# New Staggered-type Fermions on the Lattice: A Review

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Traditional approaches: Wilson fermions & staggered fermions

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# Introduction

Will discuss staggered versions of Wilson fermions, domain wall fermions and overlap fermions on the lattice.

They are theoretically novel, and the hope is that they might also be computationally more efficent than the usual Wilson-based fermions.

**Background:**
Originated from an attempt to identify and understand the would-be zero-modes and index of the staggered Dirac operator, and construct overlap fermions from staggered fermions.
[D.A., PRL (2010), PLB (2011)]

Will review the developments and recent results discussed at this workshop.

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# Traditional Approaches to Lattice Fermions

- Start from naive discretization of Dirac operator:

$$D_{\mathrm{naive}} = \gamma_\mu \nabla_\mu = \gamma_\mu \tfrac{1}{2a}(T_{\mu+} - T_{\mu-})$$

  where

$$T_{\mu+}\psi(x) = U_\mu(x)\psi(x + a\hat{\mu}) \quad , \quad T_{\mu-} = (T_{\mu+})^{-1}.$$

- Free field momentum rep is

$$\hat{D}_{\mathrm{naive}}(p) = i\gamma_\mu \tfrac{1}{a}\sin(ap_\mu)$$

  $\rightarrow$ $\hat{D}_{\mathrm{naive}}(p) = 0$ has 16 solutions for $p \in (-\frac{\pi}{a}, \frac{\pi}{a}]^4$.

  $\rightarrow$ 16 lattice fermion species $\rightarrow$ 15 spurious "doublers".

# Traditional Approaches (continued)

**Wilson fermions**: Add a term to give mass $\sim \frac{1}{a}$ to the 15 spurious species:

$$D_W = \gamma_\mu \nabla_\mu + a\frac{r}{2}\Delta \qquad\qquad \Delta = \text{lattice Laplace op.}$$

$$\rightarrow \;\; \hat{D}_W(p) = i\gamma_\mu \frac{1}{a}\sin(ap_\mu) + \frac{r}{a}\sum\nolimits_\nu (1 - \cos(ap_\nu))$$

$\rightarrow \;\; \hat{D}_W(p) = 0$ only has one solution: $p = 0$.

Disadvantages:

- The continuum chiral symm $\{D\,, \gamma_5\} = 0$ is broken by the Wilson term $a\frac{r}{2}\Delta$.
- $O(a^2)$ discretization error of naive fermion becomes $O(a)$.

# Traditional Approaches (continued)

**Staggered fermions**: The 16 species of $D_{\text{naive}} = \gamma_\mu \nabla_\mu$ can be reduced to 4 species via spin-diagonalization:

$$\Lambda\psi(x) = \gamma_1^{n_1} \cdots \gamma_4^{n_4} \psi(x) \qquad\qquad x = a(n_1, n_2, n_3, n_4)$$

gives

$$\Lambda^{-1}(\gamma_\mu \nabla_\mu)\Lambda = D_{st}\,\mathbf{1}$$

where

$$D_{st} = \eta_\mu \nabla_\mu \quad , \qquad \eta_\mu \psi(x) = (-1)^{n_1 + \cdots + n_{\mu-1}}\,\psi(x)$$

Note: $D_{st}$ is a scalar operator.

$\Rightarrow$ Naive lattice fermion $\simeq$ 4 copies of staggered fermion described by $D_{st}$.

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- Advantage: One of the flavored chiral symms holds exactly:

$$\{D_{st}\,,\Gamma_{55}\} = 0 \quad \text{where} \quad \Gamma_{55}\chi(x) = (-1)^{n_1+n_2+n_3+n_4}\,\chi(x)$$

Interpretation: $\Gamma_{55} = \gamma_5 \otimes \gamma_5$ on spin$\otimes$flavor.

