Phase structure of finite density lattice QCD by a histogram method

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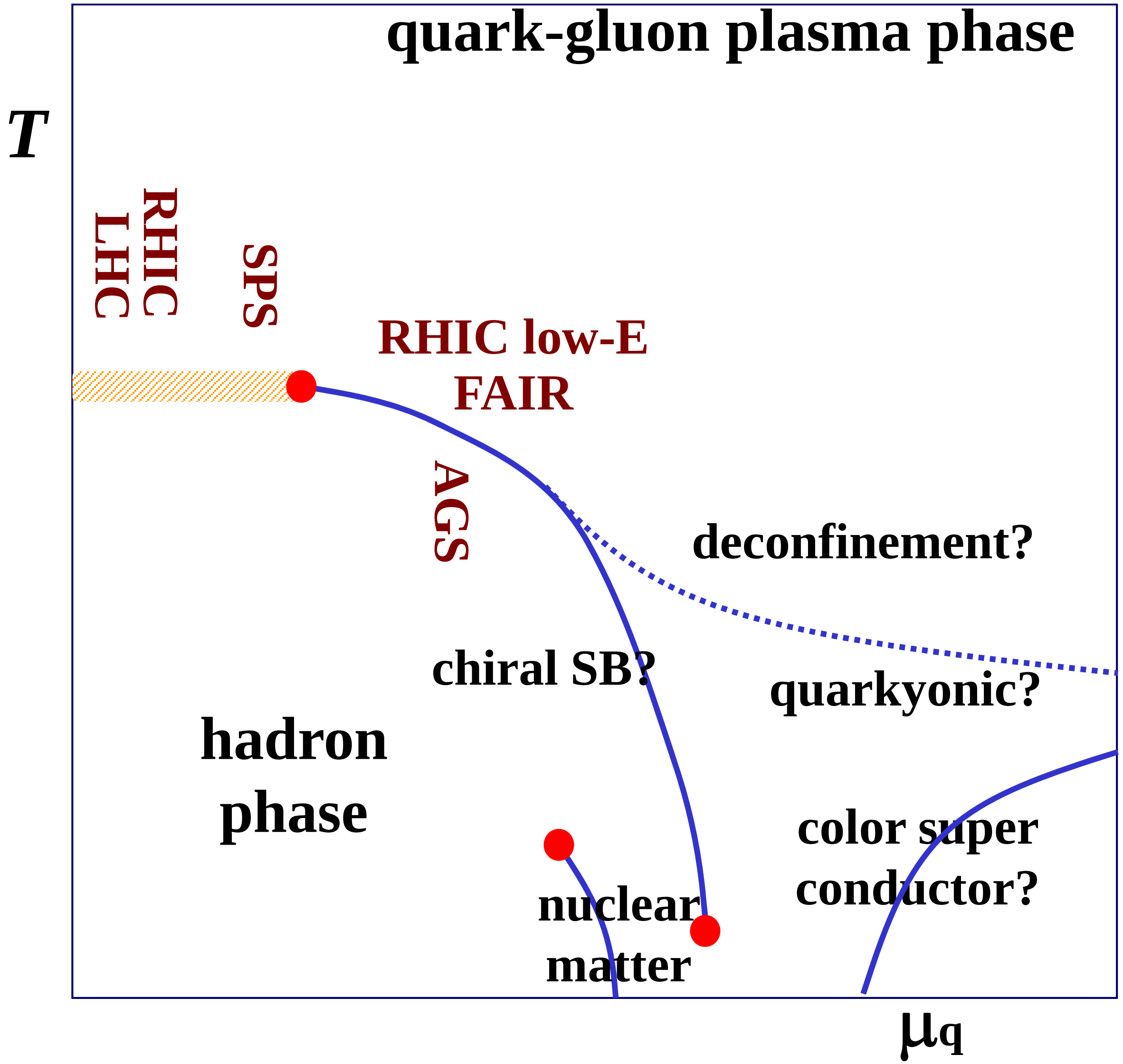
YIPQS-HPCI international molecule-type workshop on New-type of Fermions on the Lattice (YITP, Kyoto, Feb.9-24, 2012)

Phase structure of QCD at high temperature and density

Lattice QCD Simulations

- Phase transition lines
- Equation of state

- Direct simulation:
  Impossible at $\mu \neq 0$.



Probability distribution function

- Distribution function (Histogram)

  X: order parameters, total quark number, average plaquette etc.

$$Z(m,T,\mu) = \int dX\, \underbrace{W(X,m,T,\mu)}_{\text{histogram}}$$

- In the Matsubara formalism,

$$Z(m,T,\mu) \equiv \int DU \left(\det M(m,\mu)\right)^{N_{\mathrm{f}}} e^{-S_g}$$

$$W(X',m,T,\mu) \equiv \int DU \delta(X\text{-}X') \left(\det M(m,\mu)\right)^{N_{\mathrm{f}}} e^{-S_g}$$

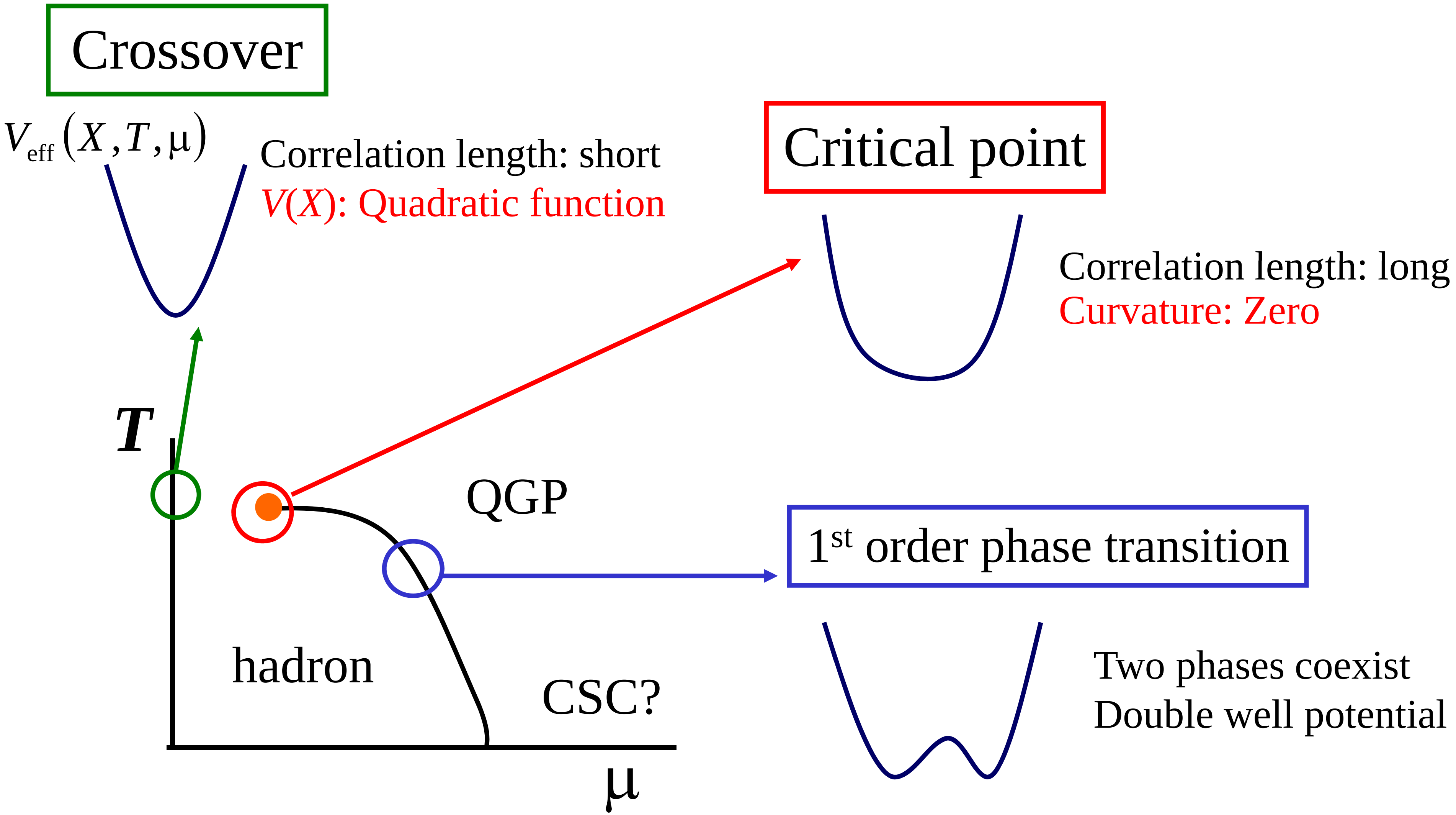
  - where det$M$: quark determinant, S_{g}: gauge action.

- Useful to identify the nature of phase transitions
  - e.g. At a first order transition, two peaks are expected in $W(X)$.

