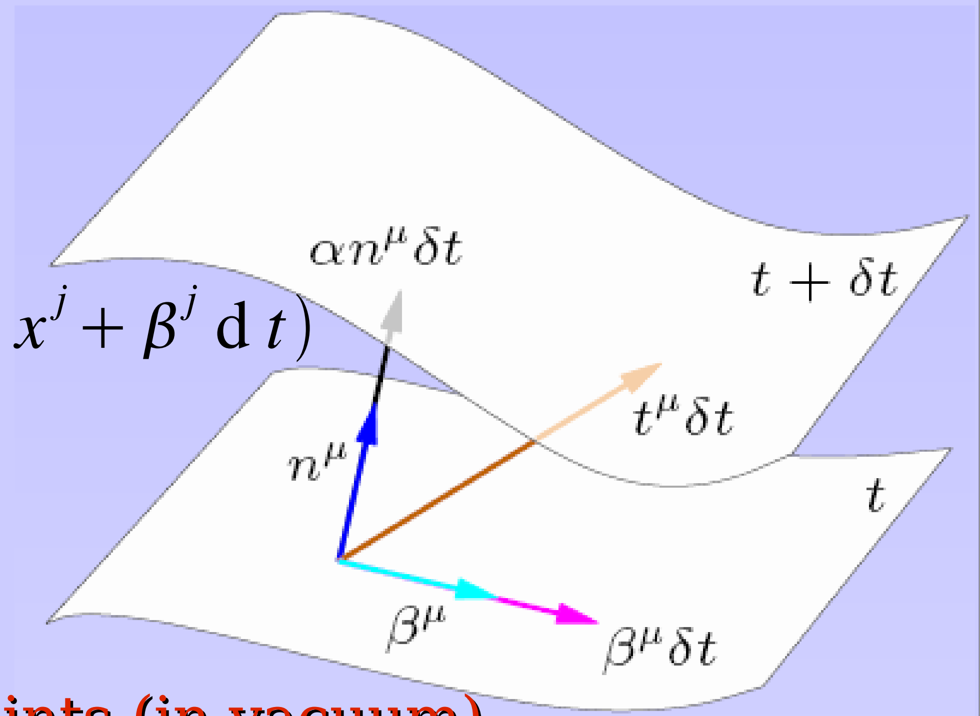
3+1 Arnowitt-Deser-Misner (ADM) formulation

- $3+1:\ {}^4M = R \times {}^3\Sigma$

- $ds^2 = -\alpha\, dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$

  $\gamma_{ij}(\mathrm{t}, \mathrm{x})$: metric

  $K_{ij}(\mathrm{t}, \mathrm{x})$: extrinsic curvature



12 evolution equations & 4 constraints (in vacuum)

Constraints: $H \equiv R + K^2 - K_{ij} K^{ij} \simeq 0$

$$M_i \equiv \nabla_j K^j{}_i - \nabla_i K \simeq 0$$

Evolution: $\dfrac{d}{dt}\gamma_{ij} = -2\,\alpha\, K_{ij} \quad \Leftarrow \quad \dfrac{d}{dt} \equiv \dfrac{\partial}{\partial t} - \mathscr{L}_{\vec{\beta}}$

$$\frac{d}{dt} K_{ij} = \alpha\,(R_{ij} - 2\, K_{im} K^m{}_j + K\, K_{ij}) - \nabla_i \nabla_j \alpha$$

Baumgarte-Shapiro-Shibata-Nakamura (BSSN) Formulation

Features:

- 1^{st} derivative in time, 2^{nd} derivative in space
- A conformal decomposition of the metric and the traceless components of the extrinsic curvature.
- has been shown to be superior to the standard ADM formulation in terms of both accuracy and stability.
- Strongly hyperbolic with suitable gauges

BSSN variables:

$$\boxed{\begin{aligned} \phi &\equiv \frac{1}{12}\ln\gamma, & \tilde{\gamma}_{ij} &\equiv e^{-4\phi}\gamma_{ij} \\ K &\equiv \gamma^{ij}K_{ij}, & \tilde{A}_{ij} &\equiv e^{-4\phi}K_{\langle ij\rangle} \\ \tilde{\Gamma}^{i} &\equiv \tilde{\gamma}^{jk}\tilde{\Gamma}^{i}_{\ jk} \end{aligned}}$$

Field equations:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\,t}\,\phi &= -\frac{1}{6}\,\alpha\,K \\
\frac{\mathrm{d}}{\mathrm{d}\,t}\,\tilde{\gamma}_{ij} &= -2\,\alpha\,\tilde{A}_{ij} \\
\frac{\mathrm{d}}{\mathrm{d}\,t}\,K &= \alpha\,(\tilde{A}_{ij}\,\tilde{A}^{ij}+\frac{1}{3}\,K^2)-\nabla^2\,\alpha \\
\frac{\mathrm{d}}{\mathrm{d}\,t}\,\tilde{A}_{ij} &= \alpha\,(K\,\tilde{A}_{ij}-2\,\tilde{A}_{ik}\,\tilde{A}^{k}{}_{j})+e^{-4\,\phi}\,(\alpha\,R_{\langle ij\rangle}-\nabla_{\langle i}\,\nabla_{j\rangle}\,\alpha) \\
\frac{\partial}{\partial\,t}\,\tilde{\Gamma}^{i} &= 2\,\alpha\,(\tilde{\Gamma}^{i}{}_{jk}\,\tilde{A}^{jk}-\frac{2}{3}\,\tilde{\gamma}^{ij}\,K_{,\,j}+6\,\tilde{A}^{ij}\,\phi_{,\,j})-2\,\tilde{A}^{ij}\,\alpha_{,\,j} \\
&\quad +\tilde{\gamma}^{jk}\,\beta^{i}{}_{,\,jk}+\frac{1}{3}\,\tilde{\gamma}^{ij}\,\beta^{k}{}_{,\,jk}+\beta^{j}\,\tilde{\Gamma}^{i}{}_{,\,j}-\tilde{\Gamma}^{j}\,\beta^{i}{}_{,\,j}+\frac{2}{3}\,\tilde{\Gamma}^{i}\,\beta^{j}{}_{,\,j}
\end{aligned}
$$

- The constraints of H & M^i were used to eliminate $\tilde{R}$ in $\frac{\mathrm{d}\,K}{\mathrm{d}\,t}$ and $\partial_j\,\tilde{A}^{i\,j}$ in $\partial_t\,\tilde{\Gamma}^i$.