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# Spin–Flavor Interpretation of Staggered Fermions

- Momentum space approach:

  $p \in [-\frac{\pi}{2a}, \frac{3\pi}{2a}]^4$ written as

  $p = q + \frac{\pi}{a} A\,, \quad q \in [-\frac{\pi}{2a}, \frac{\pi}{2a}]^4\,, \;\; A = (A_1, \ldots, A_4)\,, \;\; A_\mu \in \{0, 1\}$

  $\rightarrow \;\; \hat{\chi}(p) = \hat{\chi}(q + \frac{\pi}{a} A) \;\equiv\; \hat{\chi}_A(a).$

  $\rightarrow$ Free field momentum rep of $D_{st} = \gamma_\mu \nabla_\mu$ has the form

  $$\hat{\Gamma}_\mu \frac{i}{a} \sin(a q_\mu)$$

  where $\hat{\Gamma}_\mu = (\hat{\Gamma}_\mu)_{AB}$ are $16 \times 16$ matrices giving a 16-dim rep of Dirac algebra.

  $\rightarrow$ decomposes into 4 copies of 4-dim rep: the 4 flavors.

- There is also a free field coordinate space approach.

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# How about a Wilson term for staggered fermions?

**Goal:**

Reduce number of fermion species from 4 to 1 (or 2).

Then will have fewer "doubler" species than in usual Wilson case: 3 (or 2) versus 15.

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Natural approach is to add a momentum-dependent "flavored" mass term $W_{st}$ to staggered Dirac operator:

$$D_{st} \;\rightarrow\; D_{st} + \tfrac{1}{a} W_{st}$$

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To avoid complex fermion det, require $W_{st}^{\dagger} = \Gamma_{55} W_{st} \Gamma_{55}$.

# Flavored Mass Terms for Staggered Fermions

The possible flavoured mass terms for staggered fermions were found long ago [Golterman & Smit (1984)].

**Ingredients:**

- $\Gamma_{55} \;\simeq\; \gamma_5 \otimes \gamma_5$ $\qquad \Gamma_{55}\chi(n) = (-1)^{n_1+n_2+n_3+n_4}\,\chi(n)$
- $\Gamma_5 \;\simeq\; \gamma_5 \otimes \mathbf{1} + O(a^2)$ $\qquad \Gamma_5 = \eta_5 C$

  where

  $$\eta_5 = \eta_1\eta_2\eta_3\eta_4 \quad , \qquad \eta_5\chi(n) = (-1)^{n_1+n_3}\chi(n)$$

  $$C = (C_1 C_2 C_3 C_4)_{\text{sym}} \quad , \qquad C_\mu = \frac{1}{2a}(T_{\mu+} + T_{\mu-})$$

  Recall $T_{\mu+}\chi(n) = U_\mu(n)\chi(n+\hat{\mu})$ is parallel transporter.

# Wilson Terms for Staggered Fermions

The most general flavored mass term satisfying $\Gamma_{55}$-hermiticity is

$$W_{st} = c\mathbf{1} + \sum_{\mu<\nu} c_{\mu\nu} M_{\mu\nu} + c_5 M_5$$

where the 2-link operators $M_{\mu\nu}$ and 4-link operator $M_5$ are given by

$$\begin{aligned} M_{\mu\nu} &= i\eta_{\mu\nu} C_{\mu\nu} \quad , \qquad C_{\mu\nu} = \tfrac{1}{2}(C_\mu C_\nu + C_\nu C_\mu) \\ M_5 &= \eta_5 \Gamma_{55} \Gamma_5 \end{aligned}$$

Spin-flavor interpretations:

$$M_{\mu\nu} \sim 1 \otimes i\gamma_\mu\gamma_\nu \qquad , \qquad M_5 \sim 1 \otimes \gamma_5$$

For staggered Wilson term, choose $c$, the $c_{\mu\nu}$'s and $c_5$ so that $W_{st}$ has 1 (or 2) zero-eigenmodes.

# Wilson Terms for Staggered Fermions (cont'd)

This is quite an obvious possibility, so why didn't Golterman & Smit or others already do it a long time ago?

Answer (my guess):

1. Concern about breaking lattice rotation invariance with the $M_{\mu\nu}$'s.
2. Loses 2 key advantageous features of staggered fermions:
   - Exact flavored chiral symm $\{D_{st}\,,\Gamma_{55}\} = 0$ is broken.
   - $O(a^2)$ discretization error becomes $O(a)$.
3. Compared to usual Wilson fermion, is the gain in efficiency (if it is even realized) enough to justify the more complicated spin-flavor structure?