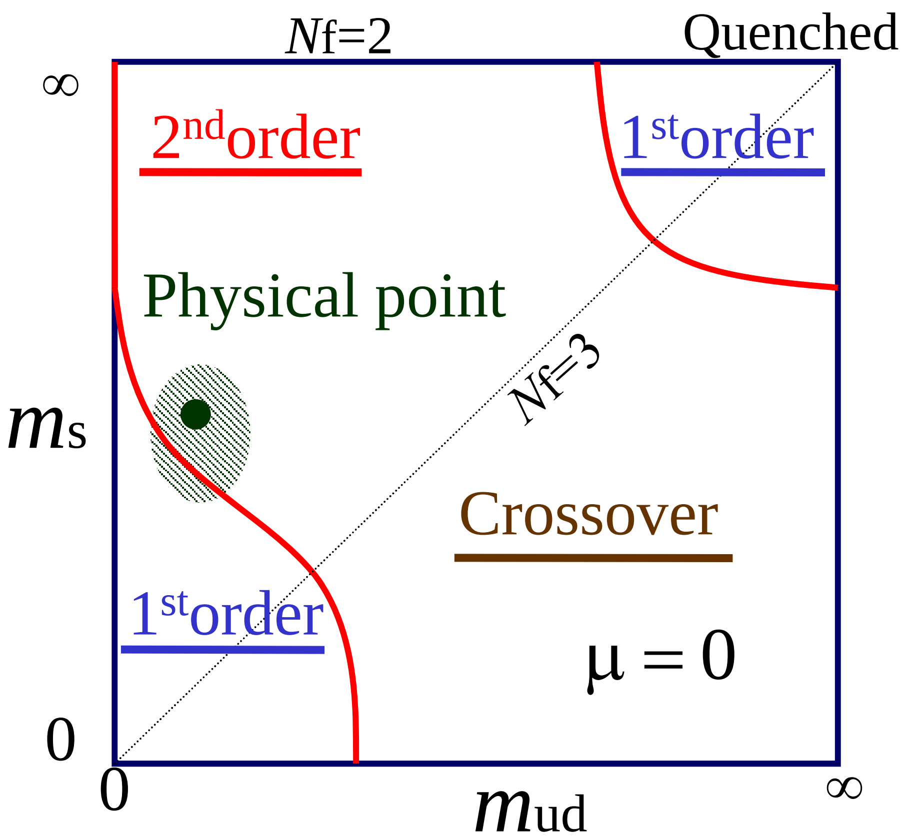
μ-dependence of the effective potential

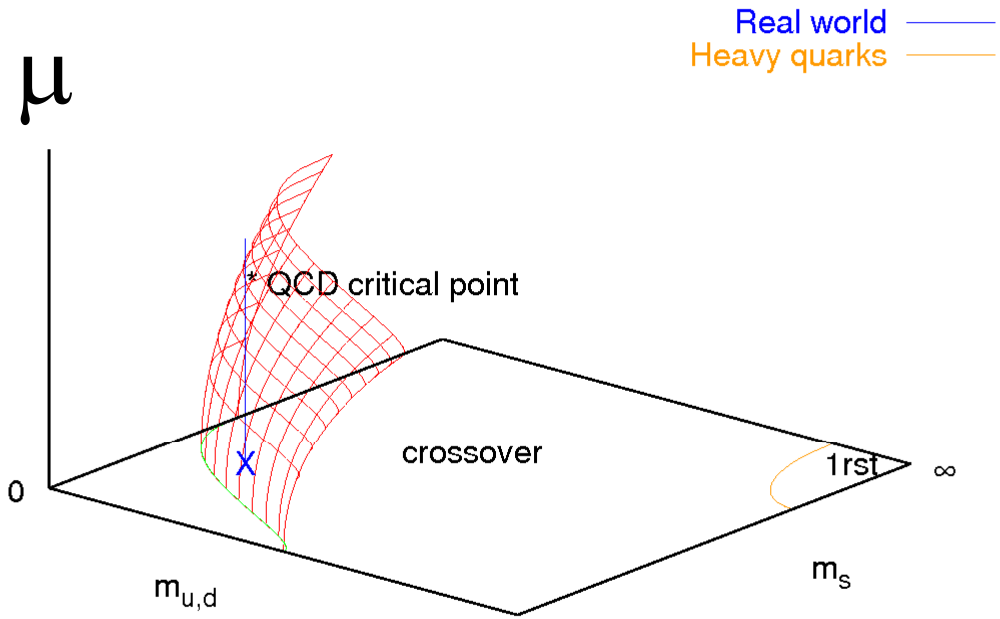
$$Z(T,\mu)=\int dX\, W(X,T,\mu), \qquad V_{\text{eff}}(X) = -\ln W(X)$$

X: order parameters, total quark number, average plaquette, quark determinant etc.



Quark mass dependence of the critical point





- Where is the physical point?
- Extrapolation to finite density
  - investigating the quark mass dependence near $\mu=0$
- Critical point at finite density?

Equation of State

- Integral method
  - Interaction measure $$\frac{\varepsilon - 3p}{T^4} = -\frac{1}{VT^3}\frac{\partial \ln Z}{\partial \ln a},$$

    computed by plaquette (1x1 Wilson loop) $\langle P \rangle$ and the derivative of det$M$.
  - Pressure at μ=0 $$\frac{p}{T^4} = \frac{1}{VT^3} \ln Z$$
    - Integral $$\left.\frac{p}{T^4}\right|_a - \left.\frac{p}{T^4}\right|_{a_0} = -\int_{a_0}^{a} \frac{\varepsilon' - 3p'}{T'^4} d\left(\ln a'\right)$$

      a0: start point p=0
- Pressure at μ≠0, $$\frac{p}{T^4}(\mu) - \frac{p}{T^4}(0) = \frac{1}{VT^3}\ln\left(\frac{Z(\mu)}{Z(0)}\right) = \left(\frac{N_t}{N_s}\right)^3 \ln\left\langle \frac{\det M(\mu)}{\det M(0)} \right\rangle_{\mu=0}$$
- $$\langle X \rangle_{(m,T,\mu)} = \frac{1}{Z}\int dX \;\, X \, W\left(X, m, T, \mu\right)$$ with $X = P$ or $\det M(\mu) / \det M(0)$

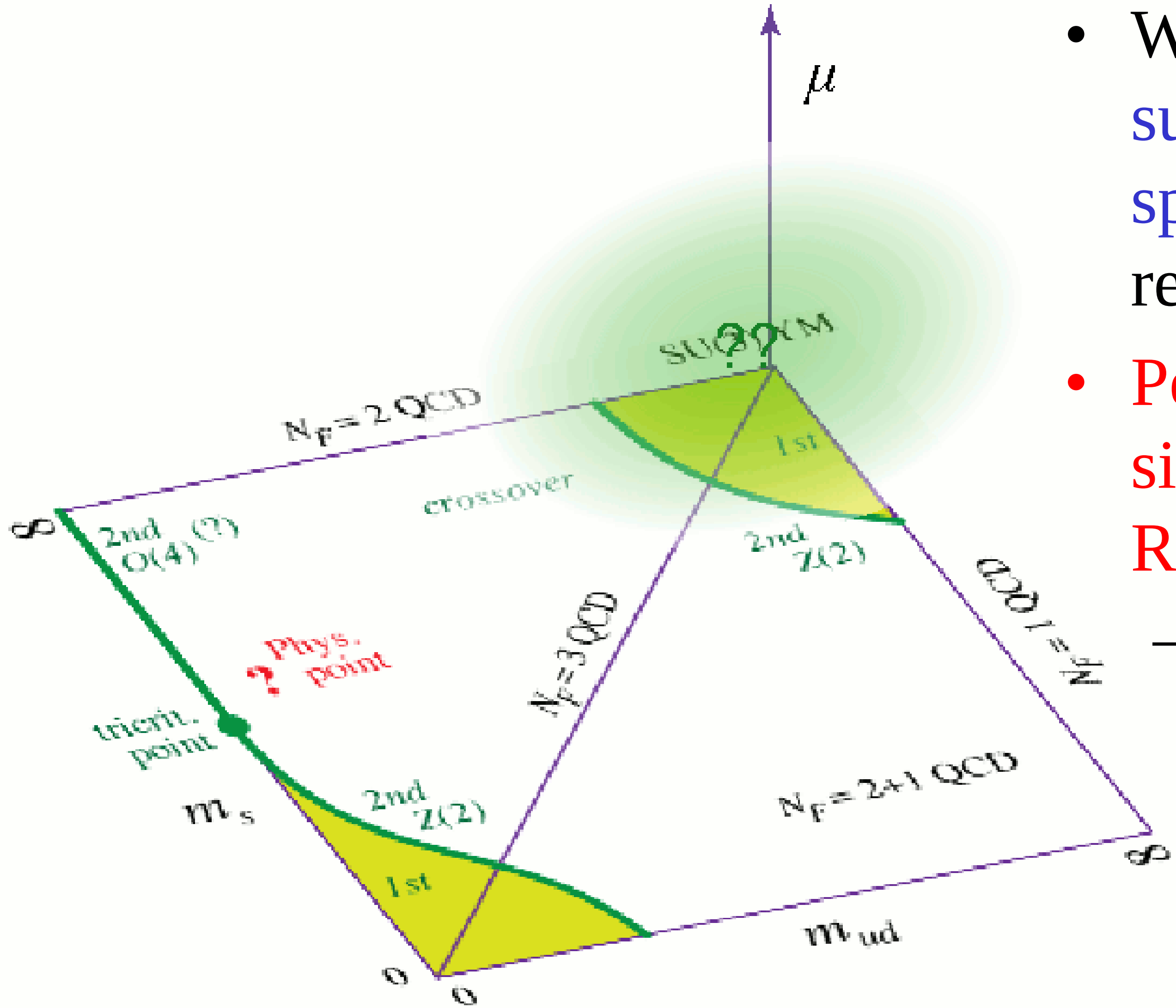
Plan of this talk

- Test in the heavy quark region
  - H. Saito et al. (WHOT-QCD Collab.), Phys.Rev.D84, 054502(2011)
  - WHOT-QCD Collaboration, in preparation

