上州人

上州無智亦無才
剛毅木訥易被欺
唯以正直接萬人
至誠依神期勝利

鑑三

Introduction to
Inflationary Cosmology

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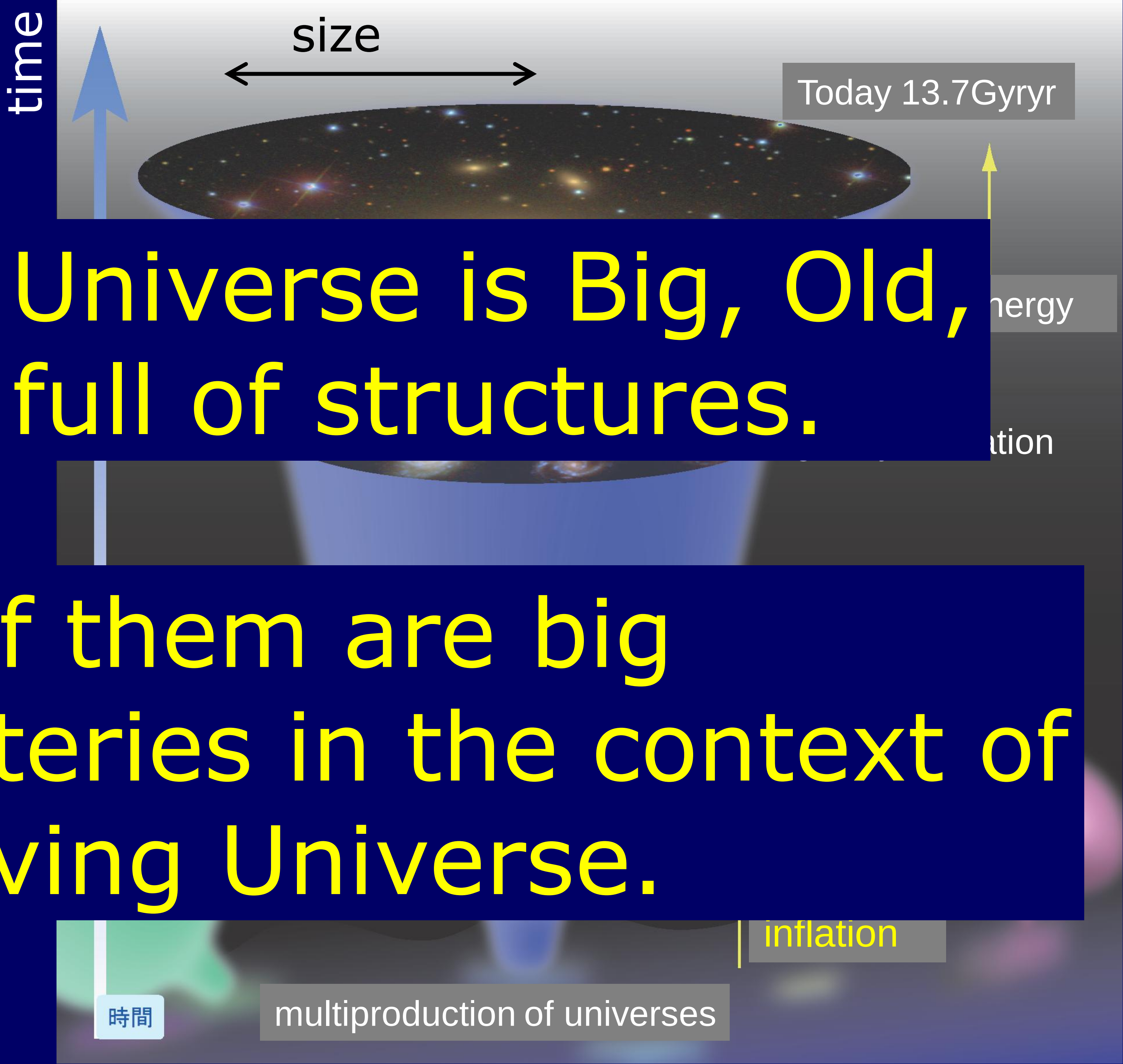
Introduction to

Inflationary Cosmology

These lectures are primarily intended for
Those who have never studied inflation
as well as for
Those who have studied inflation
but are working on bouncing cosmology
without inflation.



Our Universe is Big, Old, and full of structures.

All of them are big mysteries in the context of evolving Universe.

INFLATION in the early Universe explains

The Horizon Problem

Why is our Universe Big?

The Flatness Problem

Why is our Universe Old?

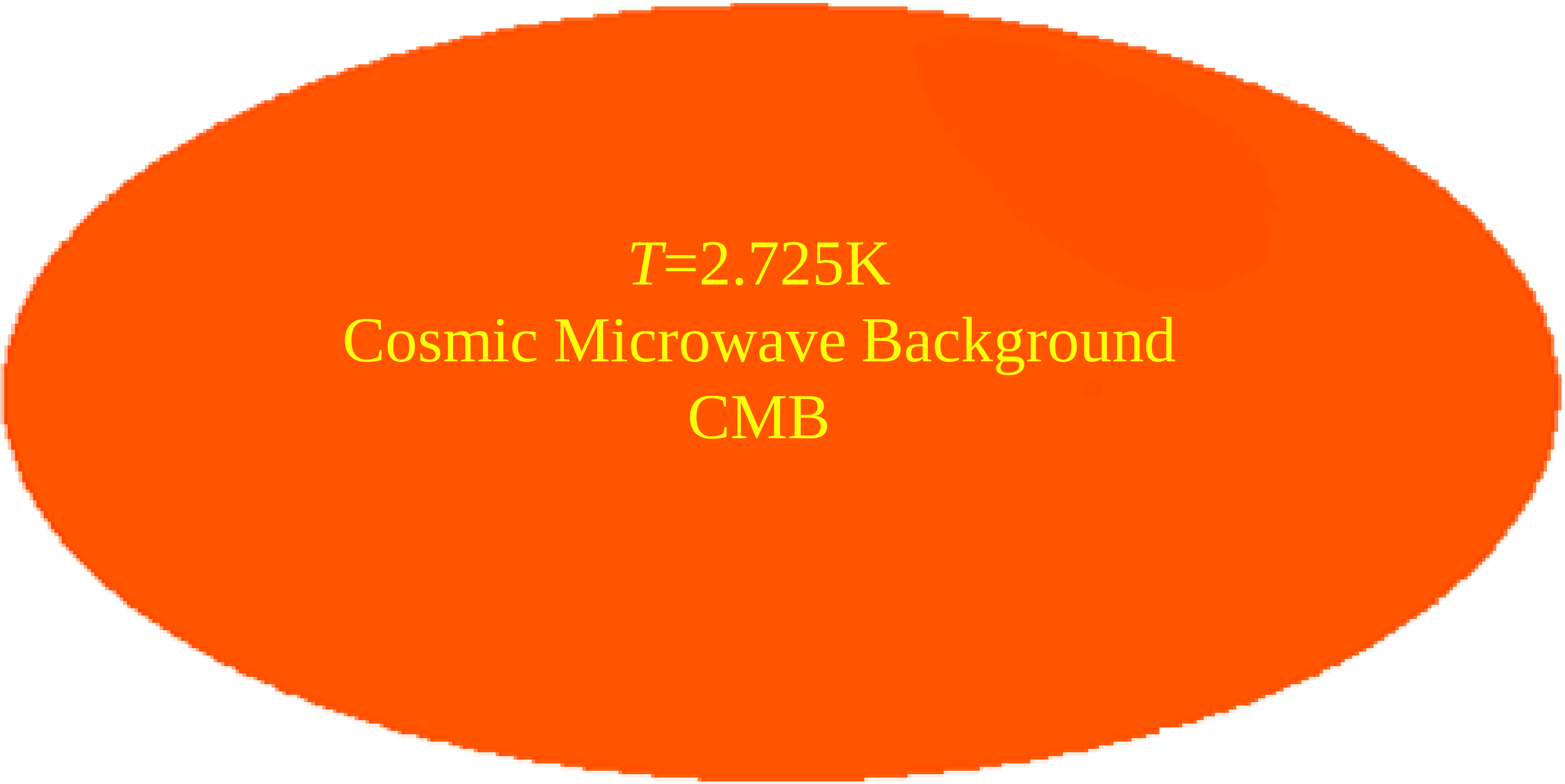
The Monopole/Relic Problem

Why is our Universe free from exotic relics?

The Origin-of-Structure Problem

Why is our Universe full of structures?

Full sky map of microwave background radiation #1



The Universe is globally isotropic and homogeneous

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right)$$

Scale factor

Curvature

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}$$

$$H = \frac{\dot{a}}{a}$$ Hubble parameter

$$\Omega = \frac{\rho}{\rho_{cr}} = \frac{8\pi G}{3H^2}\rho$$ Density parameter

$\Omega_m, \quad \Omega_b$

$$\Omega_\Lambda = \frac{\Lambda}{3H^2}$$ cosmological constant (dark energy)

$$\frac{K}{a^2} = (\Omega + \Omega_\Lambda - 1)\, H^2$$

Energy momentum tensor: Perfect Fluid

$$T^{\mu\nu} = Pg^{\mu\nu} + (\rho + P)u^{\mu}u^{\nu} \qquad u^{\mu} = (\gamma, \gamma \mathbf{v}/a)$$

Pressure (P) Energy density (ρ)

Conservation

$$T^{\mu}_{\nu\ ;\mu} = 0 \quad \Rightarrow \quad \underbrace{\frac{d}{dt}\rho a^3 = -P\frac{da^3}{dt}}_{dE = -PdV + d'Q} \quad \Rightarrow \quad \frac{d\rho}{dt} + 3H(\rho + P) = 0$$

$$d'Q = 0 \qquad d'Q = TdS = 0$$

In quasi-static processes

Comoving entropy is conserved unless some nonequilibrium processes take place.

The Einstein equations

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3P\right)$$

The Particle Horizon and The Hubble Horizon

★ Particle Horizon (Physics Horizon) $d_H(t)$

The maximum length causal interaction can reach by the time t
= The maximum length light can travel by the time t.

Light travels with $ds^2 = -dt^2 + a^2(t)d\chi^2 = 0.$

$$d_H(t) = a(t)\int d\chi = a(t)\int_{t_i}^{t} \frac{dt'}{a(t')} = \begin{cases} \boxed{\dfrac{t}{1-m}\left[1-\left(\dfrac{t}{t_i}\right)^{m-1}\right] \cong \dfrac{t}{1-m} \quad a(t)\propto t^m \text{ with } m<1 \text{ (Matter or Rad'n era)}} \\ \dfrac{t}{m-1}\left[\left(\dfrac{t}{t_i}\right)^{m-1} -1\right] \sim t^m \propto a(t) \quad a(t)\propto t^m \text{ with } m>1 \text{ (Accelerated Expansion)} \\ \dfrac{1}{H}\left(e^{H(t-t_i)} -1\right) \sim e^{Ht} \propto a(t) \quad a(t)\propto e^{Ht} \text{ (Exponential Expansion)} \end{cases}$$

The Classical Big Bang Theory has only this epoch.