In the old days, efficiency was not a pressing concern. It is more so now.

# Why is staggered-Wilson more interesting now?

- Realistic unquenched Lattice QCD simulations are now possible.
  $\rightarrow$ Much efforts to find "improved" formulations to get closer to the continuum limit and chiral limit without an excessive increase in computing cost.

  Can staggered-Wilson do significantly better than usual Wilson for this?

- Chirally improved lattice fermion formulations have been found: domain wall fermions and overlap fermions.

  – Built from Wilson fermions.

  – Attractive theoretical properties but computationally very expensive

  $\Rightarrow$ Look for more computationally efficient versions of these.

  Can staggered-Wilson give a more efficient version of domain wall and overlap fermions?

# A 2-flavor staggered Wilson term

Work on the staggered fermion index and a related staggered overlap fermion construction [D.A. PRL (2010), PLB (2011)] led to staggered Wilson fermion with Wilson term constructed with the flavored mass term $M_5 = \Gamma_{55}\Gamma_5$ :

$$W_{st} = \frac{1}{a}(\mathbf{1} - \Gamma_{55}\Gamma_5)$$

Recall $\Gamma_{55}\Gamma_5 \equiv 1 \otimes \gamma_5 + O(a^2)$

Decompose the 4 Dirac fermion speciesof the staggered fermion into 2 species with positive flavor-chirality under $1 \otimes \gamma_5$ and 2 species with negative flavor-chirality.

Then $W_{st}$ keeps the 2 positive flavor-chirality species as the physical fermions and gives mass $\sim 1/a$ to the negative flavor-chirality species; they become the "doublers".

$\rightarrow$ Get 2-flavor staggered-Wilson fermion on the lattice.

# Free field spectrum of 2-flavor staggered-Wilson

[P. de Forcrand, Lattice 2010 conf.]

**Green:** Eigenvalues of free field $D_{sW}$ (staggered Wilson)
**Blue:** Eigenvalues of free field $D_W$ (usual Wilson)

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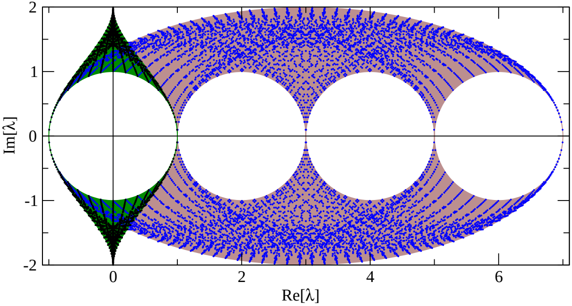



→ **Spectrum of staggered Wilson is sensible and contains much less junk than spectrum of usual Wilson.**

# Symmetries of 2-flavor staggered-Wilson

Classical action is $\bar{\chi} D_{sW} \chi$ where

$$D_{sW} = D_{st} + \tfrac{r_0}{a}(1 - \Gamma_{55}\Gamma_5) + m_0$$

$r_0$, $m_0$: bare parameters.

It is invariant under lattice rotations and all the symmetries of the original staggered fermion, except the "shift transformations" – for these have

$$\bar{\chi}\Gamma_{55}\Gamma_5\chi \;\to\; -\bar{\chi}\Gamma_{55}\Gamma_5\chi$$

Turns out only one new counter-term is possible: $\{\Gamma_{55}\Gamma_5\,, D_{st}\}$.

# Symmetries of 2-flavor staggered-Wilson (cont'd)

$\rightarrow$ The quantum effective action can be put in the form $\bar{\chi} D_{sW} \chi$ where

$$D_{sW} = (1 + c\Gamma_{55}\Gamma_5)D_{st} + \tfrac{r}{a}(1 - \Gamma_{55}\Gamma_5) + m$$

and

$$m = m_0 + \frac{c'}{a}$$

Note $\Gamma_{55}\Gamma_5 D_{st} \equiv (1 \otimes \gamma_5)D_{st}$ which coincides with $D_{st}$ on the physical species.

$\rightarrow$ Can approach massless (chiral) limit by tuning bare mass $m_0$:

$$m_0 \;\rightarrow\; -\frac{c'}{a}$$

This is basically the same situation as for usual Wilson fermions.