Constraints

- Hamiltonian constraint:

$$H = e^{-4\phi}\left(\tilde{R} - 8\,\tilde{\nabla}^2\phi - 8\,\tilde{\nabla}^i\phi\,\tilde{\nabla}_i\phi\right) + \frac{2}{3}K^2 - \tilde{A}_{ij}\tilde{A}^{ij} \simeq 0$$

- Momentum constraints: $M^i = \tilde{\nabla}_j \tilde{A}^{ij} + 6\,\tilde{A}^{ij}\phi_{,j} - \frac{2}{3}\tilde{\gamma}^{ij}K_{,j} \simeq 0$

- Traceless constraint: $A \equiv \tilde{\gamma}^{ij}\tilde{A}_{ij} \simeq 0$

- Unimodular determinant constraint: $\tilde{\gamma} \equiv \det\left(\tilde{\gamma}_{ij}\right) \simeq 1$

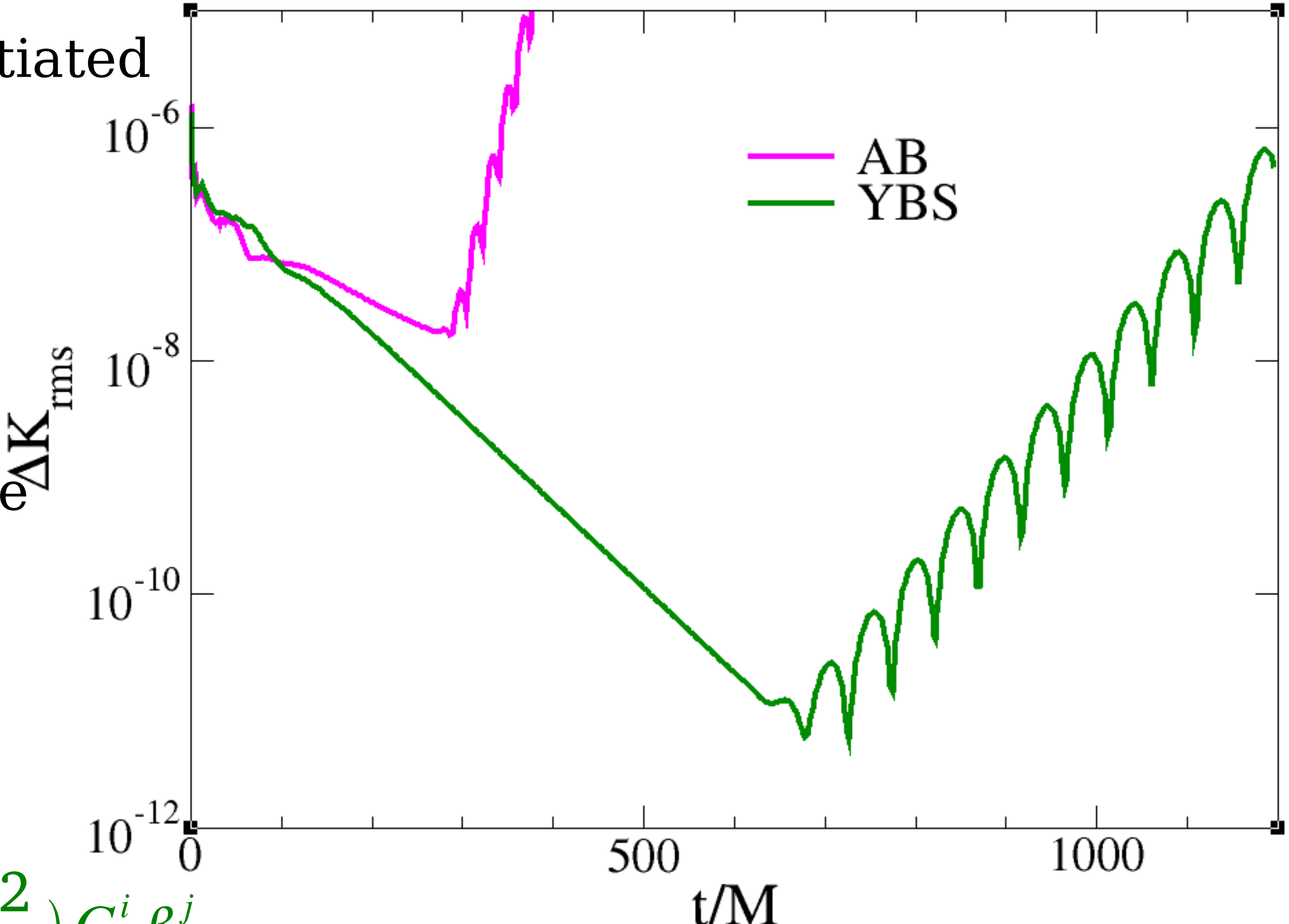
- Gamma constraint: $G^i \equiv \tilde{\Gamma}^i - \tilde{\gamma}^{jk}\tilde{\Gamma}^i{}_{jk} \simeq 0$

- Alcubierre & Brügmann [PRD '02]

(1) $\tilde{A}_{ij} \to \tilde{A}_{\langle ij \rangle}$ ($\tilde{\gamma}_{ij} \to \tilde{\gamma}^{-1/3}\,\tilde{\gamma}_{ij}$ later)

(2) substitute undifferentiated

$\tilde{\Gamma}^i$ with $\tilde{\gamma}^{jk}\,\tilde{\Gamma}^i{}_{jk}$.



- Yo, Shapiro & Baumgarte [PRD '02]

(1) $\tilde{\gamma}_{zz} \quad \Leftarrow \quad \tilde{\gamma} \simeq 1$

$\tilde{A}_{zz} \quad \Leftarrow \quad A \simeq 0$

(2) $\partial_t \tilde{\Gamma}^i \quad \to \quad \partial_t \tilde{\Gamma}^i - (\chi + \frac{2}{3})\, G^i\, \beta^j{}_{,j}$

$$= \left[(\chi + \frac{2}{3})\, \tilde{\gamma}^{jk}\, \tilde{\Gamma}^i{}_{jk} - \chi\, \tilde{\Gamma}^i \right] \beta^\ell{}_{,\ell} - \tilde{\Gamma}^j\, \beta^i{}_{,j} + \cdots$$

Modification of the formulation I

$$\partial_t \tilde{\Gamma}^i = -\frac{4}{3}\alpha \tilde{\gamma}^{ij} K_{,j} + \cdots = -\frac{4}{3}\alpha [(\tilde{\gamma}^{ij} K)_{,j} + K \tilde{\Gamma}^i] + \cdots \;\Leftarrow\; \tilde{\Gamma}^i \Leftarrow -\tilde{\gamma}^{ij}{}_{,j}$$

