# Chiral Symmetries and eigenvalue density at Finite temperature

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# 1. Introduction

**Flavor-Chiral Symmetries of QCD at low T**

$$U(N_f)_L \otimes U(N_f)_R \quad \longrightarrow \quad U(1)_B \otimes S(N_f)_L \otimes SU(N_f)_R \quad \longrightarrow \quad U(1)_B \otimes S(N_f)_V$$

flavor-chiral

chiral anomaly
(explicit breaking)

spontaneous breaking of
chiral symmetry

**QCD at high T**

deconfinement (QGP)

restoration of chiral symmetry

low T $\quad U(1)_B \otimes S(N_f)_V \quad \longrightarrow \quad$ high T $\quad U(1)_B \otimes S(N_f)_L \otimes SU(N_f)_R$

phase transition

**How about U(1)A symmetry ?**

# How about $U(1)_A$ symmetry ?

1. Analytic: $U(1)_A$ symmetry is also “restored” above $T_c$. Cohen(96)

2. Analytic: $U(1)_A$ symmetry breaking appears at $N_f$ point functions above $T_c$.
   Lee-Hatsuda(96)

3. Lattice: $U(1)_A$ symmetry is still “broken” just above $T_c$ .
   Chandrasekharan-Christ(96), Bernald, *et al.* (97)

# Cohen's argument

Meson operator and chiral symmetry( $N_f$ = 2)

color flavor

$$M_\Gamma^A(x) = \bar{\psi}^a(x)^f_\alpha (\Gamma \otimes T^A)^{fg}_{\alpha\beta} \psi^a(x)^g_\beta$$

Dirac

$$\begin{tikzcd}
\sigma \text{ meson: } \mathbf{1} \otimes \mathbf{1} \arrow[rr, leftrightarrow, "\text{chiral } SU(2)"] \arrow[d, leftrightarrow, "U(1)_A"] & & \pi \text{ meson: } \gamma_5 \otimes \tau^a \arrow[d, leftrightarrow, "U(1)_A"] \\
\eta \text{ meson: } \gamma_5 \otimes \mathbf{1} \arrow[rr, leftrightarrow, "\text{chiral } SU(2)"] & & \delta \text{ meson: } \mathbf{1} \otimes \tau^a
\end{tikzcd}$$

$$\text{chiral } SU(2)\text{: } \psi \to e^{i\pi/4\gamma_5\tau^a}\psi, \quad \bar{\psi} \to \bar{\psi}e^{i\pi/4\gamma_5\tau^a}$$

$$U(1)_A\text{: } \psi \to e^{i\pi/4\gamma_5\mathbf{1}}\psi, \quad \bar{\psi} \to \bar{\psi}e^{i\pi/4\gamma_5\mathbf{1}}$$

2-pt function $$\Pi^A_{\Gamma_1\Gamma_2}(x) = \langle M^A_{\Gamma_1}(x) M^A_{\Gamma_2}(0)\rangle$$

$$-\Pi^A_{\Gamma_1\Gamma_2}(x) = \int D[G] P_m(G)\{\mathrm{tr}[\Gamma_1 S_G(x,0)\Gamma_2 S_G(0,x)] - \delta_{A0}\mathrm{tr}[\Gamma_1 S_G(x,x)]\mathrm{tr}[\Gamma_2 S_G(0,0)]\}$$

$$P_m(G) = \frac{1}{Z} e^{-S_{YM}(G)} \det[\gamma^\mu D_\mu(G) - m]$$

$$\boxed{\chi_{U(1)_A} = \int d^4x [\Pi_\sigma(x) - \Pi_\delta(x)] = \int d^4x \int D[G] P_m[G] \mathrm{tr} S_G(x,x) \mathrm{tr} S_G(0,0)}$$

$$U(1)_A \text{ symmetry is restored} \Rightarrow \chi_{U(1)_A} = 0$$

"Spectral" representation

$$\frac{1}{V}\int d^4x \langle\bar\psi(x)\psi(x)\rangle = -\frac{1}{V}\int D[G]\sum_j\left[\frac{1}{i\lambda_j - m} + \frac{1}{-i\lambda_j - m}\right] P_m(G) = \int D[G] P_m(G) \int d\lambda \rho_G(\lambda)\frac{2m}{\lambda^2+m^2}$$

eigenvalue of $\gamma^\mu D_\mu$

density of eigenvalues $$\rho_G(\lambda) = \frac{1}{V}\sum_j \delta(\lambda - \lambda_j(G))$$