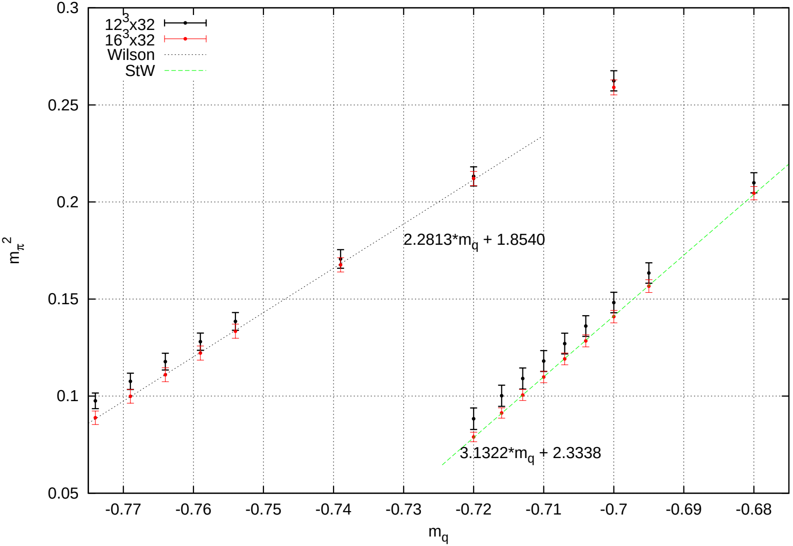
# 2-flavor staggered-Wilson (cont'd)

- Should check that a massless (chiral) limit can be approached by calculating the "pion" mass as a function of the bare quark mass in a Lattice QCD simulation.
- Also, do same with usual Wilson fermions and see if staggered-Wilson is more chiral, i.e. can reach a smaller pion mass than usual Wilson.

Simulation details: quenched simulation with 50 configs at $\beta = 6.0$ on two lattices: $12^3 \times 32$ and $16^3 \times 32$.

Done by Andriy Petrashyk, following instructions from Daniel Nogradi.

# Pion mass plots



## 2-flavor staggered-Wilson: spectrum calculations

- The spectrum of the staggered-Wilson Dirac operator in a typical quenched $\beta = 6$ background on an $8^4$ lattice was presented by Ph. de Forcrand.

  It doesn't look good, but usual Wilson spectrum was not presented for comparison.
- The spectrum looks even worse on $4^4$ lattice at $\beta = 5.6$ (results of S. Durr, presented at the workshop). Comparison with Wilson spectrum was also made in this case and staggered-Wilson looks worse.
- It would be interesting to compare staggered-Wilson and usual Wilson spectra in quenched $\beta = 6$ background on $16^3 \times 32$ lattice
  – the pion mass results suggest staggered-Wilson may be better in that case.

# 1-flavor staggered-Wilson

[C. Hoelbling, PLB (2011)]

Consider a staggered Wilson term built from the $M_{\mu\nu}$ flavored mass terms:

$$W_{st}^{H} = \frac{1}{a}(M_{12} + M_{13} + M_{14} + M_{23} + M_{24} + M_{34})$$

(or variants with different sign combinations).

This splits the flavor degeneracy to provide a 1-flavor staggered-Wilson theory with Dirac operator

$$D_{sW}^{H} = D_{st} + \frac{1}{a}(W_{st}^{H} + 1)$$

# Free field spectrum for 1-flavor staggered-Wilson

[C. Hoelbling, this workshop.]

**Blue:** 1-flavor staggered-Wilson
**Purple:** 2-flavor staggered-Wilson
**Yellow:** Usual Wilson

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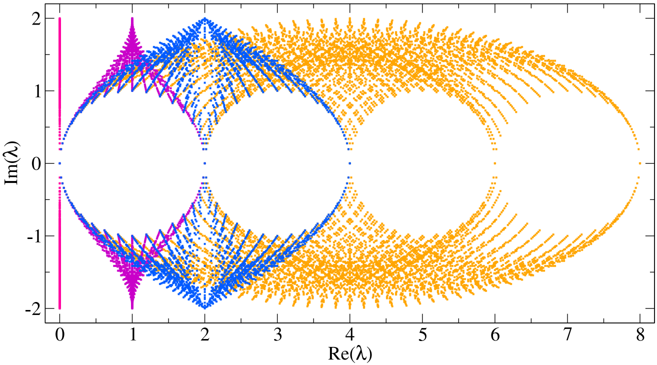

# 1-flavor staggered-Wilson: symmetries

It maintains all staggered fermion symmetries except for the shift transformations:
these change $\bar{\chi} M_{\mu\nu} \chi \;\to\; -\bar{\chi} M_{\mu\nu} \chi$ if the shift is along the $\mu-$ or $\nu-$ axes.