Advantage:

stabilize the code without the need of the substitute

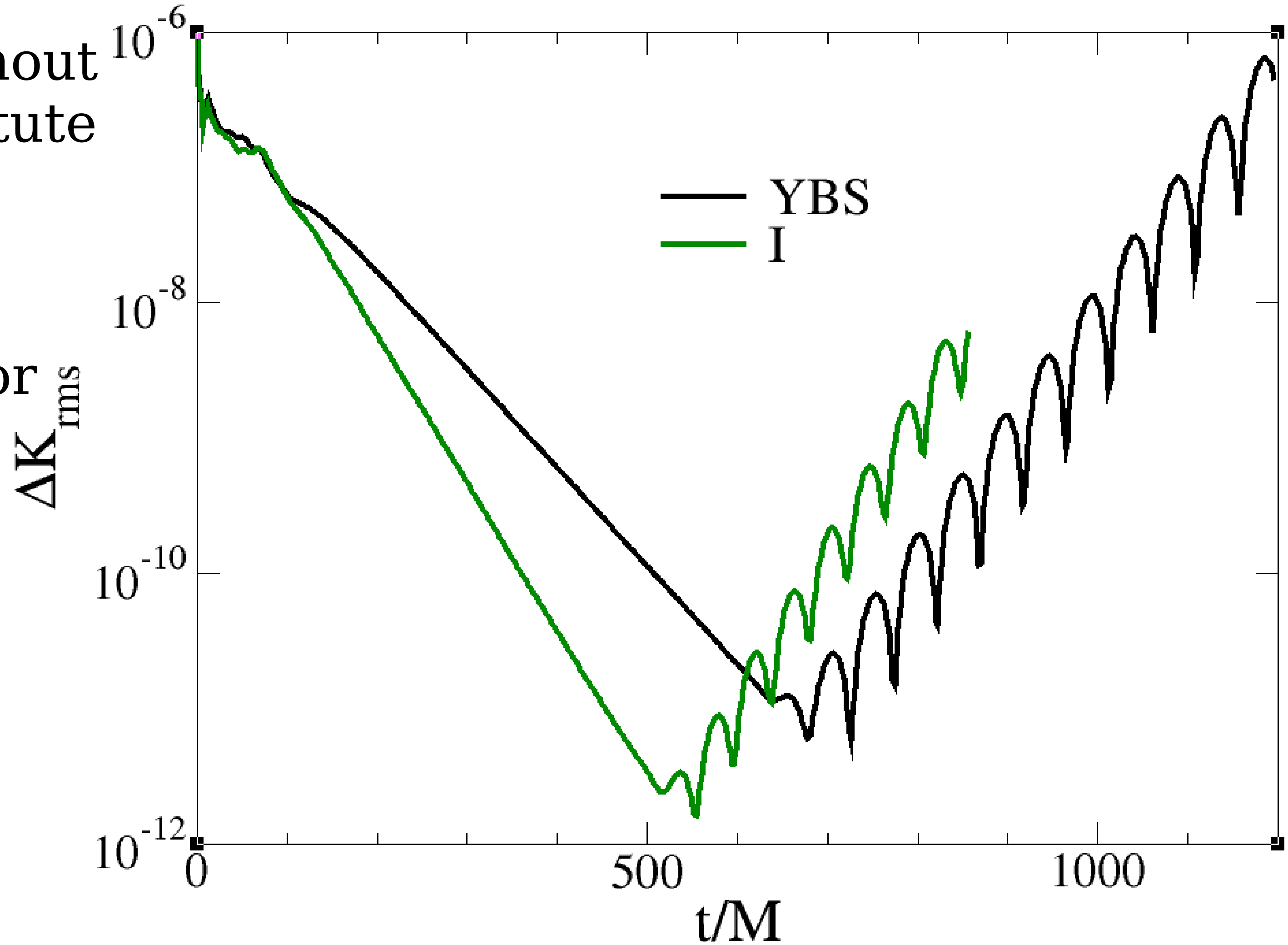
$$\tilde{\Gamma}^i \to \tilde{\gamma}^{jk} \tilde{\Gamma}^i{}_{jk}$$

Caveat:

K should be positive, or a further modification is needed.

$$\tilde{\gamma}_{zz} \quad \Leftarrow \quad \tilde{\gamma} \simeq 1$$

$$\tilde{A}_{yy} \quad \Leftarrow \quad A \simeq 0$$



Irreducible Decomposition

$$\boxed{\tilde{\Gamma}^i{}_{jk} = \tilde{\not{F}}^i{}_{jk} + V^i{}_{jk} + U^i{}_{jk} = \tilde{\not{F}}^i{}_{jk} + \frac{3}{5}\delta^i_{\langle j}\tilde{\Gamma}^\ell{}_{k\rangle\ell} - \frac{1}{5}\delta^i{}_{\langle j}\tilde{\Gamma}_{k\rangle} + \frac{1}{3}\tilde{\gamma}_{jk}\tilde{\Gamma}^i}$$

$$V^i{}_{jk} = \frac{1}{3}\tilde{\gamma}_{jk}\tilde{\Gamma}^i \quad \Leftarrow \quad \tilde{\Gamma}^i \equiv \tilde{\gamma}^{jk}\tilde{\Gamma}^i{}_{jk}$$

$$U^i{}_{jk} = \frac{3}{5}\delta^i_{\langle j}\tilde{\Gamma}^\ell{}_{k\rangle\ell} - \frac{1}{5}\delta^i{}_{\langle j}\tilde{\Gamma}_{k\rangle} \;\simeq -\frac{1}{5}\delta^i{}_{\langle j}\tilde{\Gamma}_{k\rangle} \quad \Leftarrow \quad \tilde{\Gamma}^\ell{}_{i\ell} = \partial_i \ln\sqrt{\tilde{\gamma}} \simeq 0$$

$\tilde{\not{F}}^i{}_{jk}$ = the traceless part of $\tilde{\Gamma}^i{}_{jk} = S^i{}_{jk} + A^i{}_{jk}$

$$S_{ijk} = \tilde{\not{F}}_{(ijk)} = \frac{1}{3}\left(\tilde{\not{F}}_{ijk} + \tilde{\not{F}}_{jki} + \tilde{\not{F}}_{kij}\right)$$

$$A_{ijk} = \tilde{\not{F}}_{ijk} - S_{ijk} = \frac{2}{3}\left(\tilde{\not{F}}_{ijk} - \tilde{\not{F}}_{(jk)i}\right)$$