- Application to the light quark region at finite density
  - S.E., Phys.Rev.D77, 014508(2008))
  - WHOT-QCD Collaboration, in preparation
    (Lattice 2011 proc.: Y. Nakagawa et al., arXiv:1111.2116)

Distribution function in the heavy quark region

WHOT-QCD Collab., Phys.Rev.D84, 054502(2011)



- We study the critical surface in the (m_{ud}, m_s, μ) space in the heavy quark region.
- Performing quenched simulations + Reweighting.
  - plaquette gauge action + Wilson quark action

(β, m, μ)-dependence of the Distribution function

- Distributions of plaquette P (1x1 Wilson loop for the standard action)

$$W(P', \beta, m, \mu) \equiv \int DU \delta(P-P') \left(\det M(m,\mu)\right)^{N_f} e^{6N_{site}P}$$

$$R(P, \beta, \beta_0 m, m_0, \mu) \equiv W(P, \beta, m, \mu) / W(P, \beta_0, m_0, 0)$$ (Reweight factor)

$$R(P') = e^{6N_{site}(\beta-\beta_0)P'} \frac{\left\langle \delta(P-P') \left(\frac{\det M(m,\mu)}{\det M(m_0,0)} \right)^{N_f} \right\rangle_{(\beta_0,\mu=0)}}{\left\langle \delta(P-P') \right\rangle_{(\beta_0,\mu=0)}} \equiv e^{6N_{site}(\beta-\beta_0)P'} \left\langle \left(\frac{\det M(m,\mu)}{\det M(m_0,0)} \right)^{N_f} \right\rangle_{P'}$$

Effective potential:

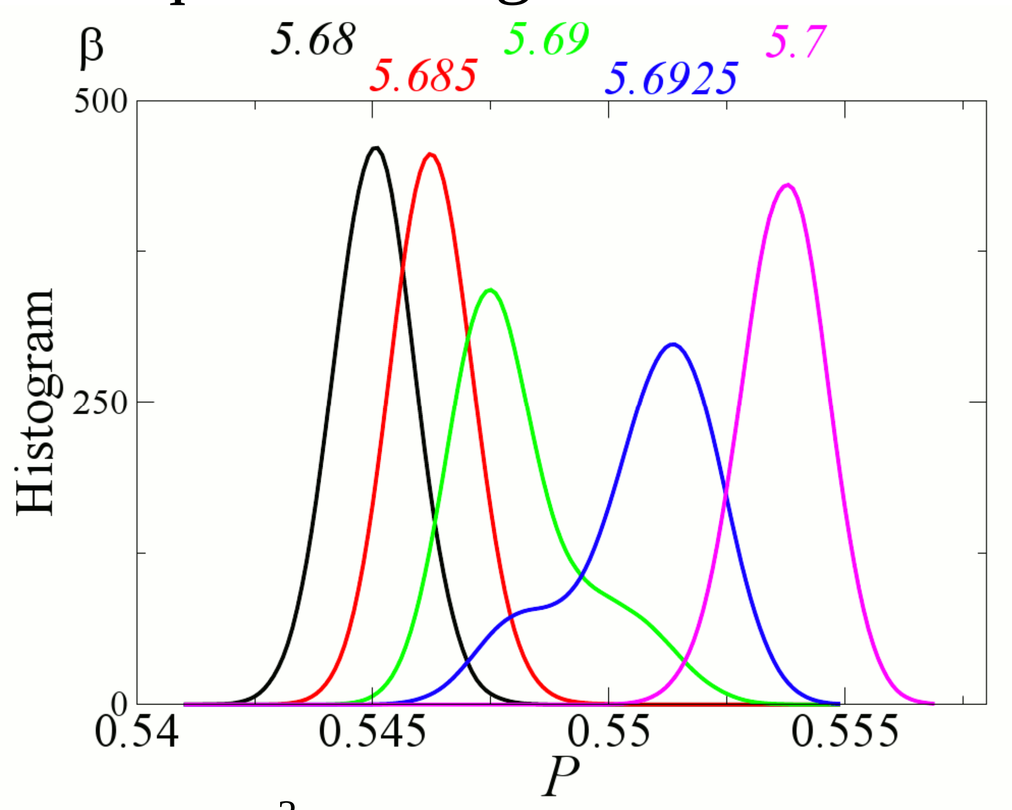
$$V_{eff}(P, \beta, m, \mu) = -\ln[W(P, \beta m, \mu)] = V_{eff}(P, \beta_0, m_0, 0) - \ln R(P, \beta, \beta_0 m, m_0, \mu)$$

$$\ln R(P) = \underline{6N_{site}(\beta - \beta_0)P} + \underline{\ln \left\langle \left(\frac{\det M(m,\mu)}{\det M(m_0,0)} \right)^{N_f} \right\rangle_P}$$

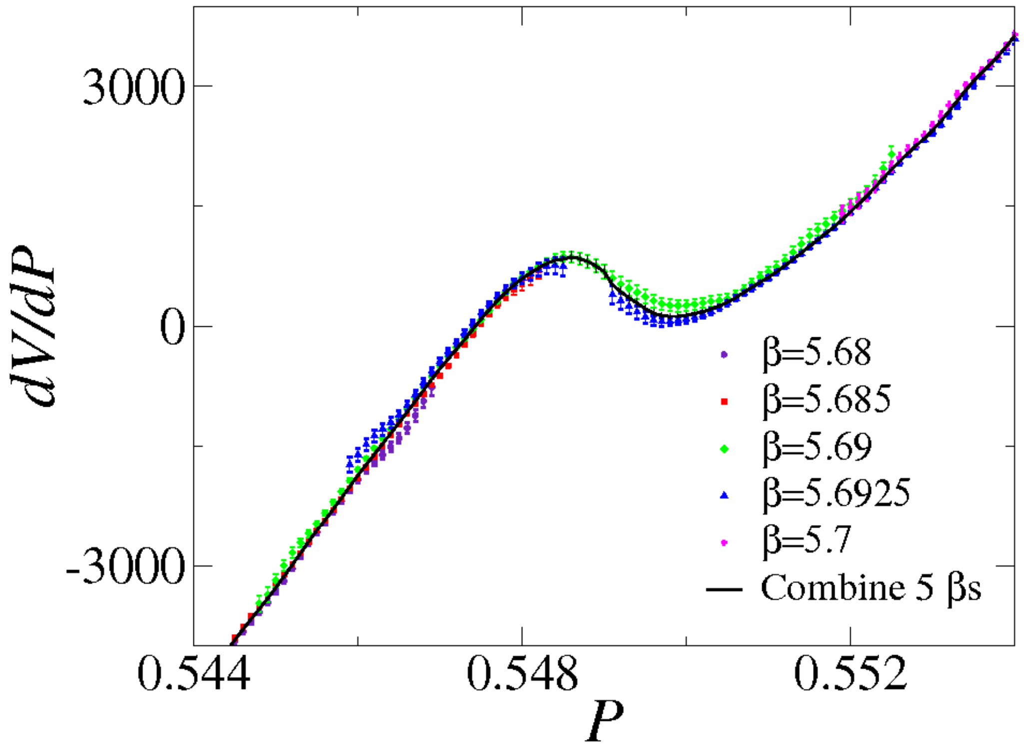
Distribution function in quenched simulations

Effective potential in a wide range of *P*: required.

Plaquette histogram at $K=1/m_q=0$.



Derivative of V_{eff} at β=5.69



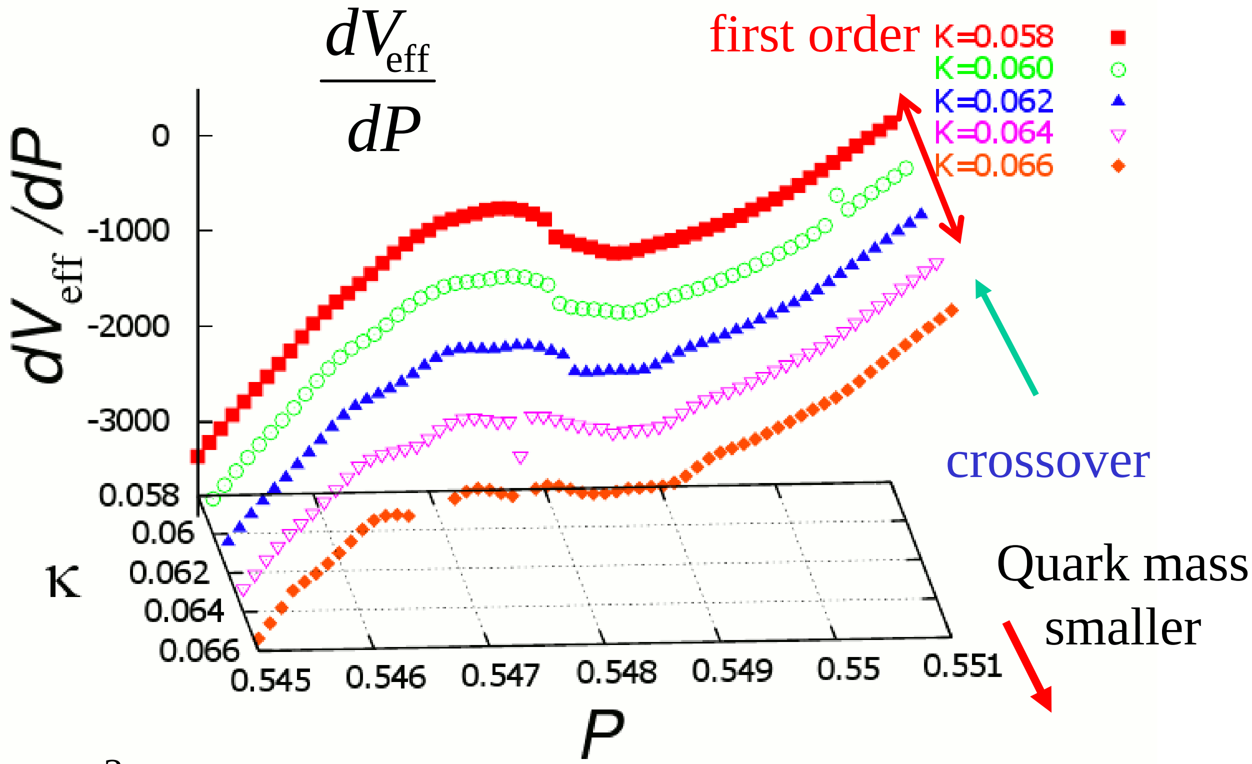
$N_{\mathrm{site}} = 24^3 \times 4$, 5 β points, quenched.

dV_{eff}/dP is adjusted to β=5.69, using
$$\frac{dV_{\mathrm{eff}}}{dP}(\beta_2) = \frac{dV_{\mathrm{eff}}}{dP}(\beta_1) - 6N_{\mathrm{site}}(\beta_2 - \beta_1)$$

These data are combined by taking the average.