## Chiral symmetry restoration

$$\langle \bar{\psi}\psi \rangle = 2\pi\rho(0) = 0, \quad m \to 0$$

$$\rho(\lambda) = \int D[G] P_m(G) \rho_G(\lambda)$$

$P_m(\forall G) \geq 0$ ⇒ $\rho_G(0) = 0 \text{ for } {}^\forall G$

⇒ $$\chi_{U(1)_A}/V = \int D[G] P_m[G] \left\{ \int d\lambda \rho_G(\lambda) \frac{2m}{\lambda^2 + m^2} \right\}^2 = O(m^2) \to 0, \quad (m \to 0)$$

⇒ $$\chi_{U(1)_A}/V = 0, \quad (m \to 0)$$

$\boxed{U(1)_A \text{ seems to be restored.}}$

However, where is “anomaly” ?

# Loophole of the argument

Lee-Hatsuda(96)

Effect of zero mode (Q=-1,1)

$$O(m^{N_f}) \text{ contribution}$$

$$\langle \bar{\psi}\psi \rangle \propto m^{N_f} \times \frac{1}{m} \to 0$$

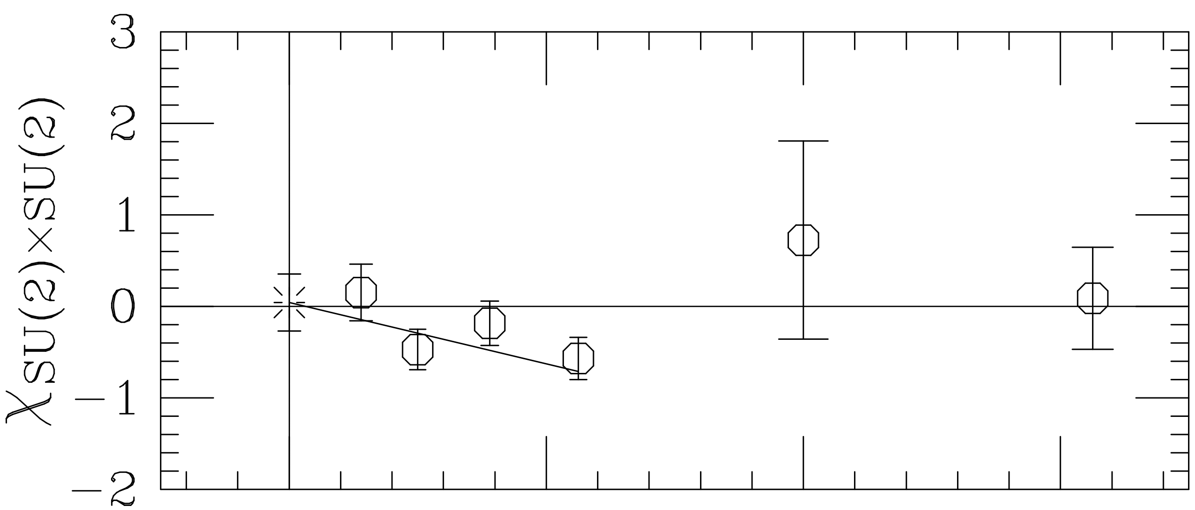
but

$$\chi_{U(1)_A} \propto m^{N_f} \times \frac{1}{m^2} \neq 0$$

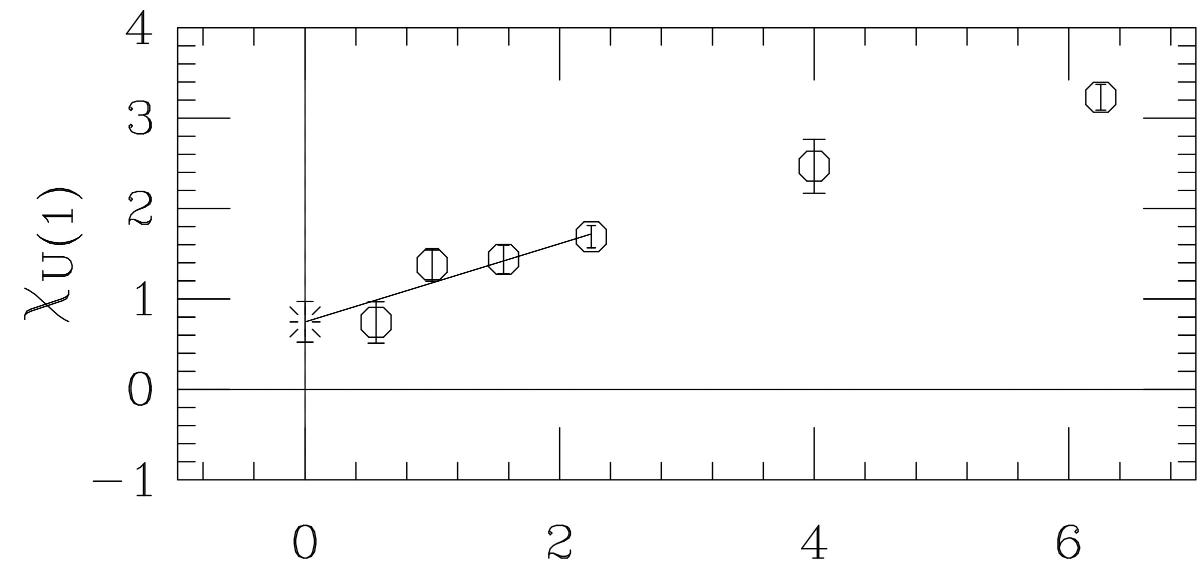
$\text{at } N_f = 2$

# Lattice Results(KS fermion)

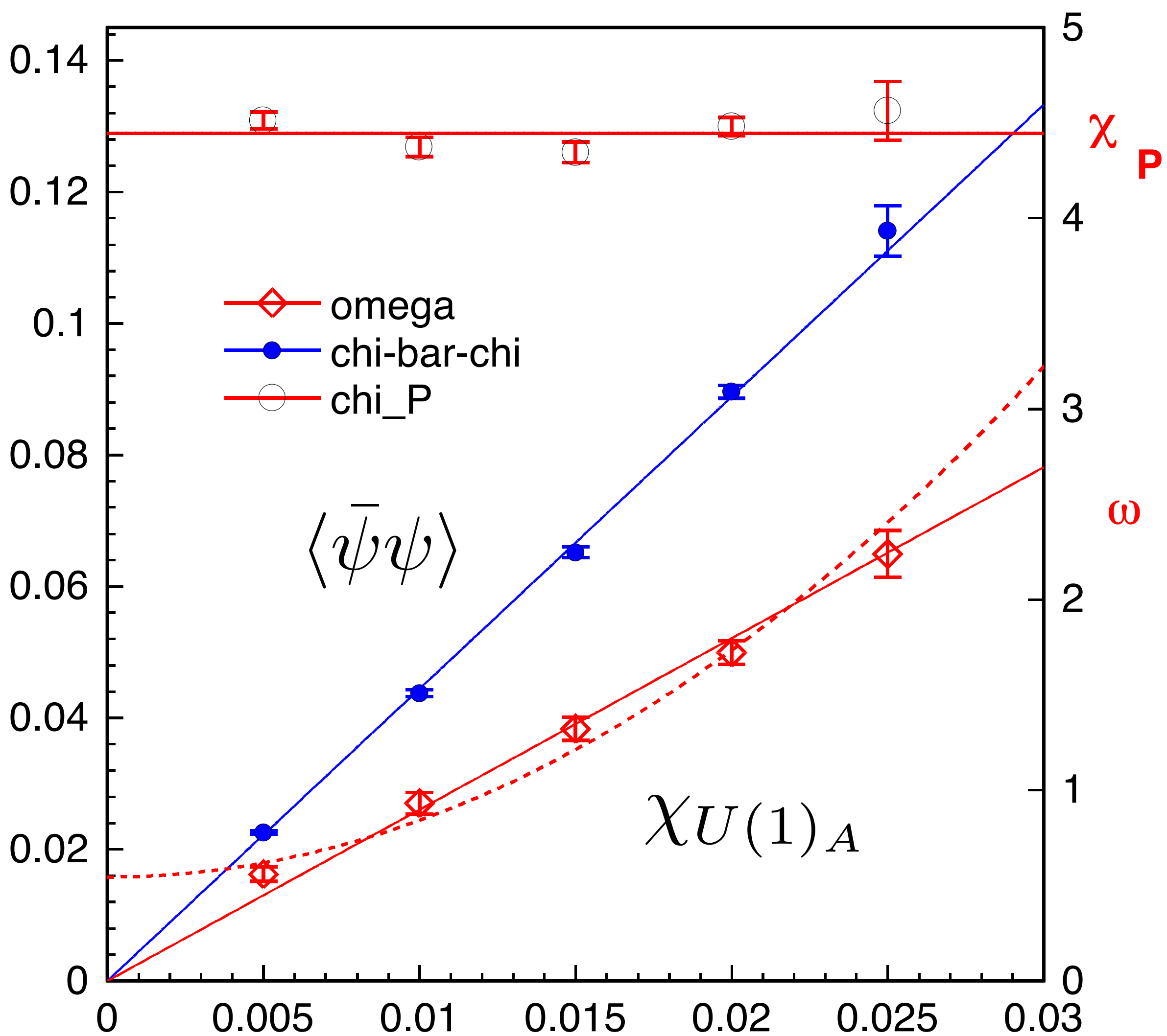
Bernald et al. (97)