Constraint Application

$$\boxed{\tilde{\Gamma}^i{}_{jk} \to \tilde{\Gamma}^i{}_{jk} + \frac{1}{3}\tilde{\gamma}_{jk} G^i - \frac{1}{5}\delta^i{}_{\langle j} G_{k\rangle} - \frac{3}{5}\delta^i_{\langle j}\tilde{\Gamma}^\ell{}_{k\rangle\ell}} \qquad (\mathrm{III})$$

Similarly,

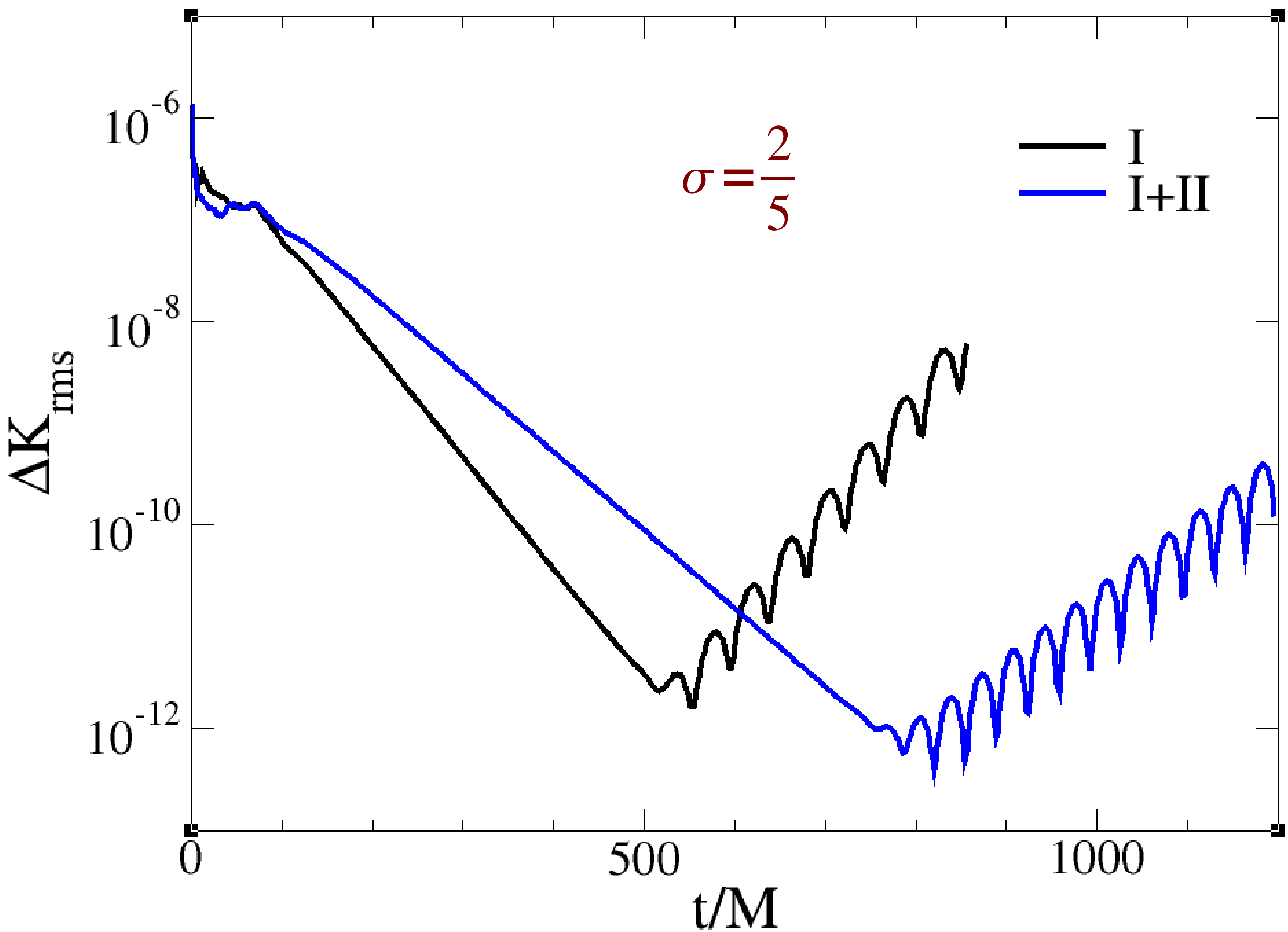
$$\partial_i \tilde{\gamma}_{jk} \to \partial_i \tilde{\gamma}_{jk} - \frac{1}{3}\tilde{\gamma}_{jk}\tilde{\Gamma}^\ell{}_{i\ell} + \frac{1}{5}\tilde{\gamma}_{i\langle j}\tilde{\Gamma}^\ell{}_{k\rangle\ell} + \frac{3}{5}\tilde{\gamma}_{i\langle j} G_{k\rangle}$$

$$\Rightarrow \boxed{\beta^i \partial_i \tilde{\gamma}_{jk} \to \beta^i \partial_i \tilde{\gamma}_{jk} + \frac{3}{5}\beta_{\langle j} G_{k\rangle} - \frac{1}{3}\tilde{\gamma}_{jk}\beta^i \tilde{\Gamma}^\ell{}_{i\ell} + \frac{1}{5}\beta_{\langle j}\tilde{\Gamma}^\ell{}_{k\rangle\ell}} \quad (\mathrm{II}')$$