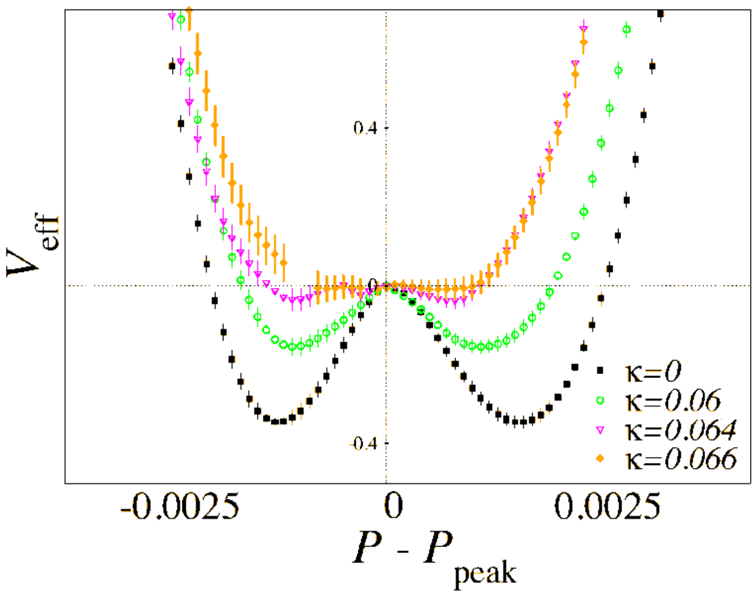
Effective potential near the quenched limit

WHOT-QCD, Phys.Rev.D84, 054502(2011)



Quenched Simulation
(mq=∞, K=0)

K~1/mq for large mq



$24^3 \times 4$ lattice, 5 β points, Nf=2

- detM: Hopping parameter expansion,

$$N_f \ln\left(\frac{\det M(K)}{\det M(0)}\right) = N_f\left(288 N_{site} K^4 P + 12 \times 2^{N_t} N_s^3 K^{N_t} \Omega_R + \cdots\right)$$

real part of Polyakov loop

Nf=2: Kcp=0.0658(3)(8)

$$\frac{T_c}{m_\pi} \approx 0.02$$

- First order transition at K = 0 changes to crossover at K > 0.

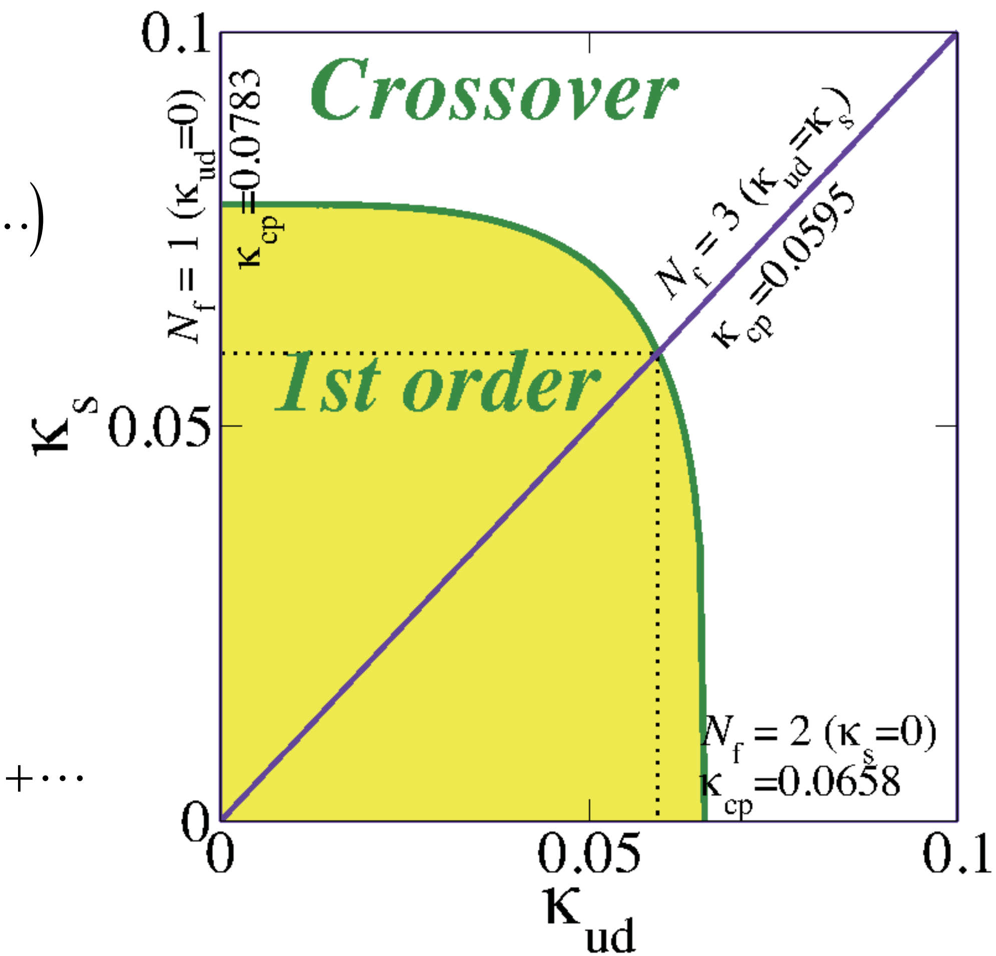
Endpoint of 1st order transition in 2+1 flavor QCD

$N_f=2$: $K_{cp}=0.0658(3)(8)$

$$2\ln\left(\frac{\det M(K)}{\det M(0)}\right)=2\left(288N_{\rm site}K^4P+12\times 2^{N_t}N_s^3K^{N_t}\Omega_R+\cdots\right)$$

$N_f=2+1$

$$\ln\left[\frac{\left(\det M(K_{ud})\right)^2\det M(K_s)}{\left(\det M(0)\right)^3}\right]$$
$$=288N_{site}\left(2K_{ud}^4+K_s^4\right)P+12\times 2^{N_t}N_s^3\left(2K_{ud}^{N_t}+K_s^{N_t}\right)\Omega_R+\cdots$$



The critical line is described by

$$\boxed{2K_{ud}^{N_t}+K_s^{N_t}=2K_{{\rm cp}(Nf=2)}^{N_t}}$$

Finite density QCD in the heavy quark region

$$\boxed{U_4(x) \Rightarrow e^{\mu_q a} U_4(x), \quad U_4^\dagger(x) \Rightarrow e^{-\mu_q a} U_4^\dagger(x)} \text{ in } \det M$$

$$\Longrightarrow \quad \boxed{\Omega \Rightarrow e^{\mu_q/T}\Omega, \quad \Omega^* \Rightarrow e^{-\mu_q/T}\Omega^*} \quad \text{Polyakov loop}$$

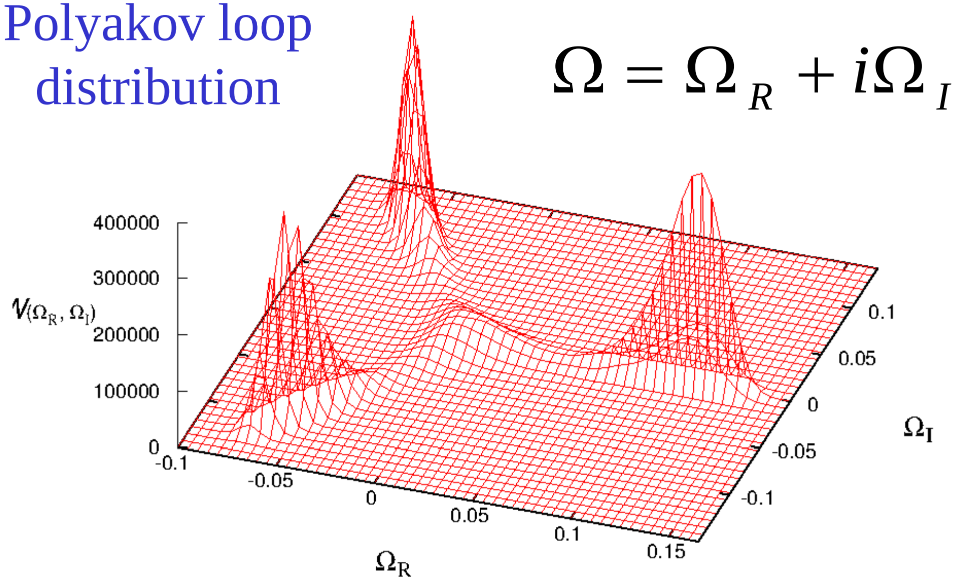
$$N_f \ln\left(\frac{\det M(K,\mu)}{\det M(0,0)}\right) = N_f\left(288 N_{site} K^4 P + 6\cdot 2^{N_t} N_s^3 K^{N_t}\left(e^{\mu/T}\Omega + e^{-\mu/T}\Omega^*\right) + \cdots\right)$$

$$= N_f\left(288 N_{site} K^4 P + 12\cdot 2^{N_t} N_s^3 K^{N_t}\left(\cosh(\mu/T)\Omega_R + \underline{\underline{i\sinh(\mu/T)\Omega_I}}\right) + \cdots\right)$$

phase

Polyakov loop distribution

$$\Omega = \Omega_R + i\Omega_I$$



- We can extend this discussion to finite density QCD.