Chiral symmetry is restored.




U(1)A is NOT.




Chandrasekharan et al. (98)

Chiral symmetry is restored at hight T,
but U(1)A is not recovered.

## This talk

give a more precise argument in continuum QCD.
give a rigorous argument in lattice QCD with overlap fermions.
consider a more general case.

## Content

# 2. Continuum QCD

"Spectral" representation of quark propagator

$$S_A(x,y) = -\frac{1}{m}\sum_k \phi_k^A(x)\phi_k^A(y)^\dagger + \sum_{\lambda_n>0}\left[\frac{\phi_n^A(x)\phi_n^A(y)^\dagger}{i\lambda_n^A - m} + \frac{\gamma_5\phi_n^A(x)\phi_n^A(y)^\dagger\gamma_5}{-i\lambda_n^A - m}\right]$$

A: gauge fields

zero modes(chiral)

non-zero modes

$$\begin{aligned}
\langle\bar\psi(x)\psi(x)\rangle &= -\frac{N_f}{V}\Big\langle -\frac{1}{m}\sum_k(\phi_k,\phi_k) \\
&+ \sum_{\lambda_n>0}\left[\frac{1}{i\lambda_n^A - m}(\phi_n^A,\phi_n^A) + \frac{1}{-i\lambda_n^A - m}(\gamma_5\phi_n^A,\gamma_5\phi_n^A)\right]\Big\rangle_A \\
&= N_f\Big\langle\frac{N_{R+L}(A)}{mV} + \int_0^\infty d\lambda\frac{2m}{\lambda^2+m^2}\frac{1}{V}\sum_{\lambda_n>0}\delta(\lambda-\lambda_n^A)\Big\rangle_A \\
&= N_f\Big\langle\frac{N_{R+L}(A)}{mV} + \int_0^\infty d\lambda\frac{2m}{\lambda^2+m^2}\rho^A(\lambda)\Big\rangle_A
\end{aligned}$$

$$N_{R+L} = N_R + N_L$$

$$\rho^A(\lambda) = \frac{1}{V}\sum_{\lambda_n>0}\delta(\lambda-\lambda_n^A)$$

$$P_m(A) = e^{-S_{YM}(A)}(-m)^{N_f N_{R+L}}\prod_{\lambda_n>0}(\lambda_n^2+m^2).$$

positive & even in $m$ if $N_f$ is even

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## Expansion

$V \to \infty$

$$\rho^A(\lambda) = \sum_{n=0}^{\infty} \rho_n^A \frac{\lambda^n}{n!} = \rho_0^A + \rho_1^A \lambda + \rho_2^A \frac{\lambda^2}{2!} + \cdots .$$