Hubble Horizon, Hubble Radius, Hubble Length

- The scale causal interaction is possible within the cosmic expansion time H^{-1}

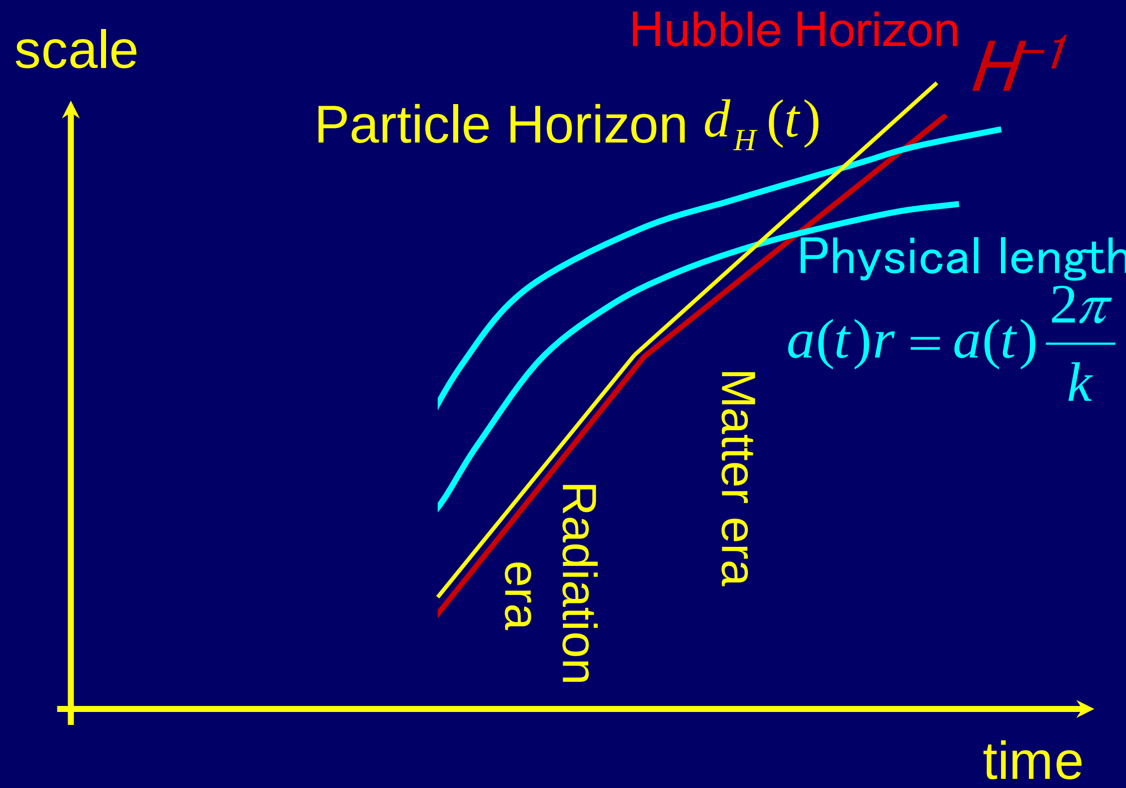
- One can neglect effects of expansion within this time scale, so one finds

$$cH^{-1} = H^{-1} = \begin{cases} \dfrac{t}{m} & a(t) \propto t^m \\ H^{-1} & a(t) \propto e^{Ht} \end{cases}$$

- In the expanding Universe, various events have taken place at different epochs of the relevant energy scales. The Hubble radius gives the maximum scale that each event can occur coherently.

- Important when particle physics is applied to cosmology.
- The term “Horizon” most likely means the Hubble horizon.
- The maximum scale we can directly observe at each time.

Evolution of scales in the Classical Big Bang Theory



- In the Classical Big Bang Theory, both the particle horizon and the Hubble horizon evolves in proportion to time, namely more rapidly than the physical length of each coordinate scale ($\propto a(t)$).
- The scales of no previous causal interaction enter the Hubble radius continuously and can be seen for the first time.

They look all the same!＝The Horizon Problem

The Universe at the Decoupling Epoch

The Hubble Radius Then
～1 angular degree

We must sum up more than 10^5 causal patches to make up the current Hubble volume.

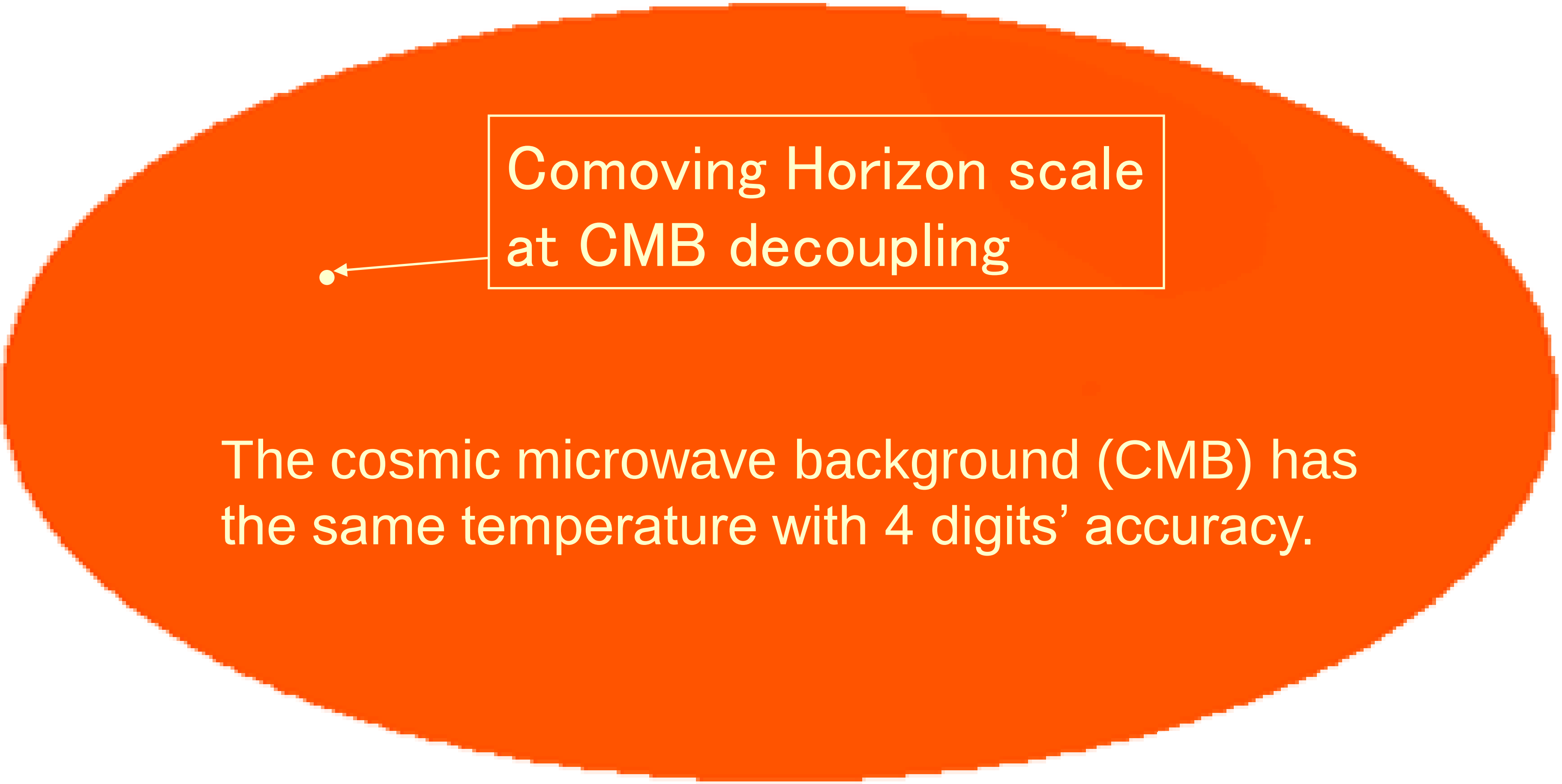
The Horizon Problem

INFLATION in the early Universe

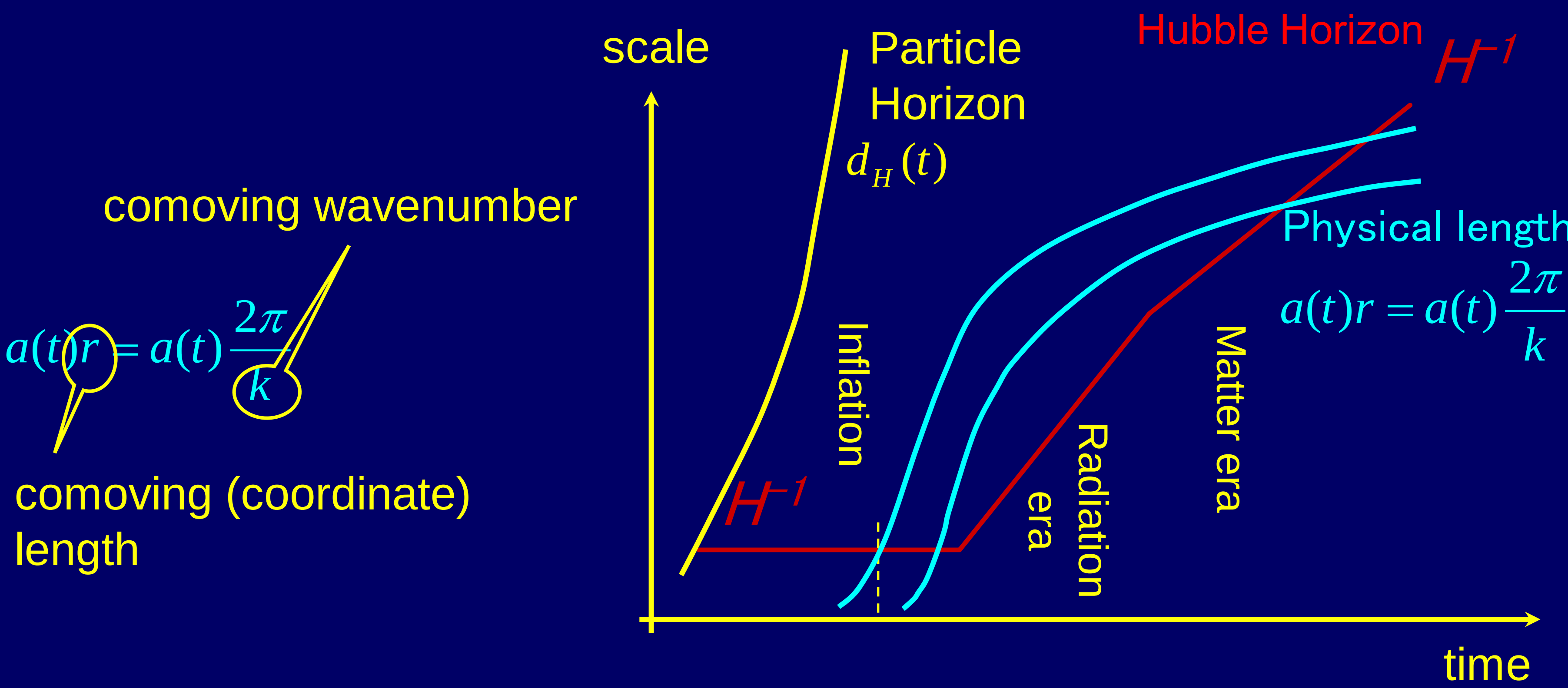
solves...