Phase quenched simulations, Isospin chemical potential

$(N_f=2)$, $\mu_u = -\mu_d$.

$$\det M(K,-\mu) = [\det M(K,\mu)]^*$$

$$\left|\det M(K,\mu)\right|^2 = \det M(K,\mu) \det M(K,-\mu)$$

$$\ln\left(\frac{\det M(K,\mu)}{\det M(0,0)}\right) = 288 N_{\rm site} K^4 P + 12\cdot 2^{N_t} N_s^3 K^{N_t}\left(\cosh(\mu/T)\Omega_R \underbrace{\cancel{+ i\sinh(\mu/T)\Omega_I}}_{\text{phase}}\right) + \cdots$$

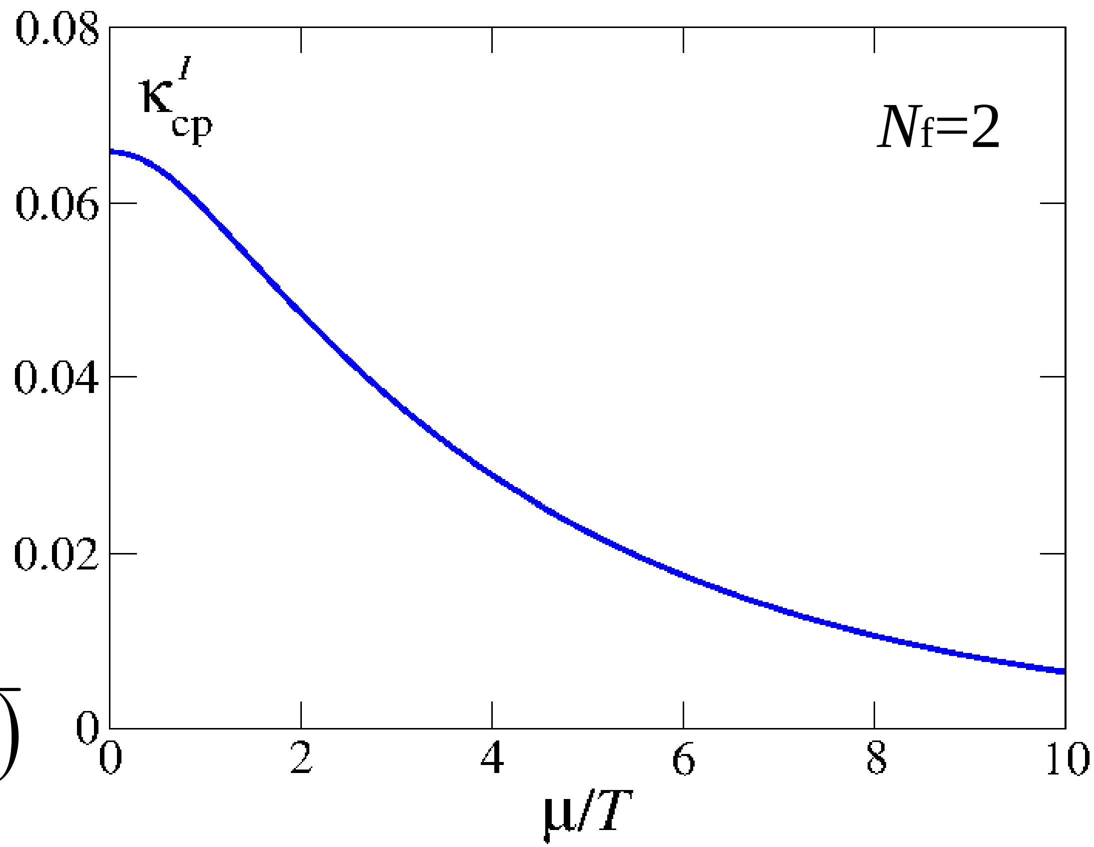
- If the complex phase is neglected,

$$\boxed{K^{N_t} \Rightarrow K^{N_t}\cosh(\mu/T)}$$

  - Critical point:

$$\underline{K_{\rm cp}^{N_t}(\mu)\cosh(\mu/T) = K_{\rm cp}^{N_t}(0)}$$

$$K_{\rm cp}(\mu) = K_{\rm cp}(0) \Big/ \sqrt[N_t]{\cosh(\mu/T)}$$



Distribution function for P and Ω_R at $\mu=0$

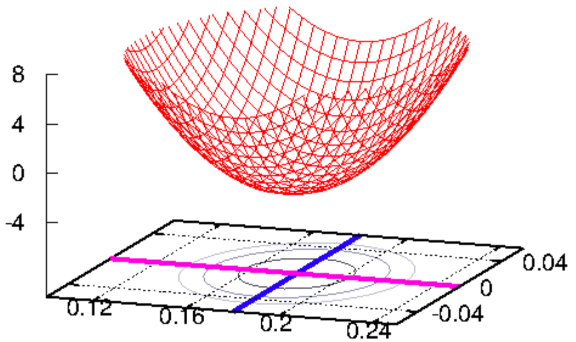
$$\boxed{W(P',\Omega_R',\beta,\kappa)=\int DU\ \delta(P-P')\delta(\Omega_R-\Omega_R')(\det M(\kappa))^{N_f}\ e^{-S_g}}$$

$$\frac{W(\beta,\kappa)}{W(\beta_0,0)}=\left\langle \exp\left(6(\beta-\beta_0)N_{\rm site}P+288N_f N_{\rm site}K^4 P+12\times 2^{N_t}N_f N_s^3 K^{N_t}\Omega_R+\cdots\right)\right\rangle_{P,\Omega_R\ \rm fixed}$$

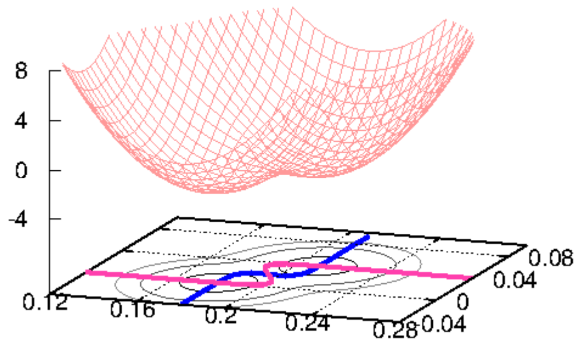
$$\approx \exp\left(\left(6(\beta-\beta_0)+288N_f K^4\right)N_{\rm site}P+12\times 2^{N_t}N_f N_s^3 K^{N_t}\Omega_R\right)$$

$$\Rightarrow\ V_{\rm eff}(\beta,\kappa)-V_{\rm eff}(\beta_0,0)=-\left(6(\beta-\beta_0)+288N_f K^4\right)N_{\rm site}P-12\times 2^{N_t}N_f N_s^3 K^{N_t}\Omega_R$$

- Peak position of W: $\dfrac{dV_{\rm eff}}{dP}=\dfrac{dV_{\rm eff}}{d\Omega_R}=0$