(possible, at least, at T > Tc )

## Integral

$$\int d\lambda \, \rho^A(\lambda) \frac{2m}{\lambda^2 + m^2} = \pi_m \rho_0^A + O(m)$$

$$\pi_m \equiv 2 \tan^{-1}\left(\frac{\Lambda}{m}\right)$$

$\Lambda$ UV cut-off

$$\lim_{m\to 0} \lim_{V\to\infty} \langle \bar{\psi}\psi \rangle = \lim_{m\to 0} N_f \left[ 2\tan^{-1}\left(\frac{\Lambda}{m}\right) \langle \rho_0^A \rangle_A + \lim_{V\to\infty} \frac{\langle N_{R+L}(A) \rangle_A}{mV} \right].$$

=0, if chiral symmetry is restored,

positive semi-definite

$$\langle \rho_0^A \rangle_A = m^2 \bar{\rho}_0,$$

$$\lim_{V\to\infty} \frac{1}{V} \langle N_{R+L}(A) \rangle_A = m^2 \bar{N}_1$$

$$N_{R+L}(A) \propto (m^{N_f})^{N_{R+L}(A)}$$

much more suppressed.

## Other conditions from

$$\lim_{m\to 0}\lim_{V\to\infty}\chi^{\sigma-\pi} = \lim_{m\to 0}\lim_{V\to\infty}\chi^{\eta-\delta} = 0,$$ (chiral)

$$\boxed{\begin{aligned} \langle \rho_0^A \rangle &= O(m^2), \\ \lim_{V\to\infty}\frac{1}{m^2 V}\langle N_{R+L}(A)\rangle_A &= O(m^2), \\ \lim_{V\to\infty}\frac{N_f^3}{m^2 V}\langle Q^2(A)\rangle_A &= 2\bar{\rho}_1 + O(m^2), \end{aligned}}$$

$\lim_{m\to 0}\langle \rho_1^A\rangle_A = \bar{\rho}_1.$

$N_R - N_L = Q$ (index theorem)

$Q(A) \propto (m^{N_f})^{Q(A)}$ much more suppressed.

$$\lim_{m\to 0}\lim_{V\to\infty}\frac{1}{m^2 V}\langle Q^2(A)\rangle = 0$$

c.f. $\langle Q^2(A)\rangle \propto m\Sigma V$ at $T=0$

# $\boxed{\text{U(1)}_\text{A} \text{ susceptibility}}$

$$\chi^{\pi-\eta} = \lim_{V\to\infty} \frac{N_f^4}{m^2 V} \langle Q^2(A) \rangle_A = 2N_f \bar{\rho}_1 + O(m^2)$$

but $$\lim_{m\to 0} \lim_{V\to\infty} \frac{1}{m^2 V} \langle Q^2(A) \rangle = 0$$

$$\boxed{\lim_{m\to 0} \chi^{\pi-\eta} = 0}$$

This does NOT necessarily mean the recovery of U(1) symmetry.

$$\bar{\rho}_1 = 0,$$

eigenvalue density

$$\lim_{m\to 0} \rho_m(\lambda) = \langle \rho_2^A \rangle_A \frac{\lambda^2}{2!} + O(\lambda^3)$$

$$\rho_m(\lambda) \equiv \langle \rho^A(\lambda) \rangle_A$$

linear term is absent !

$\boxed{\text{Remark}}$ Effect of “anomaly” is represented by “index theorem”. This can be seen in the next section.

# 3. Lattice QCD with overlap fermions

Ward-Takahashi identities under "chiral" rotation

$$\begin{aligned} \theta^a(x)\delta_x^a\psi(x) &= i\theta^a(x)T^a\gamma_5(1-RaD)\psi(x), \\ \theta^a(x)\delta_x^a\bar\psi(x) &= i\bar\psi(x)\theta^a(x)T^a\gamma_5, \end{aligned}$$

$$\langle (J_x^a - \delta_x^a S)O + \delta_x^a \mathcal{O}\rangle = 0$$

Action $S = \bar\psi D\psi - m\int d^4x\,\bar\psi F(D)\psi.$ $\qquad F(D) = 1 - \frac{R}{2}aD.$