Modification of the formulation II

1st derivative of the conformal metric

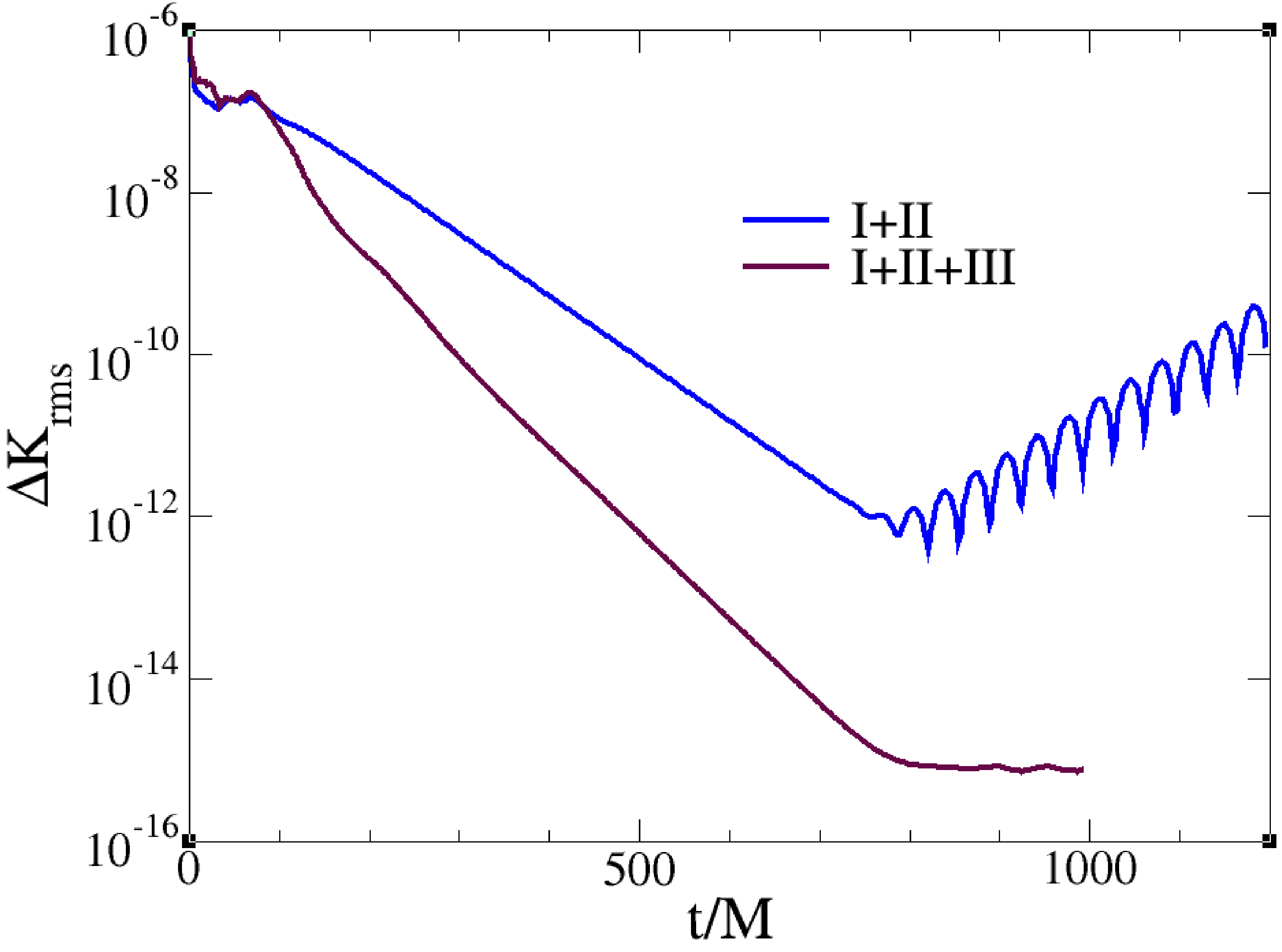
$$\beta^i \partial_i \tilde{\gamma}_{jk} \rightarrow \beta^i \partial_i \tilde{\gamma}_{jk} + \sigma \beta_{(j} G_{k)} - \frac{1}{5} \tilde{\gamma}_{jk} \beta^i G_i \quad (\mathrm{II})$$



Modification of the formulation III

connection reconstruction

$$\widetilde{\Gamma}^{i}{}_{jk} \rightarrow \widetilde{\Gamma}^{i}{}_{jk} + \frac{1}{3}\tilde{\gamma}_{jk} G^{i} - \frac{1}{5}\delta^{i}{}_{\langle j} G_{k\rangle} - \frac{3}{5}\delta^{i}_{\langle j}\widetilde{\Gamma}^{\ell}{}_{k\rangle\ell} \qquad \text{(III)}$$



Application of Momentum Constraint

- Yoneda & Shinkai ['02]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{A}_{ij} \to \frac{\mathrm{d}}{\mathrm{d}t}\tilde{A}_{ij} + \kappa_A \alpha \tilde{\nabla}_{(i} M_{j)}$$

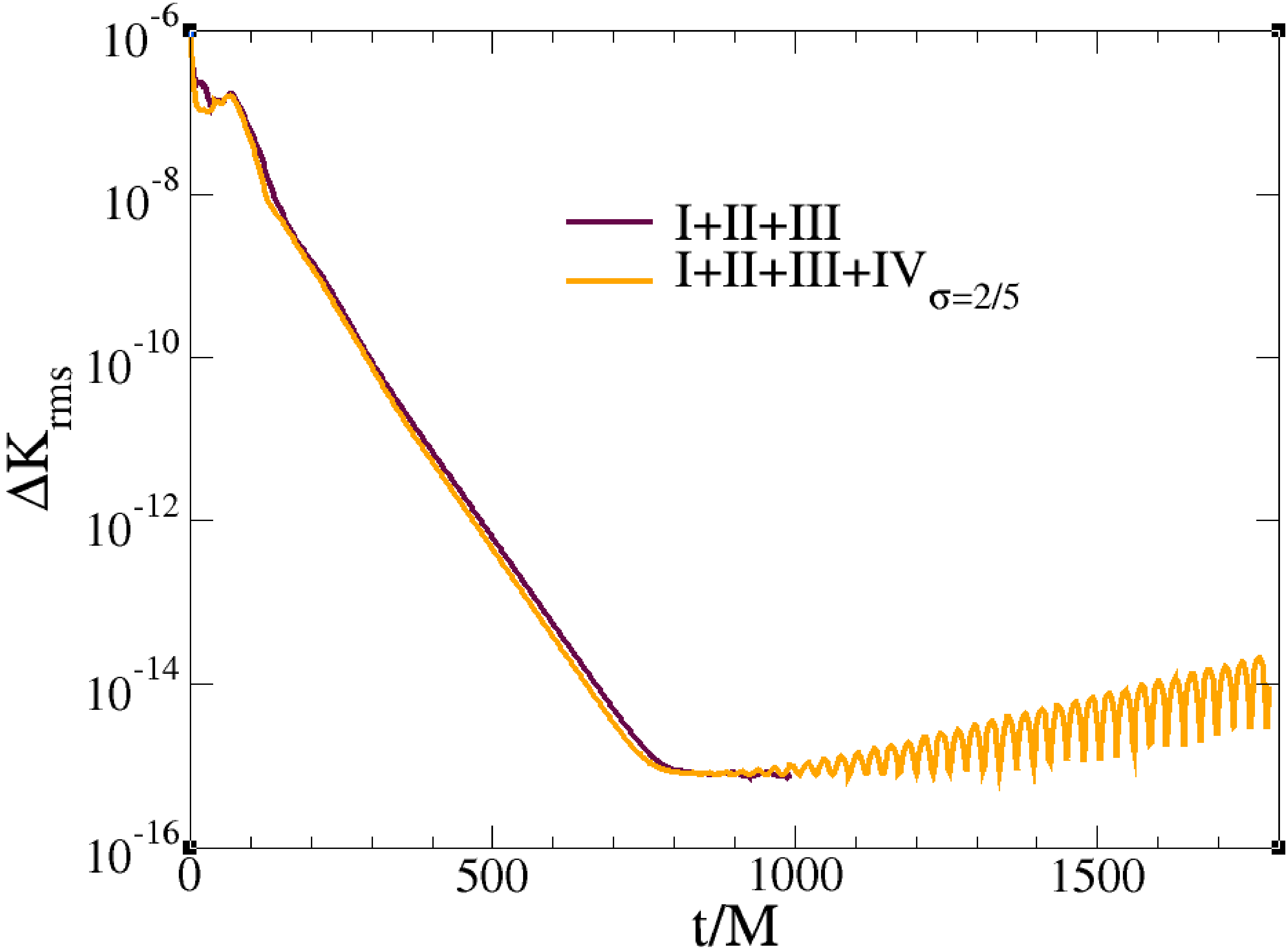
$$\Rightarrow \boxed{\frac{\mathrm{d}}{\mathrm{d}t}\tilde{A}_{ij} \to \frac{\mathrm{d}}{\mathrm{d}t}\tilde{A}_{ij} + h f(\alpha) M_{\langle i,j \rangle} \quad (\mathrm{IV})}$$