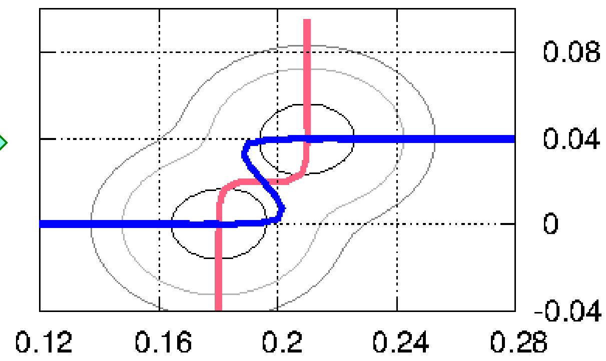


crossover
1 intersection



first order transition
3 intersections

Lines of zero derivatives for first order

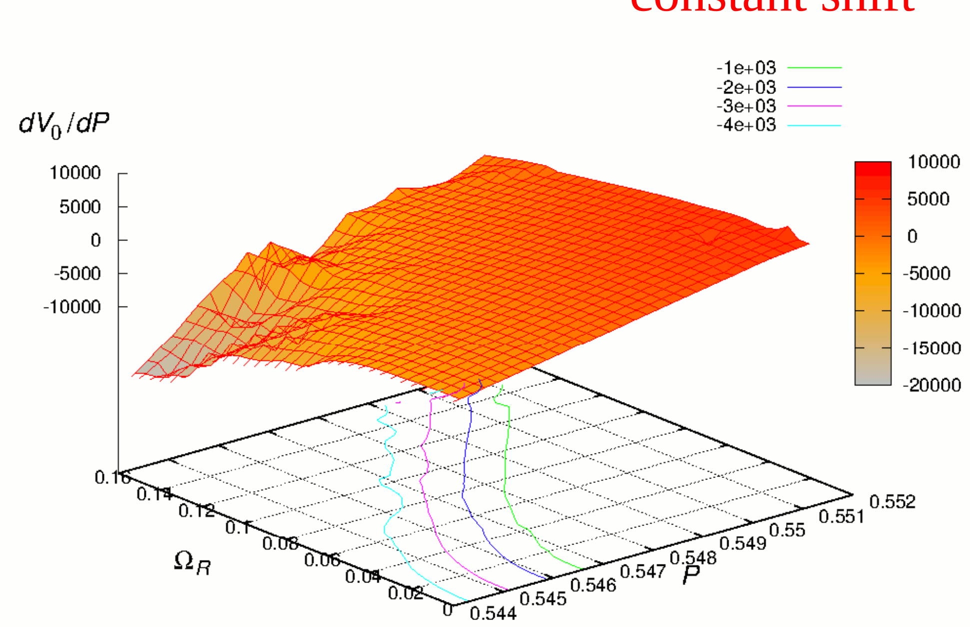


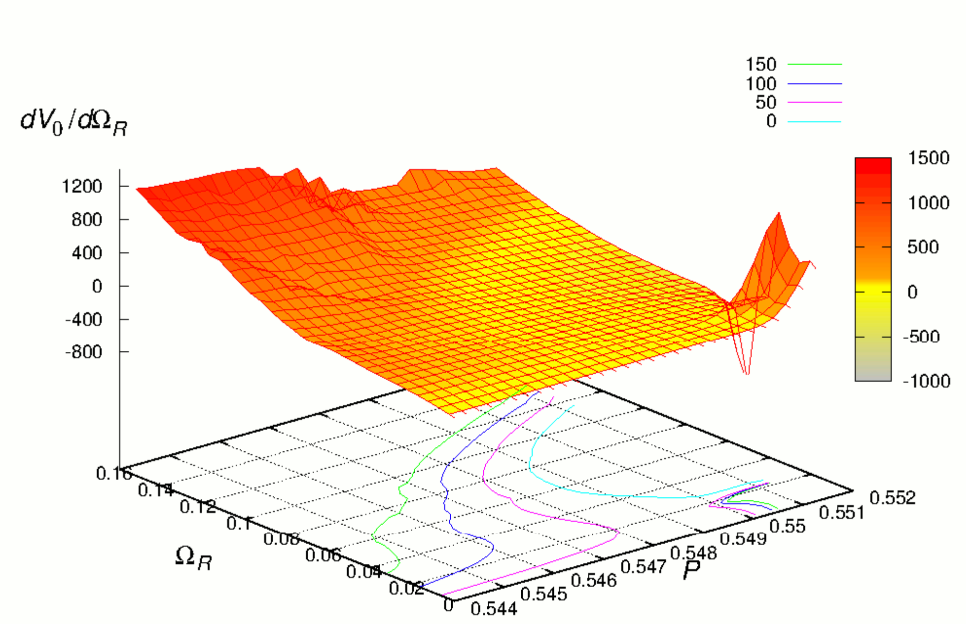
Derivatives of $V_{\rm eff}$ in terms of P and Ω_R

In heavy quark region, $24^3 \times 4$ lattice, 5 β points

$$V_{\rm eff}(\beta,\kappa) - V_{\rm eff}(\beta_0,0) = -\left(6(\beta-\beta_0) + 288 N_{\rm f} K^4\right) N_{\rm site} P - 12 \times 2^{N_t} N_{\rm f} N_s^3 K^{N_t} \Omega_R$$

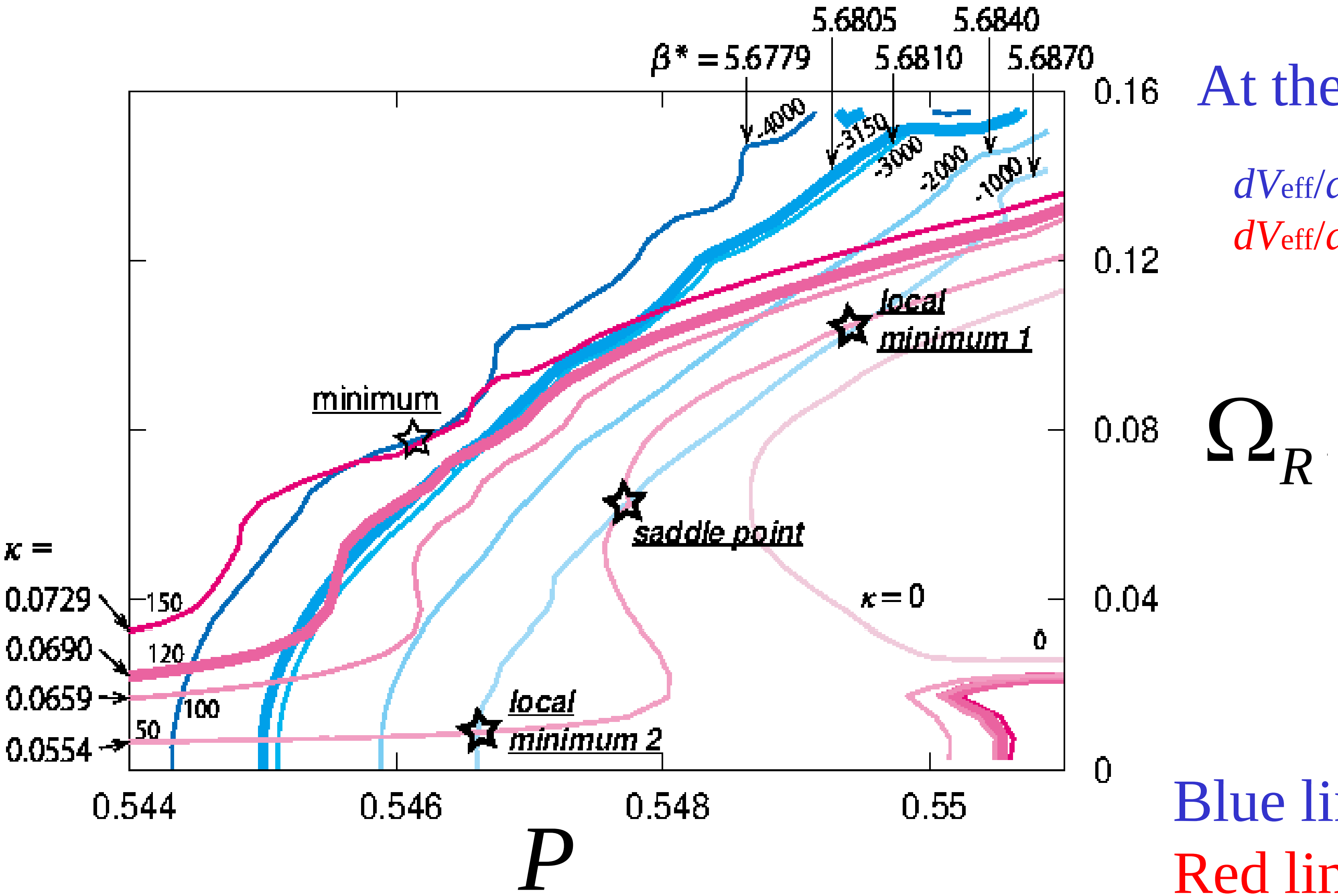
$$\frac{dV_{\rm eff}(\beta,K)}{dP} - \frac{dV_{\rm eff}(\beta_0,0)}{dP} = \underbrace{-\left(6(\beta-\beta_0) + 288 N_{\rm f} K^4\right) N_{\rm site}}_{\text{constant shift}} \qquad \frac{dV_{\rm eff}(\beta,K)}{d\Omega_R} - \frac{dV_{\rm eff}(\beta_0,0)}{d\Omega_R} = \underbrace{-12 \times 2^{N_t} N_{\rm f} N_s^3 K^{N_t}}_{\text{constant shift}}$$



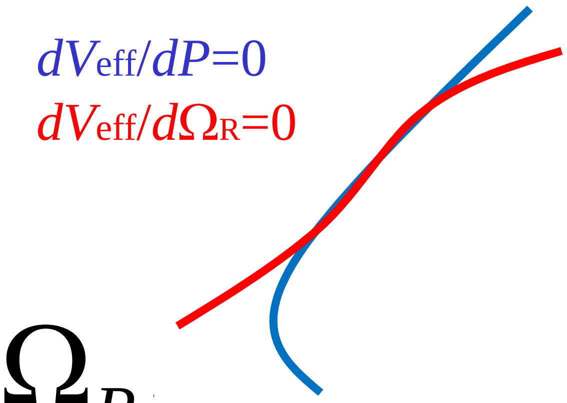


- Contour lines of $\frac{dV_{\rm eff}}{dP}$ and $\frac{dV_{\rm eff}}{d\Omega_R}$ at $(\beta,\kappa) = (\beta_0,0)$ correspond to the lines of the zero derivatives at (β,κ).