Ginsparg-Wilson relation $\qquad D\gamma_5 + \gamma_5 D = aDR\gamma_5 D$

$$\int d^4x \langle \{J_x^a + 2mP^a(x)\}\mathcal{O} + \delta_x^a\mathcal{O}\rangle = 0.$$

integrated WTI

$$\begin{aligned} S^a(x) &= \bar\psi(x)T^aF(D)\psi(x), \\ P^a(x) &= \bar\psi(x)T^a i\gamma_5 F(D)\psi(x), \end{aligned}$$

scalar

pseudo-scalar

chiral rotation

$$\begin{aligned} \delta^b S^a(x) &= 2d_c^{ab}P^c(x),\ (\delta^0 S^a(x) = 2P^a(x) = \delta^a S^0(x)) \\ \delta^b P^a(x) &= -2d_c^{ab}S^c(x),\ (\delta^0 P^a(x) = -2S^a(x) = \delta^a P^0(x)) \end{aligned}$$

$$\{T^a, T^b\} = 2d_c^{ab}T^c$$

"measure" term (anomaly)

$$J_x^a = -2i\mathrm{tr}\,T^a\gamma_5\left(1-\frac{R}{2}aD\right)(x,x) = -\delta^{a0}2iN_f\mathrm{tr}\,\gamma_5\left(1-\frac{R}{2}aD\right)(x,x)$$

anomalous WTI $\mathcal{O} = S^a(y)P^a(z)$ and $\delta^b = \delta^0$

$$\left\langle \int d^4x \, \{J_x^0 + 2mP^0\} S^a(y)P^a(z) + 2P^a(y)P^a(z) - 2S^a(y)S^a(z) \right\rangle \;=\; 0,$$

non-anomalous WTI $\mathcal{O} = S^0(y)P^a(z)$ or $S^a(y)P^0(z)$ and $\delta^a$

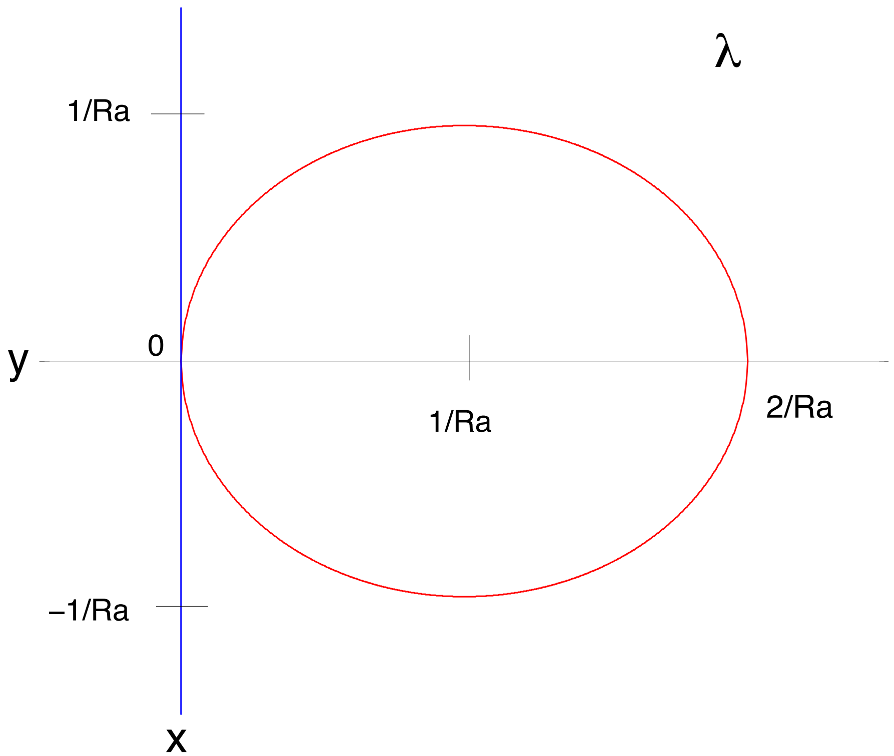
$$2\left\langle m \int d^4x \, P^a(x)S^0(y)P^a(z) + P^a(y)P^a(z) - S^0(y)S^0(z) \right\rangle \;=\; 0,$$

$$2\left\langle m \int d^4x \, P^a(x)S^a(y)P^0(z) + P^0(y)P^0(z) - S^a(y)S^a(z) \right\rangle \;=\; 0.$$