- Decomposition on extrinsic curvature

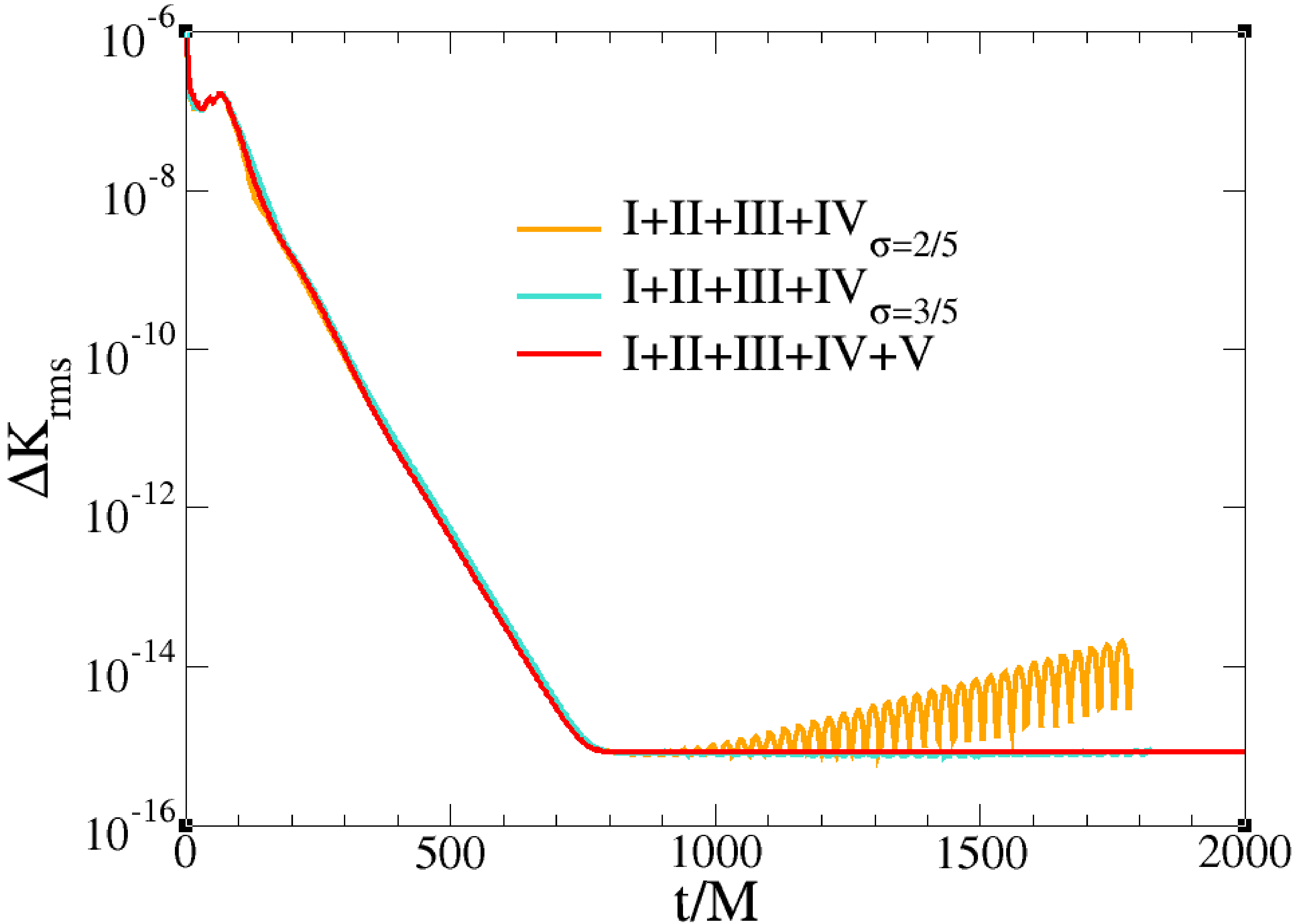
$$\partial_i \tilde{A}_{jk} \to \partial_i \tilde{A}_{jk} - \frac{1}{3}\tilde{\gamma}_{jk} A_i - \frac{1}{10}\tilde{\gamma}_{i\langle j} A_{k\rangle} - \frac{3}{5}\tilde{\gamma}_{i\langle j} M_{k\rangle}$$

$$\Rightarrow \boxed{\beta^i \partial_i \tilde{A}_{jk} \to \beta^i \partial_i \tilde{A}_{jk} - \frac{3}{5}\beta_{\langle j} M_{k\rangle} - \frac{1}{3}\tilde{\gamma}_{jk}\beta^i A_i - \frac{1}{5}\beta_{\langle j} A_{k\rangle\ell} \quad (\mathrm{V})}$$