The Horizon problem

The Universe observed by COBE



Evolution of scales in the Inflationary Cosmology



- The particle horizon is exponentially stretched.
- Each coordinate scale crosses the Hubble horizon twice, during and after inflation.
- In between two horizon crossing epochs, that scale is beyond the Hubble radius and hence invisible.

INFLATION in the early Universe

solves...

The Horizon problem
The Flatness problem

FLATNESS PROBLEM

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}$$

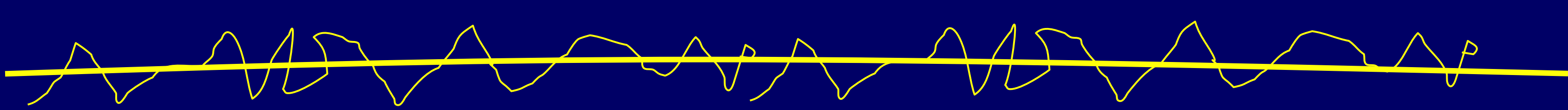
The curvature term decreases less rapidly than matter or radiation.

$\propto a^{-3}$ $\propto a^{-4}$

$$\frac{K}{a^2} = \left(\Omega_{tot} - 1\right)H^2 \quad \Rightarrow \frac{\Omega_{tot}(t_0)-1}{\Omega_{tot}(t_i)-1} = \left(\frac{a(t_i)}{a_0}\right)^2 \left(\frac{H(t_i)}{H_0}\right)^2 \sim \left(\frac{T_0}{T_i}\right)^2 \left(\frac{t_0}{t_i}\right)^2$$

$$\sim 10^{58} \;@\; t_i = t_{Pl}$$

- At the Planck time the curvature radius must have been larger than the Hubble radius by more than 10^{29} times.



INFLATION in the early Universe

solves...

The Horizon Problem
The Flatness Problem
The monopole & other relic Problems

If one monopole is created per horizon@GUT phase transition,

$$\frac{n_M}{s} \approx 10 \frac{M}{M_{Pl}} \approx 10^{-10} \frac{M}{10^{16}\,\mathrm{GeV}}$$ vs current constraint $$\frac{n_M}{s} < 10^{-24} \left(\frac{M}{10^{16}\,\mathrm{GeV}} \right)^{-1}$$

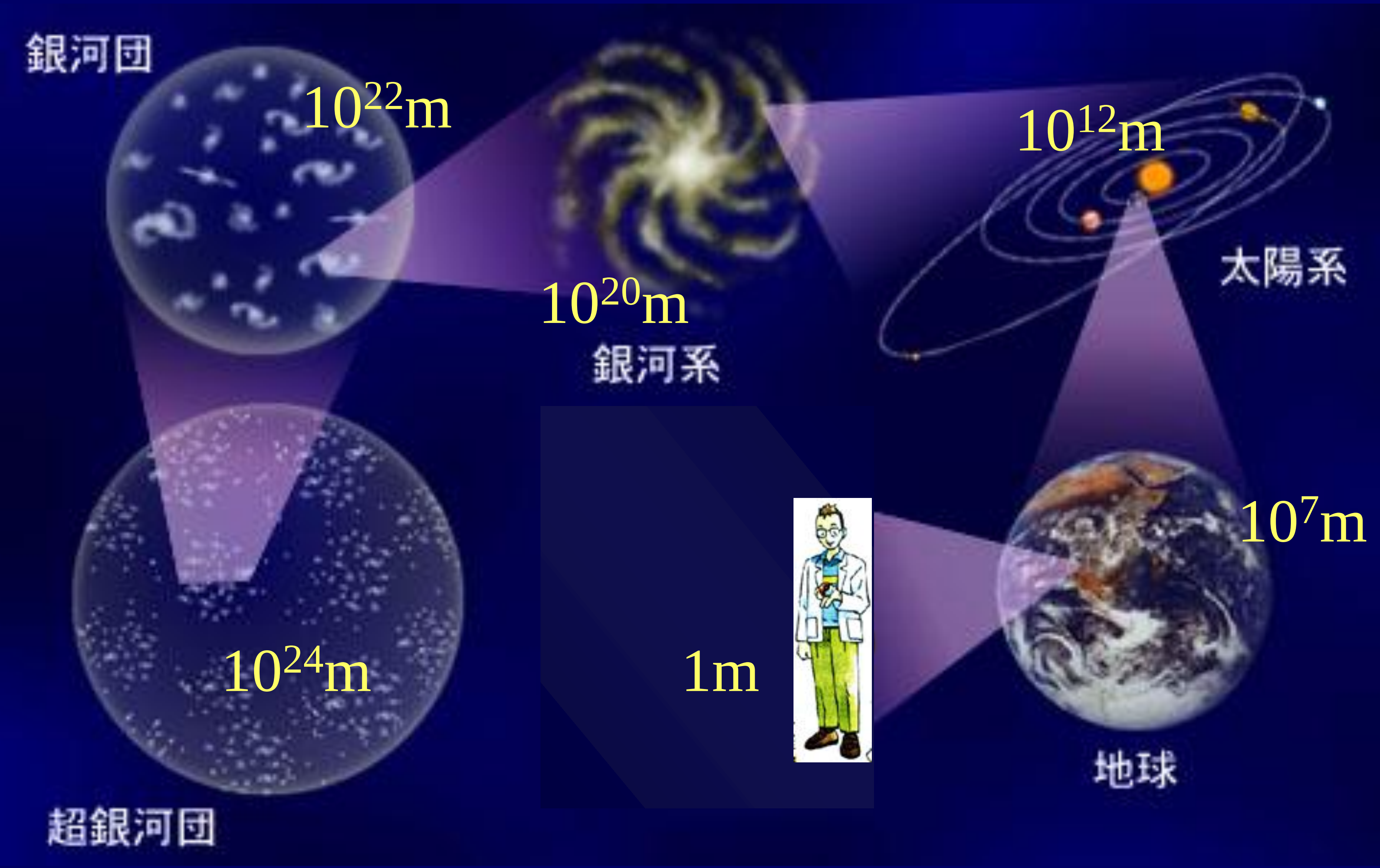
Monopoles and other relics /entropy are NOT diluted by inflationary expansion but by the subsequent entropy production at the reheating.

INFLATION in the early Universe

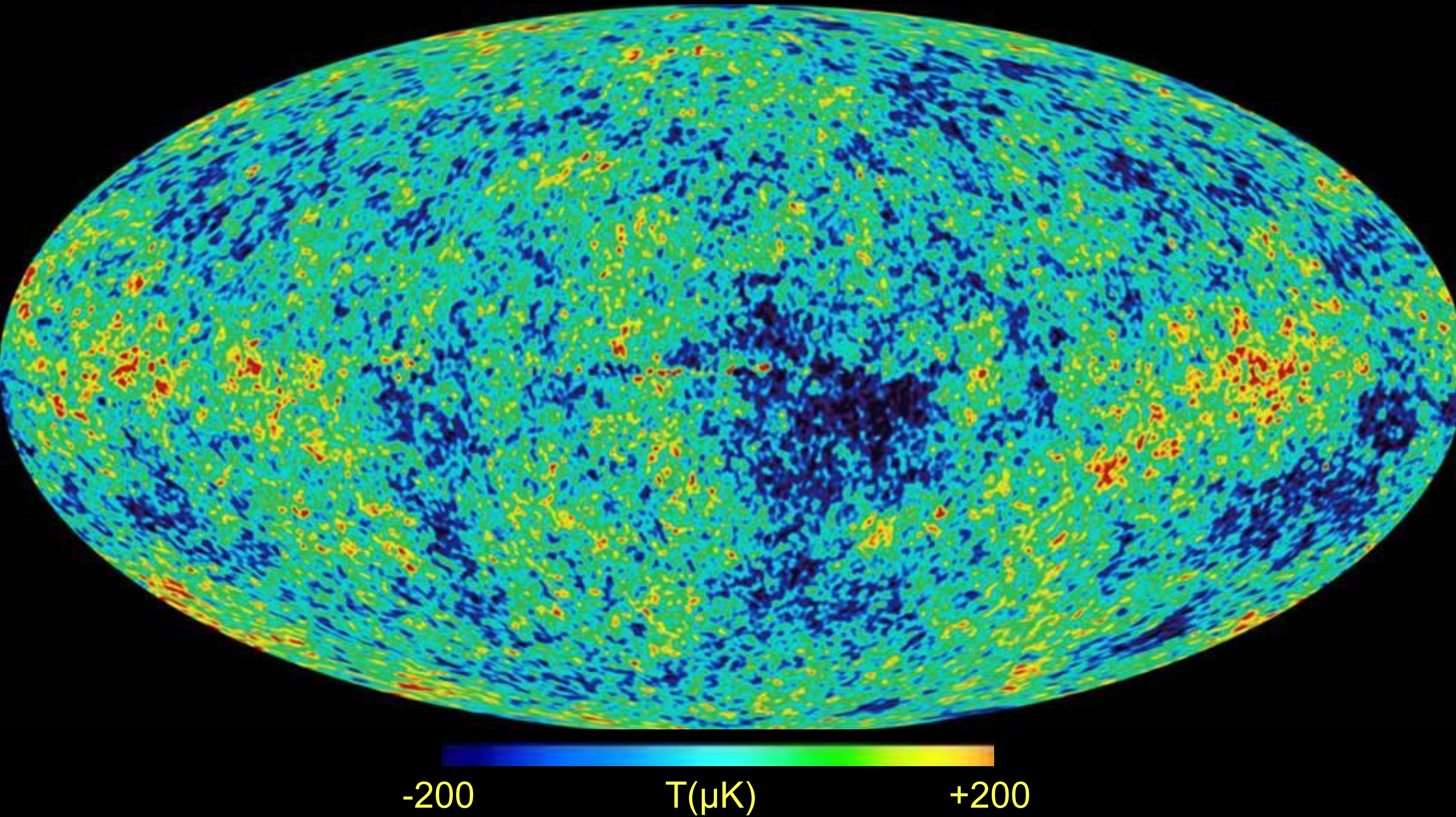
solves...

The Horizon Problem
The Flatness Problem
The monopole & other relic Problem
The origin-of-fluctuations Problem

Our Universe has hierarchical structures.