# Eigenvalue spectrum

$$\lambda_n + \bar{\lambda}_n = aR\bar{\lambda}_n\lambda_n$$

$$f_m = 1 + maR/2.$$




$$\begin{aligned} S_F(x,y) &= \sum_n \left[ \frac{\phi_n(x)\phi_n^\dagger(y)}{f_m\lambda_n - m} + \frac{\gamma_5\phi_n(x)\phi_n^\dagger(y)\gamma_5}{f_m\bar{\lambda}_n - m} \right] \\ &+ \sum_k \frac{1}{-m}\phi_k(x)\phi_k^\dagger(y) + \sum_K \frac{aR}{2}\phi_K(x)\phi_K^\dagger(y) \end{aligned}$$

zero modes(chiral) doublers(chiral)

$$Z_m^2 = 1 - (ma)^2\frac{R^2}{4}$$

$$0 < \lambda < \Lambda = \frac{2}{Ra}$$

$$\rho^A(\lambda) \equiv \frac{1}{V}\sum_n \delta(\lambda - \sqrt{\bar{\lambda}_n\lambda_n})$$

$$m_R = m/Z_m$$

$$\begin{aligned} \frac{1}{V}\int d^4x\, S^0(x) &= \frac{N_f}{mV}\sum_k(\phi_k,\phi_k) \\ &- \frac{N_f}{V}\sum_n\left[\frac{1-\frac{R}{2}a\lambda_n}{f_m\lambda_n - m}(\phi_n,\phi_n) + \frac{1-\frac{R}{2}a\bar{\lambda}_n}{f_m\bar{\lambda}_n - m}(\gamma_5\phi_n,\gamma_5\phi_n)\right] \\ &= \frac{N_f}{mV}N_{R+L}(A) + \frac{N_f}{V}\sum_n\frac{2m}{Z_m^2\bar{\lambda}_n\lambda_n + m^2}\left(1 - \frac{R^2}{4}a^2\bar{\lambda}_n\lambda_n\right), \end{aligned}$$

$$\langle S^0(x)\rangle = \frac{N_f}{mV}\langle N_{R+L}(A)\rangle_A + \frac{N_f}{Z_m}\int_0^\infty d\lambda\,\langle\rho^A(\lambda)\rangle_A\frac{2m_R}{\lambda^2 + m_R^2}\left(1 - \frac{R^2}{4}a^2\lambda^2\right),$$

continuum formula holds, except $Ra$ terms.

Others also hold.

$\Rightarrow$ $\boxed{\text{Result in the previous section can be rigorously derived in lattice QCD.}}$

## U(1)$_A$ susceptibility

$$\lim_{m\to 0}\lim_{V\to\infty}\chi^{\pi-\eta} = \lim_{m\to 0}\lim_{V\to\infty}\frac{N_f^4}{m^2V}\langle Q^2(A)\rangle_A = 2N_f\bar{\rho}_1. \quad (=0)$$

from a direct calculation

consistent

Effect of "anomaly" is represented by "index theorem".

$$\begin{aligned}
\chi^{\pi-\eta} &= -\frac{1}{2V}\int d^4xd^4yd^4z\underline{\langle J_x^0 S^2(y)P^a(z)\rangle} \\
&= \frac{N_f^2}{V}\int d^4xd^4yd^4z\,\langle \mathrm{tr}\gamma_5 F(D)(x,x)\,\mathrm{tr}F(D)S_F(z-y)\gamma_5 F(D)S_F(y-z)\rangle \\
&= \frac{N_f^2}{m^2V}\langle N_{R-L}^2(A)\rangle.
\end{aligned} \tag{107}$$

from an anomalous WTI

# 4. General cases

## 4-1. Higher order susceptibilities at $N_f = 2$

$$\mathcal{O}^{n_1,n_2,n_3,n_4} = \underbrace{P_a^{n_1} S_a^{n_2}}_{\text{non-singlet}} \overbrace{P_0^{n_3} S_0^{n_4}}^{\text{singlet}}$$