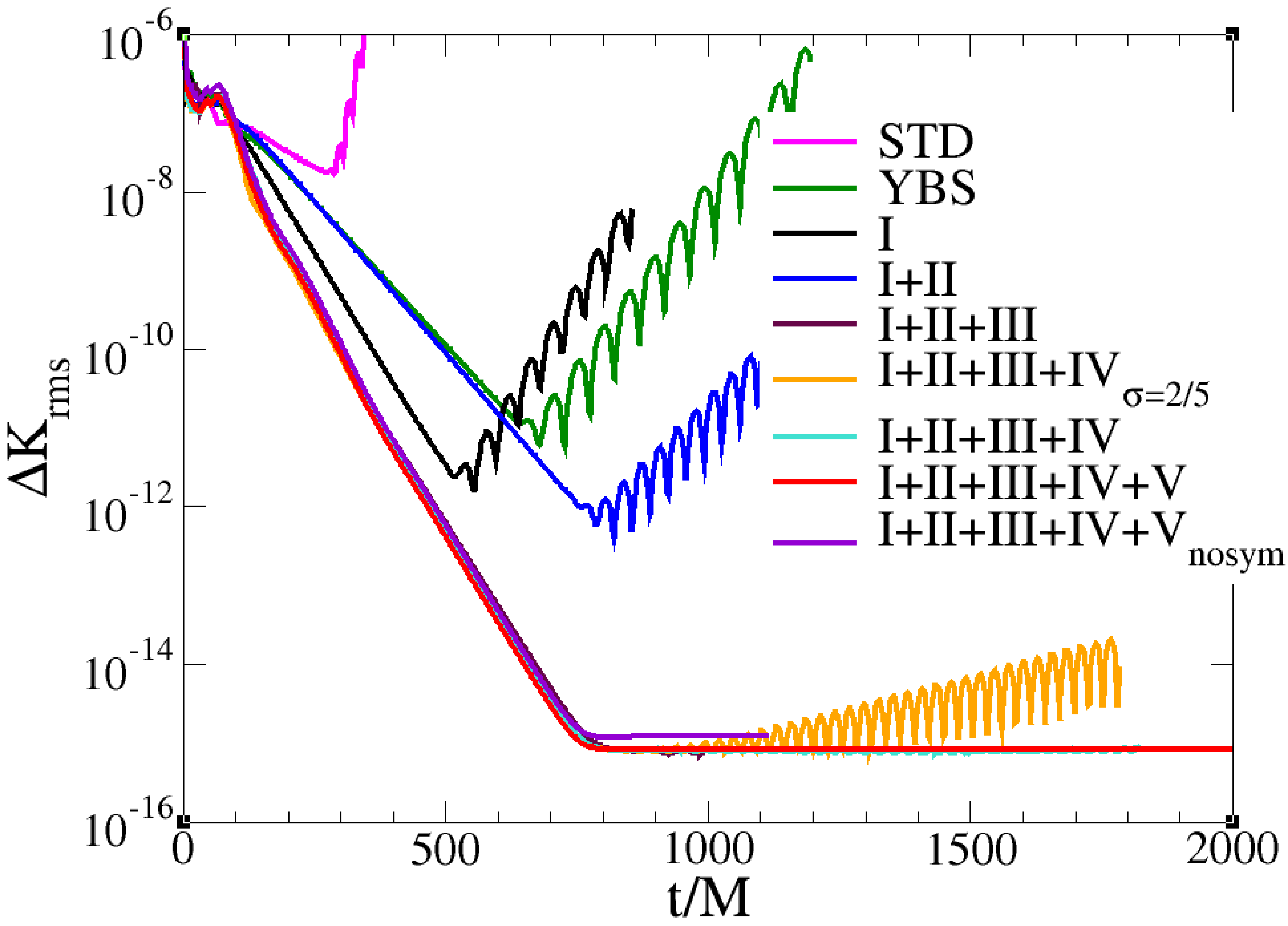
where $\quad A_i \equiv \partial_i A = \tilde{\gamma}^{jk}\tilde{A}_{jk,i} - 2\tilde{A}_{jk}\tilde{\Gamma}^{jk}{}_i \approx 0$



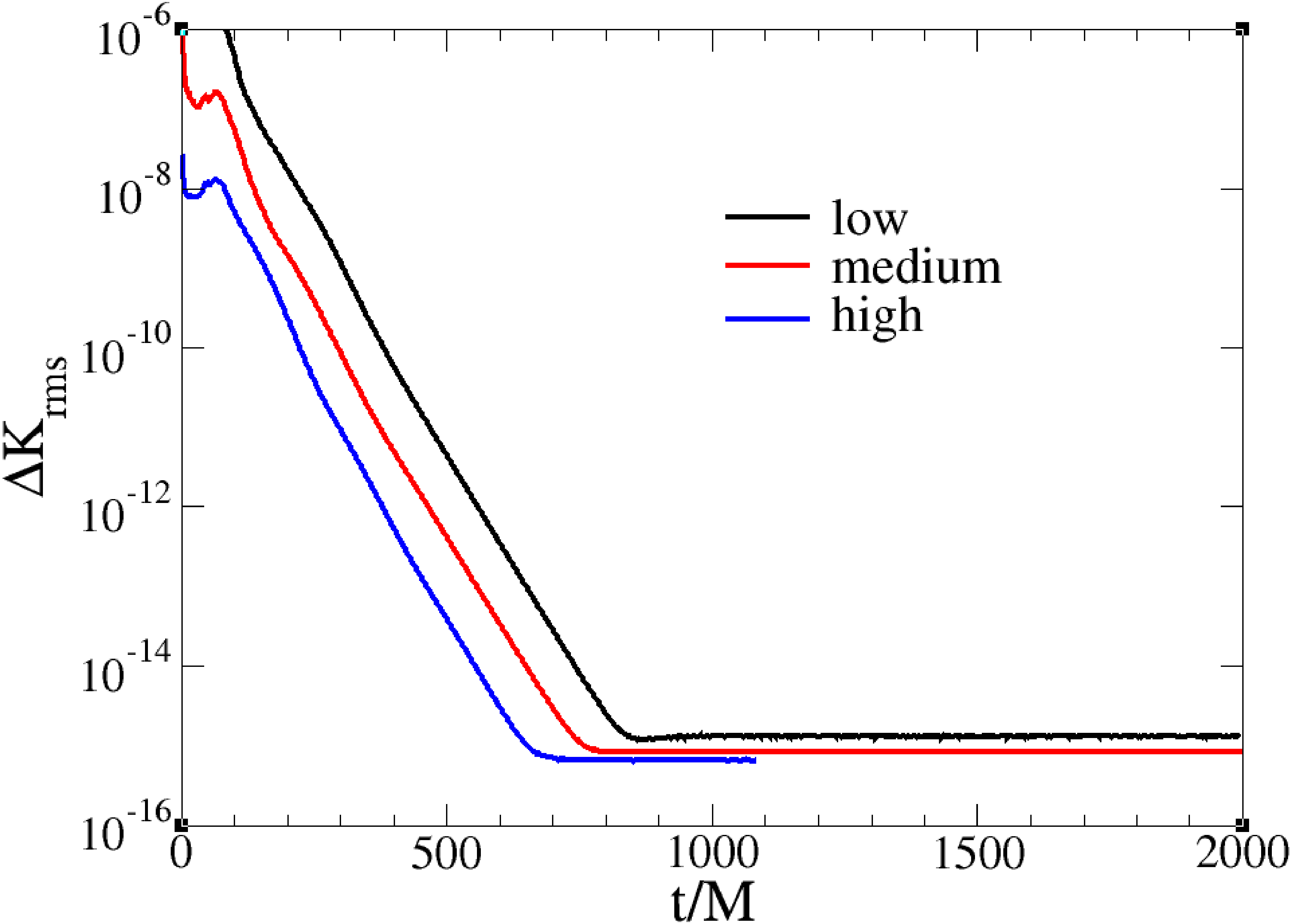
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
0
500
1000
1500
t/M
ΔK_{rms}
I+II+III
I+II+III+IV$_{\sigma=2/5}$