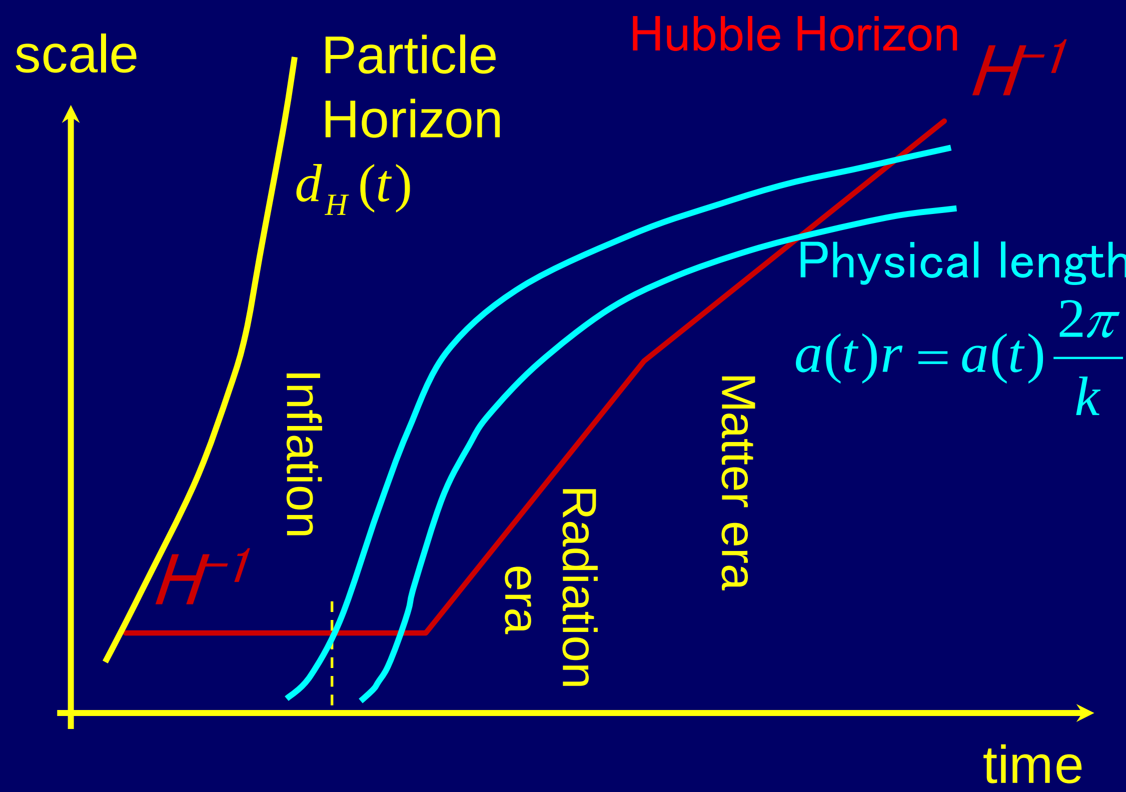


Their seed has also been observed as CMB anisotropy.



Temperature anisotropy at the level of 10^{-5}.

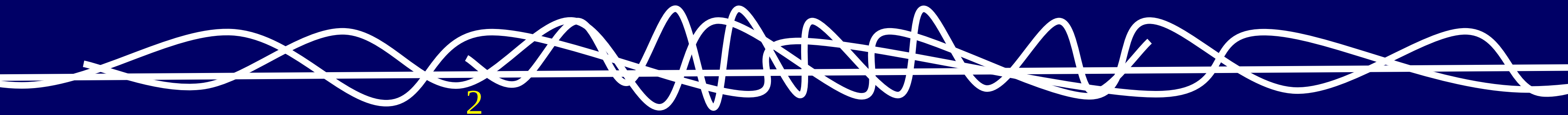
Evolution of scales in the Inflationary Cosmology



- The particle horizon is exponentially stretched.
- Each coordinate scale crosses the Hubble horizon twice, during and after inflation.
- In between two horizon crossing epochs, that scale is beyond the Hubble radius and hence invisible.
- During inflation, superhorizon fluctuations may be generated.

Inlation solves all these problems by

- Accelerated Expansion



- $a(t) \propto t^{\frac{2}{3(1+w)}}$ for $p = w\rho$

so $\boxed{w < -\frac{1}{3}}$ $\rho \propto a^{-3(1+w)}$ decreases less rapidly than the curvature term.

$$\boxed{w = -1 \Rightarrow a(t) \propto e^{Ht} \quad \rho = \text{const: } \Lambda_{\text{eff}}}$$

- Followed by Entropy Production

Reheating

Is inflation natural? Yes, if not always.

Cosmic No Hair Conjecture

If there exist a positive effective cosmological constant $\Lambda_{\rm eff}$(vacuum energy), then the Universe undergoes an exponential expansion within the Hubble time determined by the vacuum energy density.

It is easy to make counter examples, so it does not always hold.

Still there are proofs in some limited cases.

Homogeneous but anisotropic space (Wald 1983)

Bianchi type I∼VIII (spatially flat or open), inflation occurs with $\Lambda_{\rm eff}$.
Bianchi type IX (positive curvature), inflation occurs if $\Lambda_{\rm eff} > \frac{1}{2} R^{(3)}_{\rm max}$.

maximum 3-curvature with fixed spatial volume

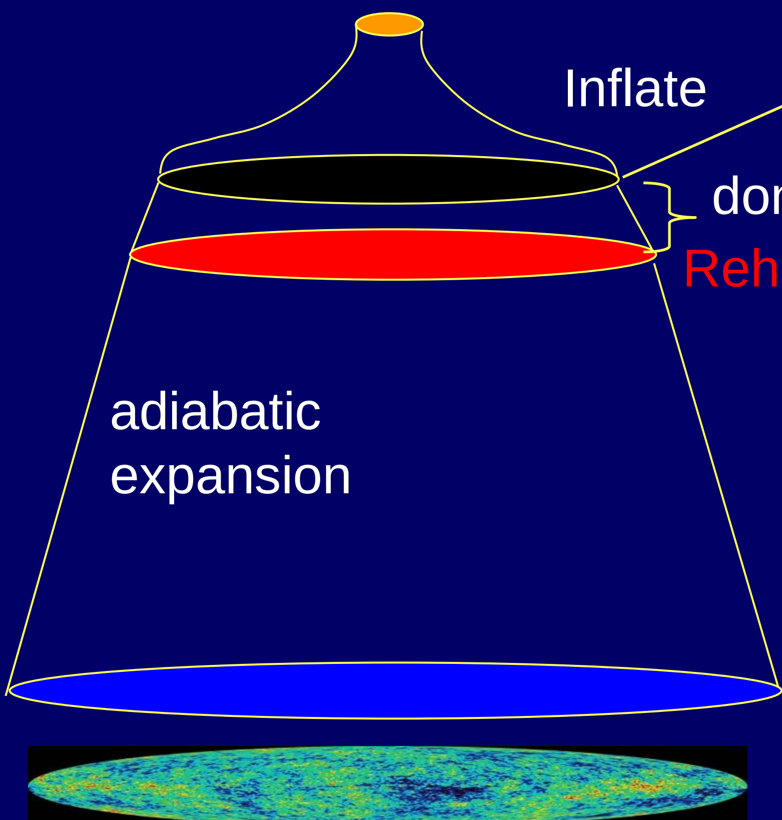
Inhomogeneous space

$\exists\Lambda_{\rm eff}$ Inflation occurs if $R^{(3)} < 0$ everywhere. (This condition is too strong.)

Numerical analysis suggests that if there exists $\Lambda_{\rm eff}$ and inhomogeneity in the corresponding Hubble volume is at most around unity, then inflation sets in for a wide class of initial conditions. (Goldwirth & Piran 92)

How much inflation is required to solve the horizon and the flatness problems?

The initial Hubble patch with radius H^{-1}



expand by $a_f / a_i \equiv e^N$ times

dominated by energy w/ EOS $P = w\rho$

Reheating

Reheating temperature T_R

The initial Hubble patch has expanded to

$$H^{-1} e^{N} \left(\frac{\pi^2 g_* T_R^4}{30 \rho_{\rm inf}} \right)^{-\frac{1}{3(1+w)}} \equiv r_H$$

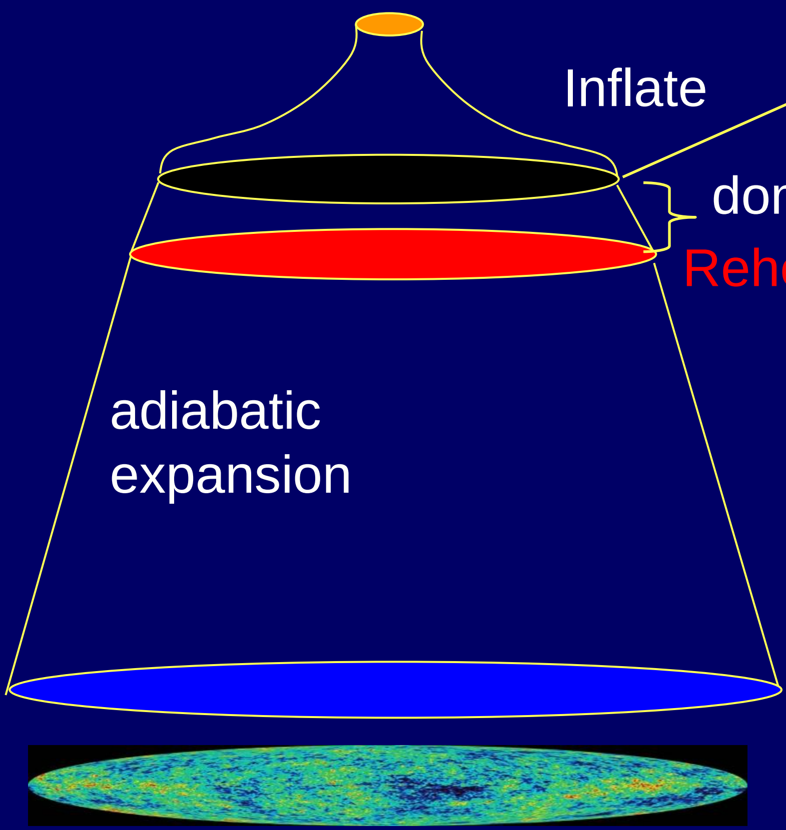
This region must be bigger than

the observable region, whose entropy is

given by $S_0 = 2.6 \times 10^{88}$ (2.7K CMB photon & 1.95K neutrinos × 3 generations in the Hubble radius $H_0^{-1} = 4.2 \times 10^3 \mathrm{Mpc}$).

How much inflation is required to solve the horizon and the flatness problems?