$$F \equiv \int d^4x\, F(x)$$

$\boxed{\text{Chiral variation}}$

non-singlet
$$\delta_a \mathcal{O}^{n_1,n_2,n_3,n_4} = -n_1 \mathcal{O}^{n_1-1,n_2,n_3,n_4+1} + n_2 \mathcal{O}^{n_1,n_2-1,n_3+1,n_4} - n_3 \mathcal{O}^{n_1,n_2+1,n_3-1,n_4} + n_4 \mathcal{O}^{n_1+1,n_2,n_3,n_4-1}$$

singlet
$$\delta_0 \mathcal{O}^{n_1,n_2,n_3,n_4} = -n_1 \mathcal{O}^{n_1-1,n_2+1,n_3,n_4} + n_2 \mathcal{O}^{n_1+1,n_2-1,n_3,n_4} - n_3 \mathcal{O}^{n_1,n_2,n_3-1,n_4+1} + n_4 \mathcal{O}^{n_1,n_2,n_3+1,n_4-1}$$

## Operator sets

$$\mathcal{O}_a^{(N)} = \{\mathcal{O}^{n_1,n_2,n_3,n_4} \,|\, n_1+n_2=\mathrm{odd},\ n_1+n_3=\mathrm{odd},\ \sum_i n_i = N\}$$

$$\mathcal{O}_0^{(N)} = \{\mathcal{O}^{n_1,n_2,n_3,n_4} \,|\, n_1+n_2=\mathrm{even},\ n_1+n_3=\mathrm{odd},\ \sum_i n_i = N\}$$

$$\delta_a\mathcal{O}_a^{(N)} \in \{\mathcal{O}^{n_1,n_2,n_3,n_4} \,|\, n_1+n_2=\mathrm{even},\ n_1+n_3=\mathrm{even},\ \sum_i n_i = N\}$$

$$\equiv \mathcal{O}^{(N)}$$

$$\delta_0\mathcal{O}_0^{(N)} \in \mathcal{O}^{(N)}$$

## WT identities

N: odd

$$\delta_a\mathcal{O}_a^{(N)} = 0 \quad \Longleftrightarrow \quad \delta_0\mathcal{O}_0^{(N)} = 0$$

equivalent

N: even

$$\delta_a\mathcal{O}_a^{(N)} = 0 \quad \not\Longleftrightarrow \quad \delta_0\mathcal{O}_0^{(N)} = 0$$

## 4-2. $N_f$ = 3 case

Non-singlet Chiral WT identities

$$P_a(x)P_a(y) - S_0(x)S_0(y) = 0 \qquad\qquad P_0(x)P_0(y) - S_a(x)S_a(y) = 0$$

$$\delta_8 S_3 = 2d_{383}P_3, \;\; \delta_8 P_3 = -2d_{383}S_3$$

special at $N_f$ = 3

$$\Rightarrow \quad \delta_8(P_3(x)S_3(y)) = 2d_{383}(P_3(x)P_3(y) - S_3(x)S_3(y)) = 0$$

$$\Rightarrow \quad \boxed{\underset{\pi^0}{P_3(x)P_3(y)} - \underset{\eta'}{P_0(x)P_0(y)} = 0}$$

Singlet WT identity is satisfied at this order.

Zero-modes $\propto (m^{N_f})^Q$ $\Rightarrow$ Effects should appear at 3-pt functions.

# 5. Conclusion

**Conclusion** If chiral symmetry is restored at T > Tc ,

$$\lim_{m\to 0} \chi^{\pi-\eta} = 0$$

$$\lim_{m\to 0} \rho_m(\lambda) = \langle \rho_2^A \rangle_A \frac{\lambda^2}{2!} + O(\lambda^3)$$

**Work in progress** constraints from higher order susceptibilities

chiral symmetry $\delta_a \mathcal{O}_a^{(N)} = 0$

constraint on eigenvalue density

eigenvalues of Dirac operator
have a gap near zero.

$$\rho(\lambda) = 0 \text{ at } |\lambda| \leq \lambda_c$$