Lines of constant derivatives obtained by quenched simulations



At the critical point,



Blue lines: dV_{eff}/dP

Red lines: $dV_{\mathrm{eff}}/d\Omega_R$

- The number of the intersection changes around κ=0.658

Finite density QCD in the heavy quark region

$$\boxed{U_4(x) \Rightarrow \mathrm{e}^{\mu_q a} U_4(x), \quad U_4^\dagger(x) \Rightarrow \mathrm{e}^{-\mu_q a} U_4^\dagger(x)}$$

$$\Longrightarrow \quad \boxed{\Omega \Rightarrow e^{\mu_q/T}\Omega, \quad \Omega^* \Rightarrow \mathrm{e}^{-\mu_q/T}\Omega^*}$$

$$N_f \ln\left(\frac{\det M(K,\mu)}{\det M(0,0)}\right) = N_f\left(288 N_{\mathrm{site}} K^4 P + 6 \cdot 2^{N_t} N_s^3 K^{N_t}\left(e^{\mu/T}\Omega + e^{-\mu/T}\Omega^*\right) + \cdots\right)$$

$$= N_f\left(288 N_{\mathrm{site}} K^4 P + 12 \cdot 2^{N_t} N_s^3 K^{N_t}\left(\cosh(\mu/T)\Omega_R + \underline{\underline{i \sinh(\mu/T)\Omega_I}}\right) + \cdots\right)$$

phase

- We can extend this discussion to finite density QCD.

Finite density QCD

$$N_{\mathrm{f}} \ln\left(\frac{\det M(K)}{\det M(0)}\right) = N_{\mathrm{f}}\left(288 N_{\mathrm{site}} K^4 P + 6 \cdot 2^{N_t} N_s^3 K^{N_t}\left(e^{\mu/T}\Omega + e^{-\mu/T}\Omega^*\right) + \cdots\right)$$

$$= N_{\mathrm{f}}\left(288 N_{\mathrm{site}} K^4 P + 12 \cdot 2^{N_t} N_s^3 K^{N_t}\left(\cosh(\mu/T)\Omega_R + \underline{\underline{i \sinh(\mu/T)\Omega_I}}\right) + \cdots\right)$$

phase

$$\frac{W(\beta,\kappa)}{W(\beta_0,0)} = \left\langle \exp\left(6(\beta-\beta_0)N_{\mathrm{site}}P + 288 N_{\mathrm{f}} N_{\mathrm{site}} K^4 P + 12 \times 2^{N_t} N_{\mathrm{f}} N_s^3 K^{N_t}\left(\cosh(\mu/T)\Omega_R + \underline{i \sinh(\mu/T)\Omega_I}\right) + \cdots\right)\right\rangle_{P,\Omega_R \text{ fixed}}$$

$$\approx \exp\left(\left(6(\beta-\beta_0) + 288 N_{\mathrm{f}} K^4\right) N_{\mathrm{site}} P + 12 \times 2^{N_t} N_{\mathrm{f}} N_s^3 K^{N_t} \cosh(\mu/T)\Omega_R\right) \times \underline{\left\langle e^{i\theta}\right\rangle_{P,\Omega_R \text{ fixed}}}$$

$$\theta = 12 \cdot 2^{N_t} N_{\mathrm{f}} N_s^3 K^{N_t} \sinh(\mu/T)\Omega_I$$

$$\underline{V_{\mathrm{eff}}(\beta,\kappa) - V_{\mathrm{eff}}(\beta_0,0) = -\left(6(\beta-\beta_0) + 288 N_{\mathrm{f}} K^4\right) N_{\mathrm{site}} P - 12 \times 2^{N_t} N_{\mathrm{f}} N_s^3 K^{N_t} \cosh(\mu/T)\Omega_R - \ln\left\langle e^{i\theta}\right\rangle_{P,\Omega_R \text{ fixed}}}$$

- Reweighting: $\ln\left\langle e^{i\theta}\right\rangle_{P,\Omega_R \text{ fixed}}$ is a non-linear term in $V_{\mathrm{eff}}(\beta,\kappa) - V_{\mathrm{eff}}(\beta_0,0)$.

Sign problem.

Avoiding the sign problem

(SE, Phys.Rev.D77,014508(2008), WHOT-QCD, Phys.Rev.D82, 014508(2010))

θ: complex phase $\quad \theta \equiv \mathrm{Im} \ln \det M = 12 \cdot 2^{N_t} N_f N_s^3 K^{N_t} \sinh(\mu/T) \Omega_I$

- Sign problem: If $e^{i\theta}$ changes its sign,

$$\left\langle e^{i\theta} \right\rangle_{P,\Omega_R \text{ fixed}} \ll \text{(statistical error)}$$

- Cumulant expansion

$<..>_{F,P}$: expectation values fixed F and P.

$$\left\langle e^{i\theta} \right\rangle_{P,\Omega_R} = \exp\left[\underset{\to 0}{\cancel{i\langle\theta\rangle_C}} - \frac{1}{2}\langle\theta^2\rangle_C \underset{\to 0}{\cancel{- \frac{i}{3!}\langle\theta^3\rangle_C}} + \frac{1}{4!}\langle\theta^4\rangle_C \cancel{+ \cdots} \right]$$

cumulants

$$\langle\theta\rangle_C = \langle\theta\rangle_{P,\Omega_R}, \quad \langle\theta^2\rangle_C = \langle\theta^2\rangle_{P,\Omega_R} - \langle\theta\rangle_{P,\Omega_R}^2, \quad \langle\theta^3\rangle_C = \langle\theta^3\rangle_{P,\Omega_R} - 3\langle\theta^2\rangle_{F,\Omega_R}\langle\theta\rangle_{P,\Omega_R} + 2\langle\theta\rangle_{P,\Omega_R}^3, \quad \langle\theta^4\rangle_C = \cdots$$

- – Odd terms vanish from a symmetry under $\mu \leftrightarrow -\mu$ ($\theta \leftrightarrow -\theta$)

  Source of the complex phase

If the cumulant expansion converges, No sign problem.

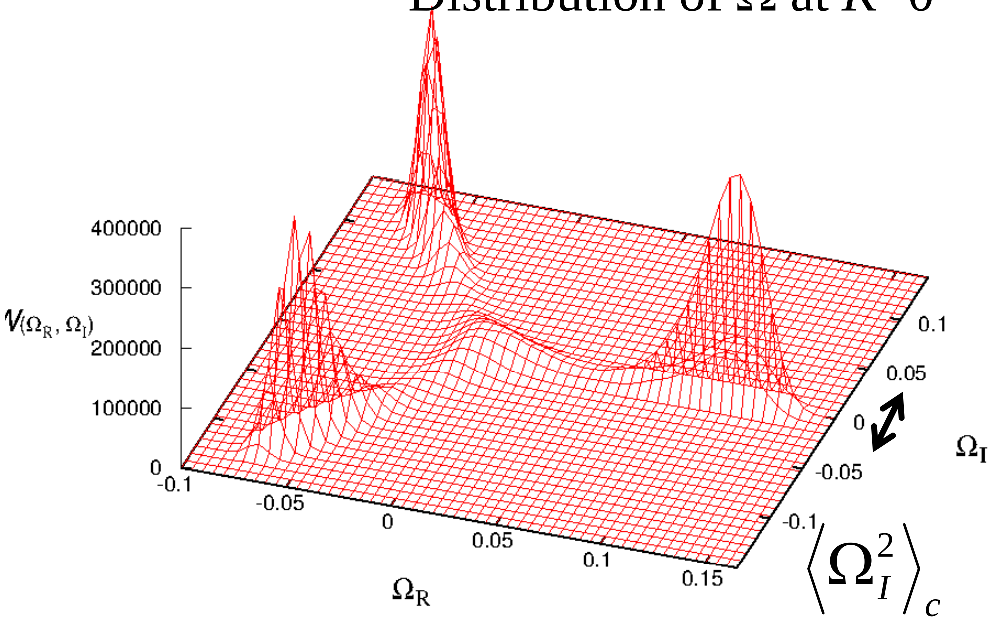
Cumulant expansion of the complex phase

- When the distribution of θ is perfectly Gaussian, the average of the complex phase is give by the second order (variance),

$$\left\langle e^{i\theta}\right\rangle_{P,F} = \exp\left[-\frac{1}{2}\left\langle \theta^2\right\rangle_C\right]$$

$$\theta = 12 \cdot 2^{N_t} N_{\mathrm{f}} N_s^3 K^{N_t} \sinh(\mu/T)\Omega_I$$

Distribution of Ω at K=0



$$\left\langle \Omega_I^2\right\rangle_c$$

- Because $W(\mu)$ is enhanced by the factor,

$$W(\beta,\kappa) \sim \exp\left(12\times 2^{N_t} N_{\mathrm{f}} N_s^3 K^{N_t} \cosh(\mu/T)\Omega_R\right)$$

the region of large Ω_R is important.

- The distribution of Ω_I seems to be of Gaussian at $\Omega_R>0$.
- The higher order cumulants $\left\langle \Omega_I^4\right\rangle_c$ etc. might be small.

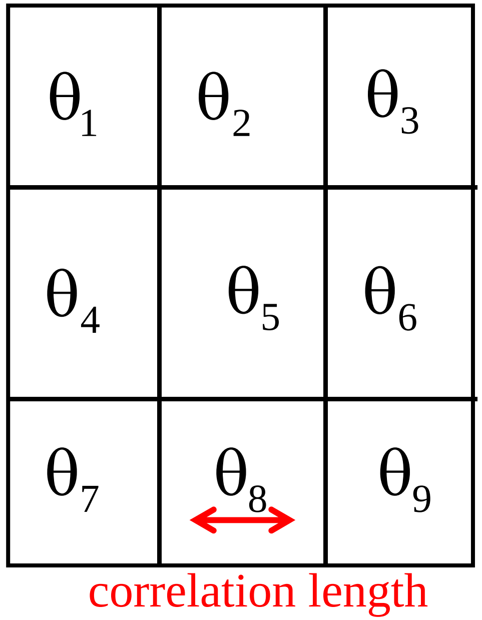
Convergence in the large volume (V) limit

- Because $\theta \sim O(V)$, Naïve expectation: $\langle \theta^n \rangle_C \sim O(V^n)$?
  - If so, the cumulant expansion does not converge.