The initial Hubble patch with radius H^{-1}



expand by $a_f/a_i \equiv e^N$ times

dominated by a field w/ EOS $P = w\rho$

Reheating

Reheating temperature T_R

The initial Hubble patch has expanded to

$$H^{-1}e^{N}\left(\frac{\pi^2 g_* T_R^4}{30\rho_{\rm inf}}\right)^{-\frac{1}{3(1+w)}} \equiv r_H$$

The entropy contained in this region is given by

$$S=\frac{4\pi^2 g_*}{90}T_R^3\times\frac{4\pi}{3}r_H^3=\frac{16\pi^3}{270}\left(\frac{45}{4\pi^3}\right)^{\frac{1}{1+w}} g_*^{\frac{w}{1+w}}\left(\frac{H}{M_{Pl}}\right)^{-\frac{1+3w}{1+w}}\left(\frac{T_R}{M_{Pl}}\right)^{\frac{-1+3w}{1+w}} e^{3N}$$

$$S=\frac{4\pi^2 g_*}{90}T_R^3\times\frac{4\pi}{3}r_H^3=\frac{16\pi^3}{270}\left(\frac{45}{4\pi^3}\right)^{\frac{1}{1+w}}g_*^{\frac{w}{1+w}}\left(\frac{H}{M_{Pl}}\right)^{-\frac{1+3w}{1+w}}\left(\frac{T_R}{M_{Pl}}\right)^{\frac{-1+3w}{1+w}}e^{3N}$$

must be larger than $S_0=2.6\times10^{88}$, so the number of e-folds N must satisfy

$$N>67.7-\frac{12.5-10.8w}{1+w}-\frac{w}{3+3w}\ln\left(\frac{g_*}{106.75}\right)+\frac{1+3w}{6(1+w)}\ln\left(\frac{r}{0.01}\right)$$
$$+\frac{1-3w}{3+3w}\ln\left(\frac{T_R}{10^8\mathrm{GeV}}\right)\equiv N_{\min},\quad r\equiv 0.01\left(\frac{H}{2.4\times10^{13}\mathrm{GeV}}\right)^2$$

However, the above is merely a condition that the initial Hubble patch should have expanded larger than the current Hubble patch whose fluctuation is only at the level of 10^{-5}.

If the initial Hubble patch had fluctuations of order of unity, then it must expand by $\left(10^{-5}\right)^{-\frac{1}{2}}\sim500$ times longer than $N_{\min}$.

So the minimal condition for the number of e-folds reads

$N > N_{\min} + \ln 500 = N_{\min} + 6.2$, namely,

$$N > 55 + \frac{1}{6}\ln\left(\frac{r}{0.01}\right) + \frac{1}{3}\ln\left(\frac{T_R}{10^8\,\mathrm{GeV}}\right)$$

for $w = 0$.
(standard inflation)

and

$$N > 67 - \frac{1}{6}\ln\left(\frac{g_*}{106.75}\right) + \frac{1}{3}\ln\left(\frac{r}{0.01}\right) - \frac{1}{3}\ln\left(\frac{T_R}{10^8\,\mathrm{GeV}}\right)$$

for $w = 1$. (k-inflation or G inflation.)

Flatness of the Universe

$$\frac{K}{a^2} = H^2(\Omega_{\rm tot} - 1)$$

$$\frac{\Omega_{\rm tot}(t_0) - 1}{\Omega_{\rm tot}(t_i) - 1} = \left(\frac{a(t_i)H_{\rm inf}}{a_0 H_0}\right)^2 = e^{-2(N - N_{\rm min})} < 500^{-2} = 4 \times 10^{-6}$$

initial value at the onset of inflation

Prediction of Inflation I

If inflation solves the horizon problem, it predicts that our Universe is spatially flat with

$$\left|\Omega_{tot\,0} - 1\right| < 10^{-5}$$

Once inflation sets in, the Universe rapidly becomes homogeneous & isotropic, and almost spatially flat.

$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho$$

rapidly decreases

Anisotropic space

$$\bar{H}^2 + \underbrace{\frac{K(\ldots)}{\bar{a}^2} + \boxed{\frac{S(\ldots)}{\bar{a}^6}}}_{} = \frac{8\pi G}{3}\rho$$

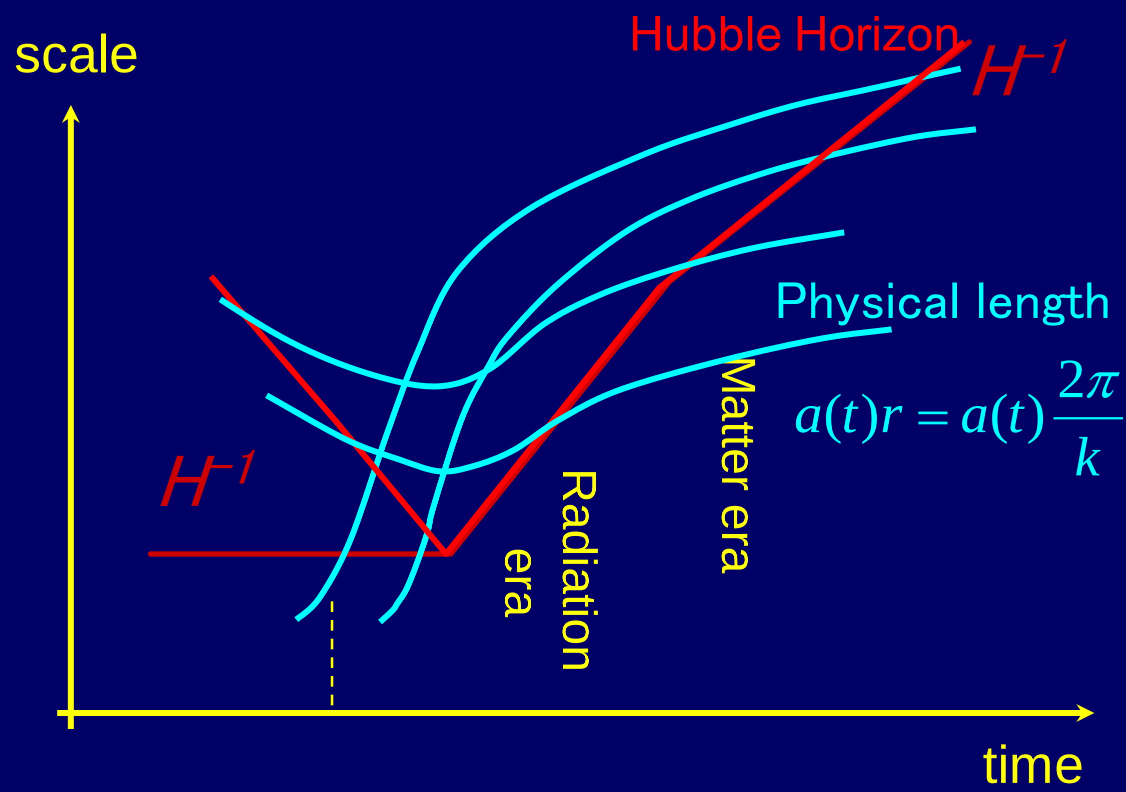
$$\bar{a} \equiv \sqrt[3]{\text{spatial volume factor}}$$

$$\bar{H} = \frac{\dot{\bar{a}}}{\bar{a}}$$

decreases with the same rate
as the spatial curvature
in the expanding phase.

Increases very rapidly
in a contraction phase
Problem for a
bouncing cosmology

Evolution of scales in the Bouncing Cosmology



-
- Each coordinate scale crosses the Hubble horizon twice, contraction and expansion stages.
- In between two horizon crossing epochs, that scale is beyond the Hubble radius and hence invisible.
- During bounce, superhorizon fluctuations may be generated.

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鑑三

What drives INFLATION?

Energy density of a scalar field

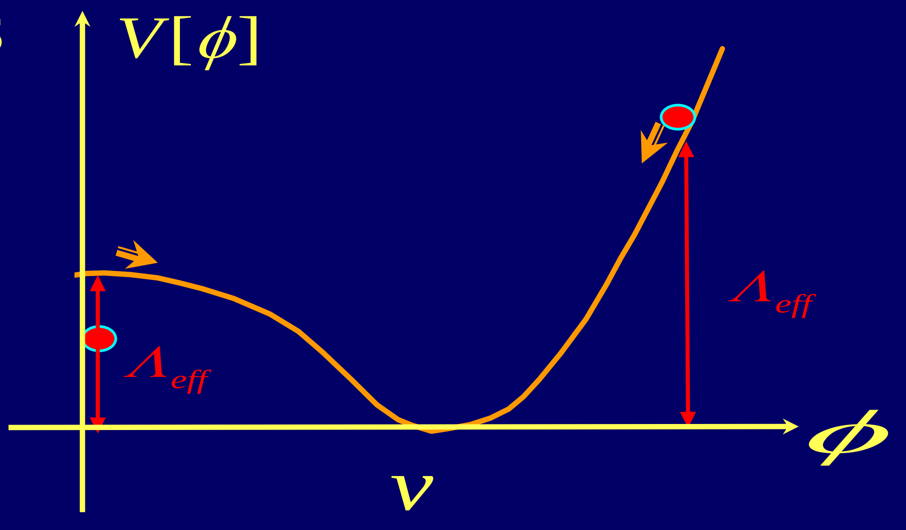
- A. Canonical Scalar Field

$$S=\int \mathcal{L}\sqrt{-g}\,d^4x=\int\left[-\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-V[\phi]\right]\sqrt{-g}\,d^4x$$

$$T_{\mu\nu}(x)=-\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}(x)}=\partial_\mu\phi\partial_\nu\phi+\left[-\tfrac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi-V[\phi]\right]g_{\mu\nu}$$

$$\boxed{\rho=\tfrac{1}{2}\dot{\phi}^2+V[\phi],\quad P=\tfrac{1}{2}\dot{\phi}^2-V[\phi]}$$

Inflation with $w=P/\rho=-1$ is realized if potential energy dominates.



Inflation driven by a canonical scalar field

$$\mathcal{L} = -\frac{1}{2} g_{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V[\phi] \Rightarrow \rho = \frac{1}{2}\dot{\phi}^2 + V[\phi], \quad P = \frac{1}{2}\dot{\phi}^2 - V[\phi]$$

Einstein equation

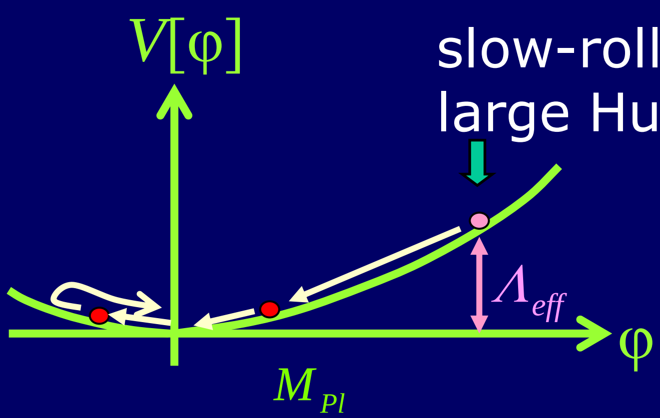
$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V[\phi]\right)$$

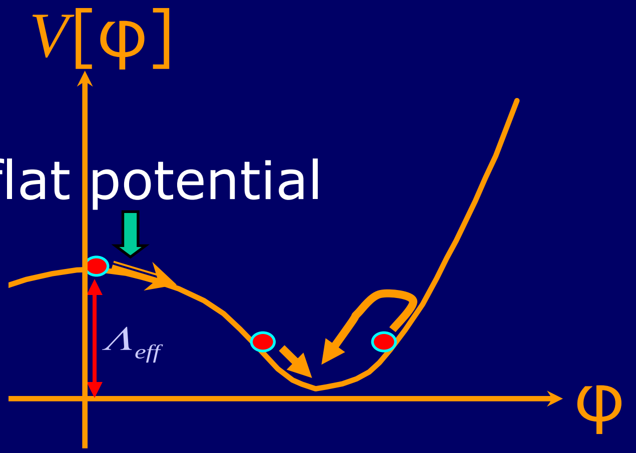
Field equation

$$\ddot{\phi} + 3H\dot{\phi} + V'[\phi] = 0$$

$P = -\rho$

If energy density is dominated by the potential, inflation occurs.