However, lattice rotation symmetry is broken.

E.g. under $R^{(12)}$ have

$$W_{st}^{H} \;\to\; \frac{1}{a}(M_{12} - M_{13} + M_{14} + M_{23} - M_{24} + M_{34})$$

But still have a residual lattice rotation symmetry:

$W_{st}^{H}$ is invariant under double rotations $R^{(\mu\nu)} R^{(\sigma\rho)}$ when $\mu, \nu, \sigma, \rho$ are all different.

# Is 1-flavor staggered-Wilson viable?

- Spacetime rotation symmetry along with gauge invariance is essential fr renormalizability of continuum QCD. –excludes new counterterms from arising.
- In lattice QCD the rotation symmetry is broken down to the discrete subgroup of hypercubic lattice rotations.

  Miraculously, this is still enough to exclude new counterterms and maintain renormalizability [T. Reisz].
- For 1-flavor staggered-Wilson the spacetime rotation symmetry is broken even further, down to the double rotations $R^{(\mu\nu)}R^{(\sigma\rho)}$ for $\mu, \nu, \sigma, \rho$ all different.
  Can we hope for a further miracle?

  (Doubtful IMHO)

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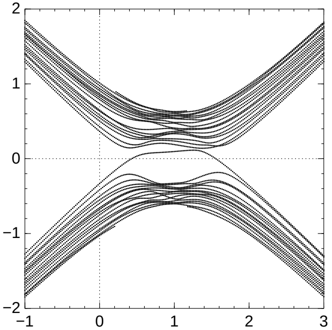
# Index via Spectral Flow

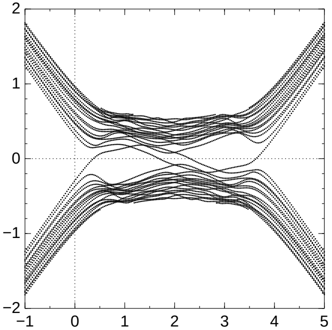
Spectral flow of $H(m)$ in a U(1) background with $\mathbf{Q} = \mathbf{1}$ on $12 \times 12$ lattice:

$m$ on horizontal axis; eigenvalues of $H(m)$ on vertical axis




staggered Wilson

$H_{sW}(m) = \Gamma_{55}(D_{sW} - m)$

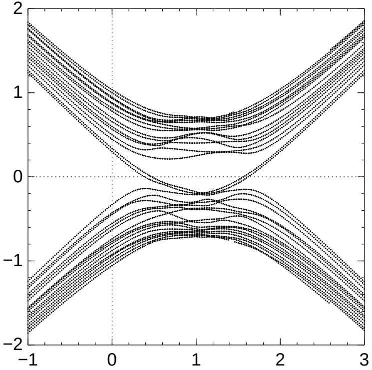



usual Wilson

$H_W(m) = \gamma_5(D_W - m)$

# Index via Spectral Flow (continued)

Spectral flow of $H(m)$ in another U(1) background with $\mathbf{Q} = -\mathbf{2}$ on $12 \times 12$ lattice:



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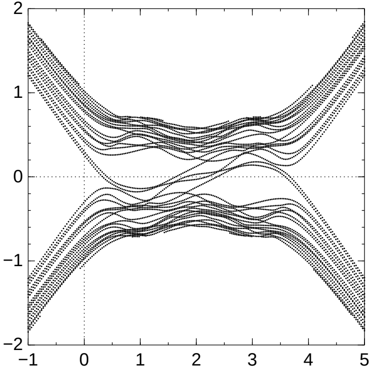

usual Wilson

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