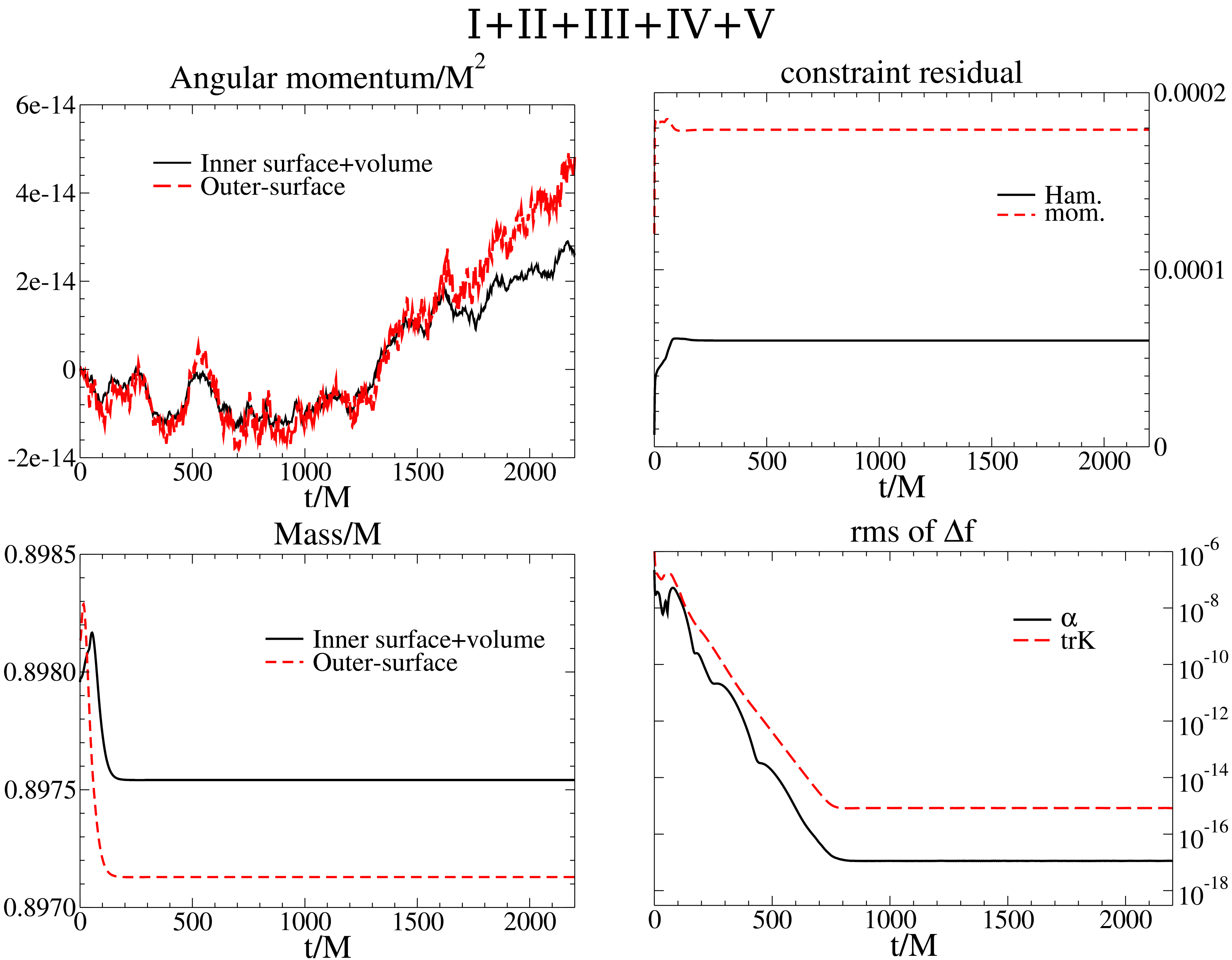
ΔK_{rms}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
0
500
1000
1500
2000
t/M
$I+II+III+IV_{\sigma=2/5}$
$I+II+III+IV_{\sigma=3/5}$
I+II+III+IV+V



10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
ΔK_{rms}
0
500
1000
1500
2000
t/M
STD
YBS
I
I+II
I+II+III
I+II+III+IV$_{\sigma=2/5}$
I+II+III+IV
I+II+III+IV+V
I+II+III+IV+V$_{nosym}$



low
medium
high
ΔK_rms
t/M
10^-6
10^-8
10^-10
10^-12
10^-14
10^-16
0
500
1000
1500
2000


I+II+III+IV+V
Angular momentum/M^2
6e-14
4e-14
2e-14
0
-2e-14
0
500
1000
1500
2000
t/M
Inner surface+volume
Outer-surface
constraint residual
0.0002
0.0001
0
Ham.
mom.
Mass/M
0.8985
0.8980
0.8975
0.8970
rms of Δf
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
10^{-18}
α
trK

Summary

- The modifications focus on the BSSN physical variables

- The recipes are able to suppress instability efficiently (at least in single Kerr-Schild BH).

- Need to understand further the behavior of these modifications in the field equations.

- Need to test these recipes in rotating BHs or BBH.