Slow-roll equations of motion

$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V[\phi]\right)$$

$$\ddot{\phi} + 3H\dot{\phi} + V'[\phi] = 0$$

Inflation driven by a canonical scalar field

$$\mathcal{L} = -\frac{1}{2} g_{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - V[\phi] \quad \Rightarrow \quad \rho = \frac{1}{2}\dot{\phi}^2 + V[\phi], \quad P = \frac{1}{2}\dot{\phi}^2 - V[\phi]$$

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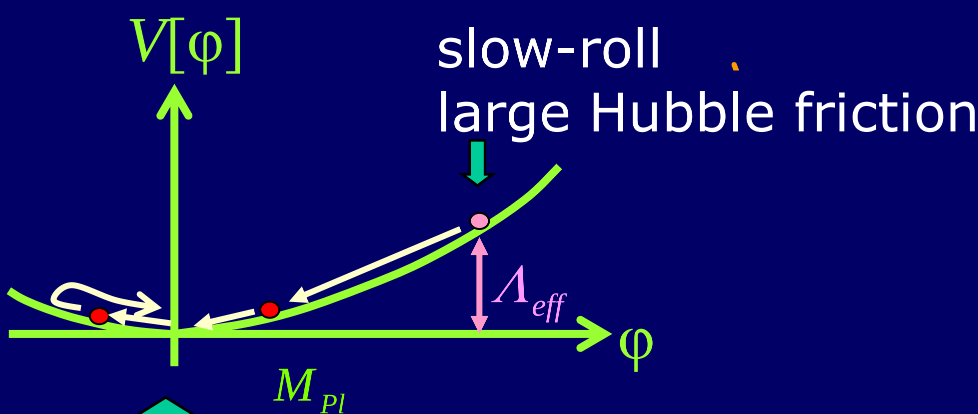
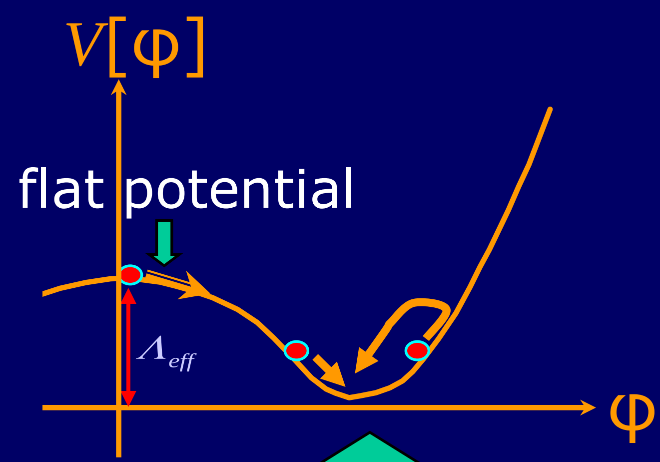




field oscillation around the minimum

→particle production→reheating

But it is not a mandatory requirement…

What drives INFLATION?

Energy density of a scalar field

- B. Non-Canonical Scalar Field (k-inflation)

(Armendariz-Picon, Damour, & Mukhanov 99)

$$S=\int \mathcal{L}\sqrt{-g}\,d^4x=\int K(X,\phi)\sqrt{-g}\,d^4x, \quad X\equiv -\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

$$T_{\mu\nu}(x)=Kg_{\mu\nu}-K_X(-\partial_\mu\phi\partial_\nu\phi)\equiv Pg_{\mu\nu}+(\rho+P)u_\mu u_\nu$$

$$\boxed{\rho=2XK_X-K, \quad P=K} \qquad K_X=\frac{\partial K}{\partial X}$$

Inflation is possible if $K<XK_X$.
Exponential inflation is realized if $K_X=0$.
For $K(X,\phi)=K_1(\phi)X+K_2(\phi)X^2$, $K_1(\phi)K_2(\phi)<0$ is required.

- C. with a Higher derivative term (G-inflation)

(Kobayashi, Yamaguchi, & JY 10)

$$\mathcal{L}_\phi=K(\phi,X)-G(\phi,X)\Box\phi\ldots,$$

The original references to Inflation

Scientific Background on the Nobel Prize in Physics 2011

THE ACCELERATING UNIVERSE

compiled by the Class for Physics of the Royal Swedish Academy of Sciences

In order to explain how the Universe can be so homogeneous with different parts that seemingly cannot have been in causal contact with each other, the idea of an inflationary phase in the early Universe was put forward [30].

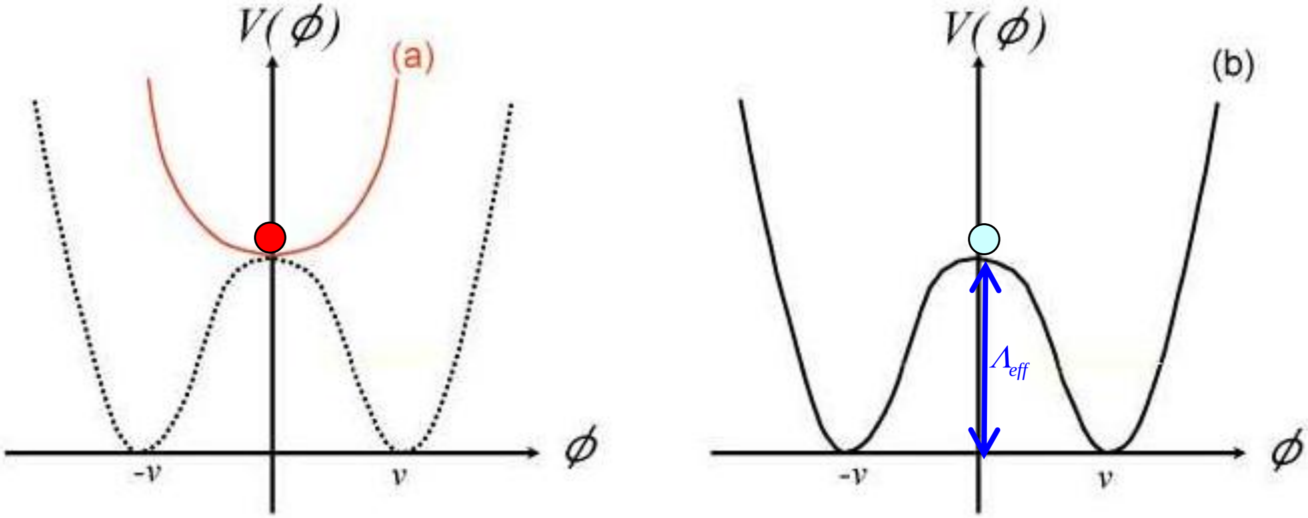
Old inflation (1^{st} order phase transition)

R^2 inflation (still viable)

[30] A. Starobinsky, “A new type of isotropic cosmological models without singularity”, Phys. Lett., **B91**, 99-102, (1980);
K. Sato, “First order phase transition of a vacuum and expansion of the Universe”, MNRAS, **195**, 467-479, (1981);
A.H. Guth, “The inflationary universe: A possible solution to the horizon and flatness problems”, Phys. Rev., **D23**, 347-356, (1980);
A.D. Linde, “A new inflationary scenario: A possible solution to the horizon, flatness, homogeneity, isotropy and primordial monopole problems”, Phys. Lett., **B108**, 389-393, (1981);
A. Albrecht and P.J. Steinhardt, “Cosmology for Grand Unified Theories with radiatively induced symmetry breaking”, Phys. Rev. Lett., **48**, 1220-1223, (1982),

New inflation (slow-roll model)

Both old and new inflation models were based on the high-temperature symmetry restoration of grand unified theories in the early universe at around $T = 10^{15}$ GeV.



(a) $T >> v$

(b) $T = 0$

Was the early Universe in a thermal equilibrium state?

Two bodv reaction rate with a massless gauge particle

$$\Gamma_2 = \langle n\sigma c \rangle \simeq \frac{NT^3}{\pi^2}\frac{\alpha^2}{T^2}$$

N Number of reaction channel

α Gauge coupling constant

g_* # Relativistic degrees of freedom

must have been larger than

$$H = \left(\frac{8\pi}{3M_{PL}^2}\frac{\pi^2}{30}g_* T^4\right)^{\frac{1}{2}} .$$

Namely, $\Gamma \gg H$.

This imposes an upper bound on the radiation temperature,

$$T \ll 10^{15}\left(\frac{\alpha}{0.05}\right)^2\left(\frac{N}{10}\right)\left(\frac{g_*}{200}\right)^{-1/2}\mathrm{GeV} \equiv T_{\rm eq}$$

Thermal phase transition at the GUT scale was impossible.

Some nonthermal mechanisms to set up the initial condition for inflation must be invoked.

Example 1: Large-field model, Chaotic Inflation

(Linde 83)

★ Consider the simplest Lagrangian

$$\mathcal{L}_\phi = -\frac{1}{2}(\partial\phi)^2 - V[\phi], \quad V[\phi] = \frac{1}{2}m^2\phi^2, \qquad m \ll M_{Pl}$$

★ With a natural initial condition at the Planck epoch when the Universe was presumably dominated by large quantum fluctuations:

$$-\frac{1}{2}(\partial\phi)^2 \lesssim M_{Pl}^4, \qquad \frac{1}{2}m^2\phi^2 \lesssim M_{Pl}^4,$$

typical initial field amplitude

$$\frac{\phi^2}{2L^2} \le M_{Pl}^4$$

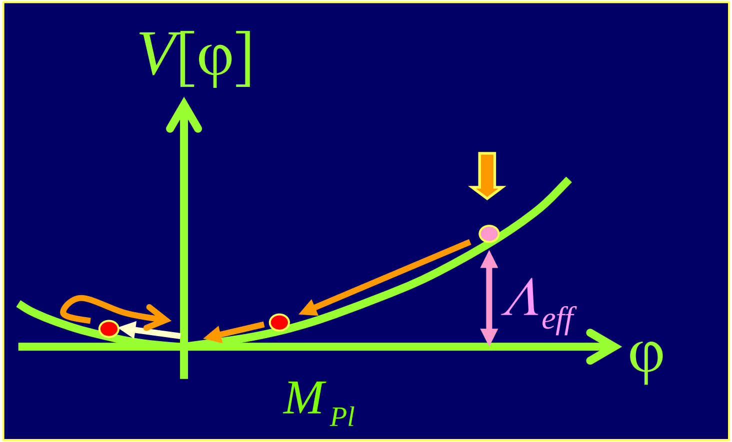
$$\phi \sim M_{Pl}^2/m \gg M_{Pl}$$

typical size of coherent domain

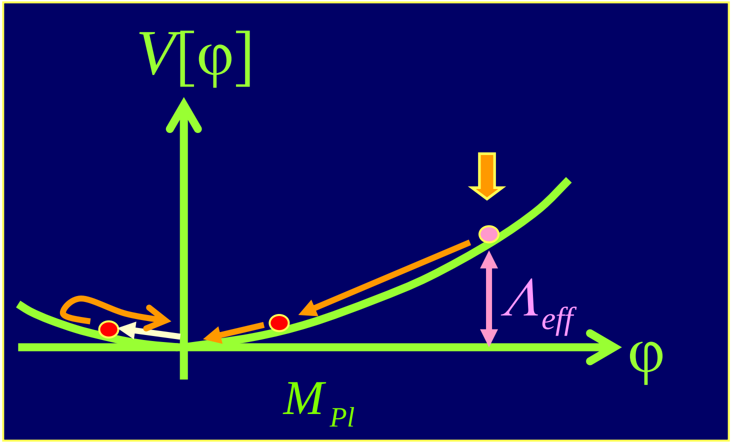
$$L \sim m^{-1} \gg M_{Pl}^{-1}$$ Horizon@ t_{Pl}

Compton wavelength of the field

"The Universe was smaller than a particle."



★This is a potential just for simple harmonic motion with a period $\tau = 2\pi/m$.
But when $\phi \gtrsim M_{Pl}$, we find $H \gtrsim m$
so the dynamics is friction dominated.



$$\ddot{\phi} + 3H\dot{\phi} + V'[\phi] = 0,$$

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi\rho_\phi}{3M_{Pl}^2} = \frac{\rho_\phi}{3M_G^2} \qquad \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V[\phi], \qquad M_G = M_{Pl}/\sqrt{8\pi}$$

★ Slow-roll equations can be solved as

$$3H\dot{\phi} + V'[\phi] = 0, \qquad \left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi V[\phi]}{3M_{Pl}^2}$$

$$\begin{aligned} \phi(t) &= \phi_i - \frac{mM_{Pl}}{2\sqrt{3\pi}}(t-t_i), \\ a(t) &= a_i \exp\left[\sqrt{\frac{4\pi}{3}}\frac{m}{M_{Pl}}\phi_i(t-t_i) - \frac{m^2}{6}(t-t_i)^2\right] \end{aligned}$$

★ Quasi-exponential inflation ends at $\phi \approx M_{Pl}/\sqrt{4\pi}$ when time variation rate $|\dot{\phi}/\phi|$ becomes as large as the cosmic expansion rate H .

★ After that the Universe is dominated by coherent field oscillation of φ.

☆ Number of e-folds

$$a(t) = a_i \exp\left[\frac{2\pi}{M_{Pl}^2}\left(\phi_i^2 - \phi^2(t)\right)\right] \Rightarrow N = \frac{2\pi}{M_{Pl}^2}\left(\phi_i^2 - \frac{M_{Pl}^2}{4\pi}\right)$$

$\phi_i \gtrsim 3M_{Pl}$ would be sufficient to solve the horizon problem.

☆ However, it is not trivial to have a flat enough potential for the field range beyond the Planck scale.

(Example) Supergravity inflation

$$D^i W \equiv \frac{\partial W}{\partial \phi_i} + \frac{3}{M_G^2}\frac{\partial K}{\partial \phi_i} W$$

$$V[\phi_i] = e^{\frac{K}{M_G^2}}\left[D^i W K_{i\bar{j}} D^{\bar{j}} W^* - \frac{3}{M_G^2}|W|^2\right] \qquad K_{i\bar{j}} = \left(\frac{\partial^2 K}{\partial\phi_i \partial\phi_{\bar{j}}^*}\right)^{-1} = \delta_{i\bar{j}}$$

Kahler potential also generates kinetic term.

$$\mathcal{L} = -K^{i\bar{j}}\partial\phi_i \partial\phi_{\bar{j}}^* - V[\phi_i] \qquad K^{i\bar{j}} \equiv \frac{\partial^2 K}{\partial\phi_i \partial\phi_{\bar{j}}^*}$$

For the minimal kinetic term $K^{i\bar{j}} = \delta^{i\bar{j}}$

$$K = \sum_i \phi_i \phi_i^* = \sum_i |\phi_i|^2 \Rightarrow$$

Exponentially steep potential beyond M_G.

First phenomenologically successful model in SUGRA (Murayama, Suzuki, Yanagida, JY 93)

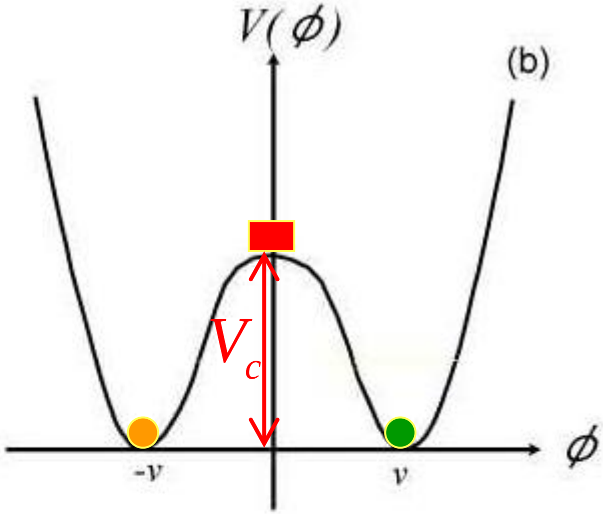
Shift symmetry (Kawasaki, Yanagida, Yamaguchi, 00)

☆ Stringy realization: Monodromy model (Silverstein & Westphal 08)

Example 2: Small-field model, Topological Inflation

(Vilenkin 94, Linde 94)

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - V[\phi], \quad V[\phi] = \frac{\lambda}{4}(\phi^2 - v^2)^2$$

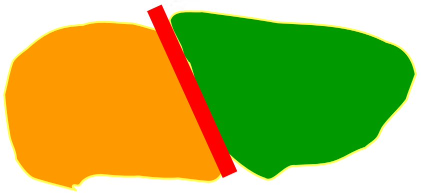


- ☆ This model has a domain wall solution.
  (Example) xy symmetric solution.

$$\phi(\boldsymbol{x}) = v \tanh\left(\sqrt{\frac{\lambda}{2}} v z\right)$$

- ☆ Thickness of the wall is determined by the balance of $V[0] \equiv V_c$ and $(\partial\phi)^2 \sim \left(\frac{v}{d_0}\right)^2$ as

$$d_0 \approx v V_c^{-1/2} \approx \frac{1}{\sqrt{\lambda} v}$$



- ☆ Comparing it with the Hubble radius corresponding to the energy density V_c

$$H_c^{-1} = \left(\frac{8\pi G}{3} V_c\right)^{-1/2} = M_{Pl}\left(\frac{3}{8\pi V_c}\right)^{1/2}$$

- ☆ We find $d_0 \gtrsim H_c^{-1}$ if $v \gtrsim M_{Pl}$, that is, the domain wall is thicker than the Hubble horizon.

★ Inside the domain wall is dominated by a large potential energy $V \sim V_c$ of almost homogeneous field in the Hubble scale.

★ Such a region would inflate without respect to outside the domain wall.

Evolution of an inflating domain wall

★ Near the core of the wall, one can expand as $\phi(\boldsymbol{x}, t_c) \simeq kz$.

★ Since the spatial gradient is small here, one can solve the slow roll eqs at each point independently assuming $\mu^2 \equiv \lambda v^2 \ll H^2$ to yield

$$\phi(\boldsymbol{x}, t) = \phi(\boldsymbol{x}, t_c) \exp\left[\frac{\mu^2}{3H_c}(t - t_c)\right] = kz \exp\left[\frac{\mu^2}{3H_c}(t - t_c)\right]$$

$$a(t) \simeq a_c \exp[H_c(t - t_c)]$$

★ The coordinate, $z_*(t)$, where $\phi =^\exists \phi_*(\ll v)$ is given by

$$z_*(t) = k^{-1}\phi_* \exp\left[-\frac{\mu^2}{3H_c}(t - t_c)\right]$$

Any point with $z \neq 0$ will eventually reach $|\phi| > \phi_*$ and terminate inflation.

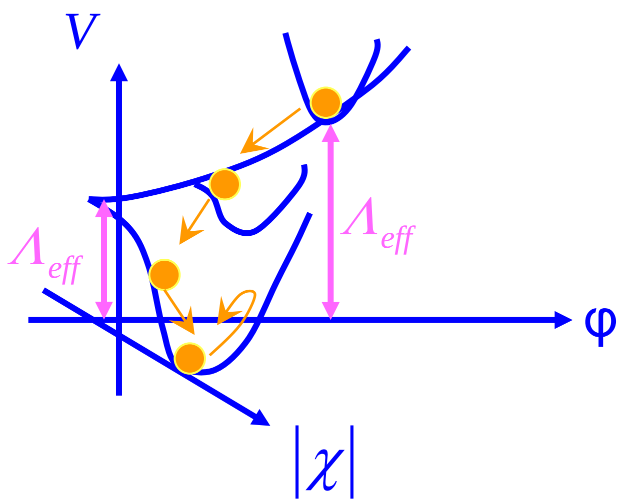
★ Its physical size will be exponentially stretched.

$$d(t) = a(t) z_*(t) = a_c k^{-1} \phi_* \exp\left[\left(H_c - \frac{\mu^2}{3H_c}\right)(t - t_c)\right]$$

Example 3: Vacuum dominated model, Hybrid Inflation

(Linde 94)

$$\mathcal{L} = -(\partial\chi)^\dagger(\partial\chi) - \frac{1}{2}(\partial\phi)^2 - V[\chi,\phi],$$



★ Symmetry restoration with another field

$$V[\chi,\phi] = \frac{\lambda}{2}(|\chi|^2 - v^2)^2 + g^2\phi^2|\chi|^2 + \frac{1}{2}m^2\phi^2$$

★ $g^2\phi^2 > \lambda v^2$: symmetry of χis restored and false vacuum energy can drive inflation.

$$V[\chi=0,\phi] = \frac{\lambda}{2}v^4 + \frac{1}{2}m^2\phi^2$$

$$\frac{\partial^2 V}{\partial\chi\partial\chi^\dagger} \equiv M_\chi^2 = \lambda(2|\chi|^2 - v^2) + g^2\phi^2$$

★ $\phi < \frac{\sqrt{\lambda}v}{g}$: phase transition and inflation ends.

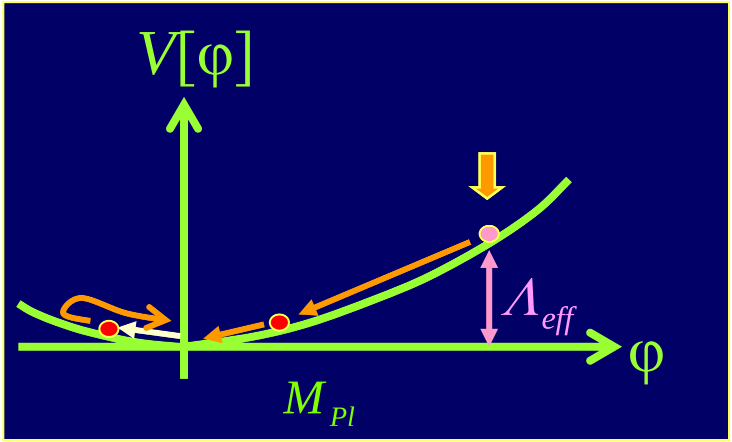
★ Inflation can occur with field amplitudes much smaller than M_{Pl} .

★ Fine tuning of initial condition of two fields is necessary.

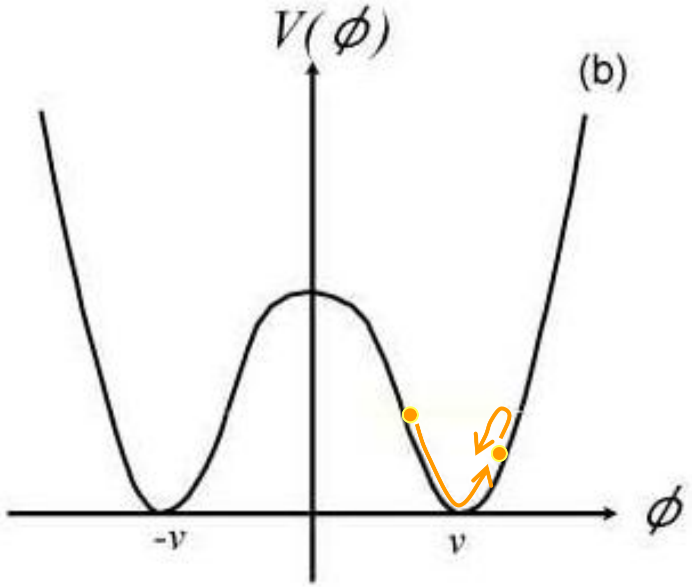
(Tetradis 98, Menders & Liddle 00)

After Inflation: Coherent Field Oscillation

- ★ After potential-driven inflation, the scalar field oscillates around the global minimum.



- ★ In some circumstances, *e.g.*, the case the minimum is at the origin and the inflaton is coupled to bosons, explosive particle production known as *preheating* takes place when the field amplitude is large.



- ★ The coherent field oscillation is equivalent to the zero-mode condensate of the inflaton, and it decays with the decay rate of the inflaton particle eventually.
  The final stage of reheating is governed by such a perturbative decay.

- ★ For example, the Yukawa coupling $h\phi\bar{\psi}\psi$ gives the decay rate

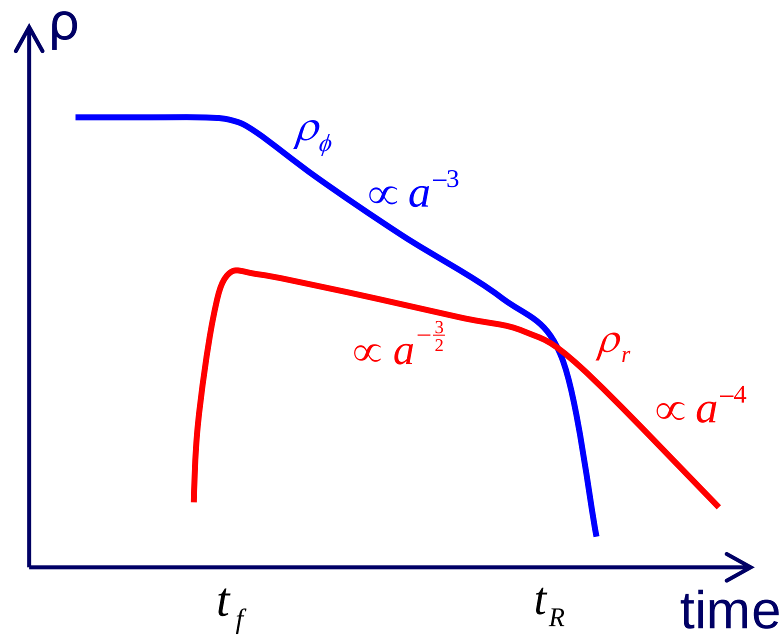
$$\Gamma_\phi = \frac{h^2}{8\pi} m$$

★ When $H \gg \Gamma_\phi$, multiplying $\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0$ by $\dot{\phi}$ one finds

$$\frac{d}{dt}\left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2\right) = -3H\dot{\phi}^2$$

$\dot{\phi}^2$ can be replaced by $\overline{\dot{\phi}^2}$ due to rapid oscillation.

$$\rho_\phi = \overline{\dot{\phi}^2}$$



★ The energy transfer equations read

$$\frac{d\rho_\phi}{dt} = -(3H + \Gamma_\phi)\rho_\phi$$

$$\frac{d\rho_{\mathrm{r}}}{dt} = -4H\rho_{\mathrm{r}} + \Gamma_\phi\rho_\phi$$

★ The solutions are

$$\rho_\phi(t) = \rho_\phi(t_f)\left[\frac{a(t)}{a(t_f)}\right]^{-3}\exp[-\Gamma_\phi(t - t_f)] \qquad \rho_{\mathrm{r}}(t) = \Gamma_\phi\int_{t_f}^{t}\left[\frac{a(t)}{a(\tau)}\right]^{-4}\rho_\phi(\tau)d\tau$$

★ The Universe is dominated by radiation around $t_R \approx \Gamma_\phi^{-1}$ with the reheating temperature

$$T_R = 0.1 \left(\frac{200}{g_*}\right)^{1/4} \sqrt{M_{Pl}\Gamma_\phi} \cong 10^{11} \left(\frac{200}{g_*}\right)^{1/4} \left(\frac{\Gamma_\phi}{10^5\mathrm{GeV}}\right)^{1/2} \mathrm{GeV}$$

which is derived from an equality $H = \left(\frac{8\pi}{3M_{Pl}^2}\rho_{\mathrm{r}}\right)^{1/2} = \left(\frac{8\pi}{3M_{Pl}^2}\frac{\pi^2 g_*}{30}T_R^4\right)^{1/2} \cong \frac{1}{2t} \cong \frac{1}{2}\Gamma_\phi$

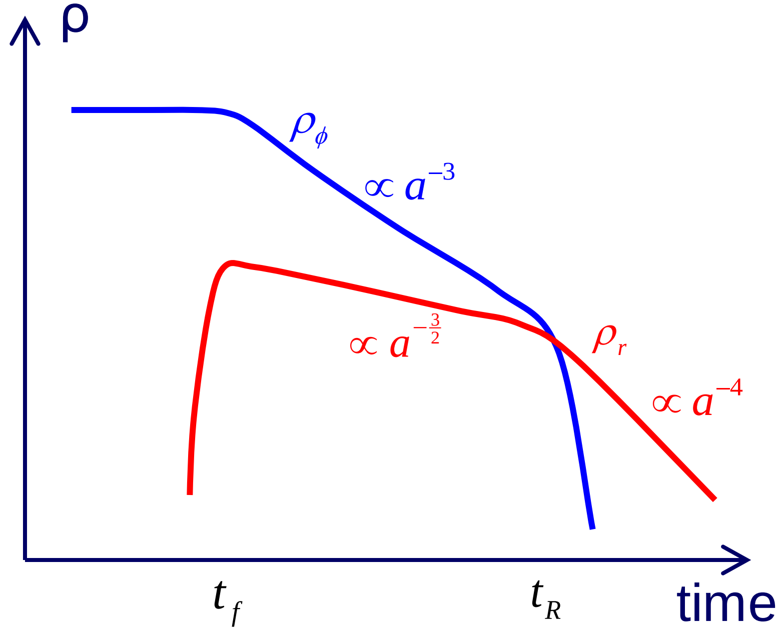
★ NB. The reheating temperature is not the maximum temperature after inflation but the temperature at the onset of radiation domination after significant entropy production.

$$\rho_{\mathrm{r}}(t) = \Gamma_\phi \int_{t_f}^{t} \left[\frac{a(t)}{a(\tau)}\right]^{-4} \rho_\phi(\tau) d\tau$$

is solved as

$$\rho_r(t) = \frac{3}{5}\Gamma_\phi t\left[\frac{a(t)}{a(t_f)}\right]^{-3}\rho_\phi(t_f) \cong \frac{6}{5}\Gamma_\phi H M_G^2,$$

in the field oscillation regime.



★ The temperature decreases as

$$T \cong \left(\frac{36}{\pi^2 g_*}\Gamma_\phi H M_G^2\right)^{1/4} \searrow T_R$$

in the field oscillation regime, if the decay product is rapidly thermalized.

★ For the Yukawa coupling $h\phi\bar{\psi}\psi$ to a fermion with mass m_ψ, the pertubative decay rate is

$$\Gamma_\phi = \frac{h^2}{8\pi} m_\phi \left[1 - \left(\frac{2m_\psi}{m_\phi} \right)^2 \right]^{\frac{3}{2}} \equiv \Gamma_{\phi pert}$$

★ When the amplitude of oscillation is large, $h\phi_{amp} > m_\phi$, it is suppressed as

$$\Gamma_\phi = \frac{4\Gamma_{\phi pert}}{\pi^{5/2}} \left[\frac{m_\phi}{h\phi_{amp} \ln(h\phi_{amp}/m_\phi)} \right]^{\frac{1}{2}}$$

(Dolgov & Kirilova 90)

★ If the decay products are thermalized in the perturbative regime, the decay rate is modified as

$$\Gamma_\phi = \Gamma_{\phi pert} \left[1 - 2n_F \left(\frac{m_\phi}{2} \right) \right]$$ to fermions

or

$$\Gamma_\phi = \Gamma_{\phi pert} \left[1 + 2n_B \left(\frac{m_\phi}{2} \right) \right]$$ to bosons

After k-Inflation and G Inflation

ex $K(X,\phi)=K_1(\phi)X+K_2(\phi)X^2$

★ Inflation ends when both coefficients turn to have positive sign.

★ After inflation the Universe is dominated by the kinetic energy of ϕ , which now behaves as a free massless field,

$$\boxed{\rho=\frac{\dot{\phi}^2}{2}\propto a^{-6}(t).} \qquad w=1$$

★ Reheating occurs through gravitational particle production due to the change of the cosmic expansion law: $a(t)\propto e^{H_{\inf}t}\rightarrow a(t)\propto t^{\frac{1}{3}}$.

At the end of inflation, radiation is created with its energy density corresponding at least to the Hawking temperature $T_H=H_{\inf}/2\pi$.

(Ford 87)

After k-Inflation and G Inflation

$$\frac{\rho_r}{\rho} \approx \frac{\frac{\pi^2}{30} g_* \left(\frac{H_{\rm inf}}{2\pi}\right)^4}{3M_{Pl}^2 H_{\rm inf}^2} \sim \left(\frac{H_{\rm inf}}{M_{Pl}}\right)^2 \ll 1$$ at the end of inflation.

$$\frac{\rho_r \propto a^{-4}(t)}{\rho \propto a^{-6}(t)} \nearrow \propto a^2(t)$$

★ The Universe will eventually be dominated by radiation.

$$\boxed{T_R \approx 0.01 \frac{H_{\rm inf}^2}{M_{Pl}} = 2\times 10^7 \left(\frac{r}{0.1}\right) \mathrm{GeV}}$$

r : tensor-to-scalar ratio

★ Massive particles with mass up to $\sim H_{\rm inf}$ are also copiously produced.

Baryogenesis through leptogenesis is possible if the mass of the lightest right-handed Majorana neutrino is smaller than the Hawking temperature.