However, this problem is solved in the following situation.

- The phase is given by $\theta = \sum_x \theta_x$
  - No correlation between θ_x.
  - In the heavy quark region, the phase is the imaginary part of the Polyakov loop average, Ω_I.

$$\theta = 12 \cdot 2^{N_t} N_f N_s^3 K^{N_t} \sinh(\mu/T) \Omega_I$$



  - If the spatial correlation length is short, the distribution of the imaginary part of the Polyakov loop is expected to be Gaussian by the central limit theorem.

Convergence in the large volume (V) limit

This problem is solved in the following situation.

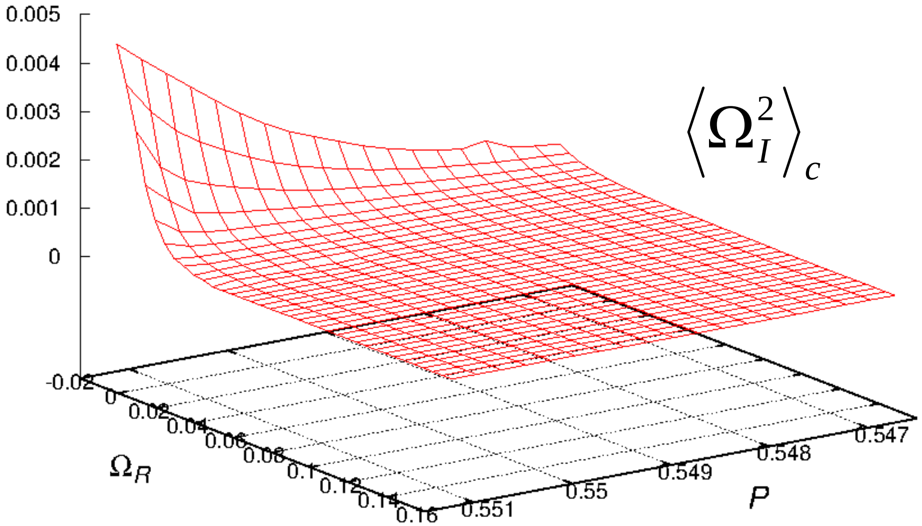
- The phase is given by $\theta = \sum_x \theta_x$
  - No correlation between θ_x.

$$\left\langle e^{i\theta} \right\rangle_{F,P} = \left\langle e^{i\sum_x \theta_x} \right\rangle_{F,P} \approx \prod_x \left\langle e^{i\theta_x} \right\rangle_{F,P} = \exp\left[\sum_x \sum_n \frac{i^n}{n!} \left\langle \theta_x^n \right\rangle_C \right]$$

$$\left\langle e^{i\theta} \right\rangle_{F,P} = \exp\left[\sum_n \frac{i^n}{n!} \left\langle \theta^n \right\rangle_C \right] \Rightarrow \left\langle \theta^n \right\rangle_C \approx \sum_x \left\langle \theta_x^n \right\rangle_C \sim O(V)$$

  - Ratios of cumulants do not change in the large V limit.
  - Convergence property is independent of V although the phase fluctuation becomes larger as V increases.

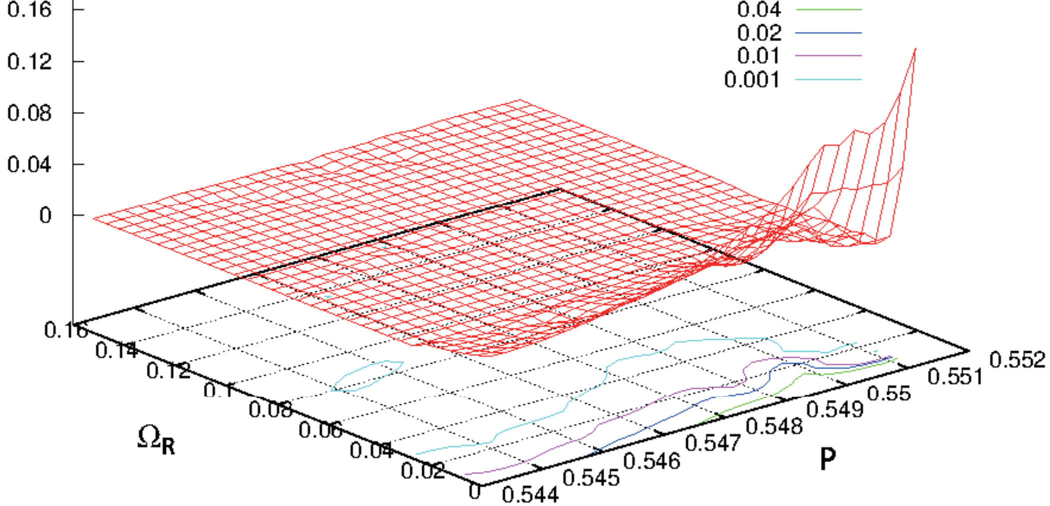
Effect from the complex phase



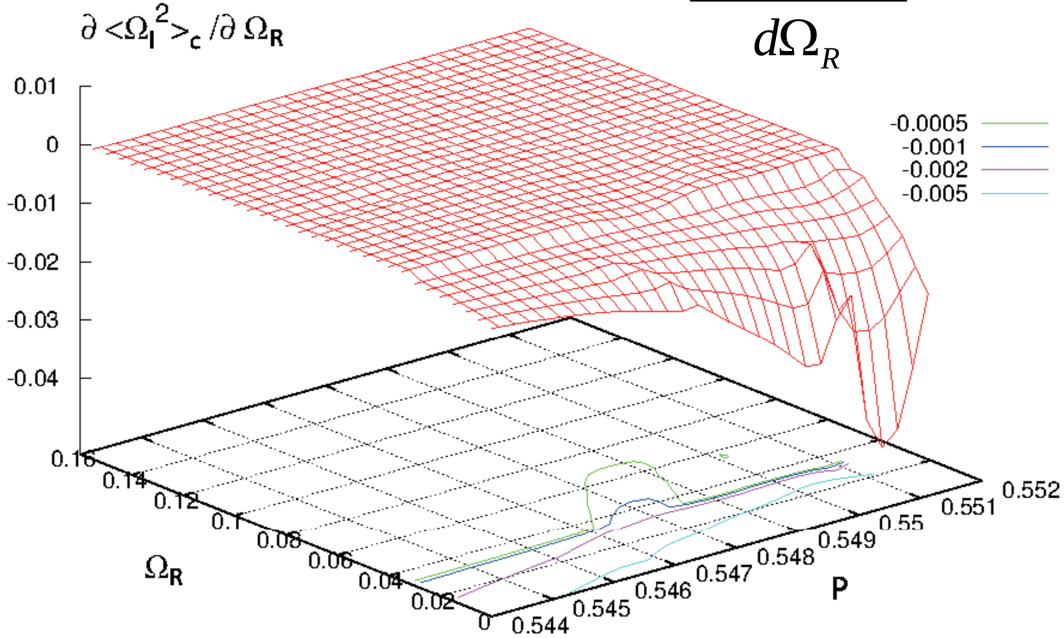
$$\left\langle \Omega_I^2 \right\rangle_c$$

- The complex phase fluctuation is large around μ=0 and μ<0.
- However, the region around μ=0 is less important for finite κ.

$$\frac{d\left\langle \Omega_I^2 \right\rangle_{P,\Omega_R}}{dP}$$



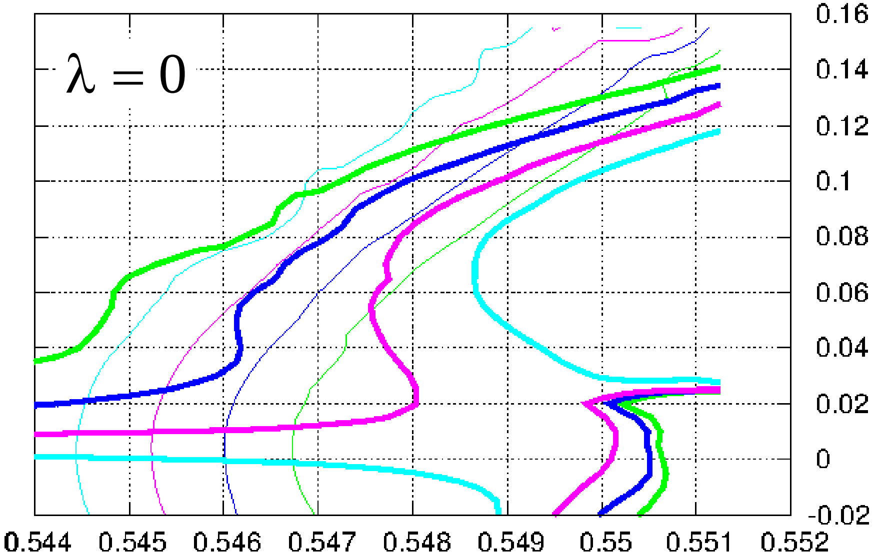
$$\frac{d\left\langle \Omega_I^2 \right\rangle_{P,\Omega_R}}{d\Omega_R}$$

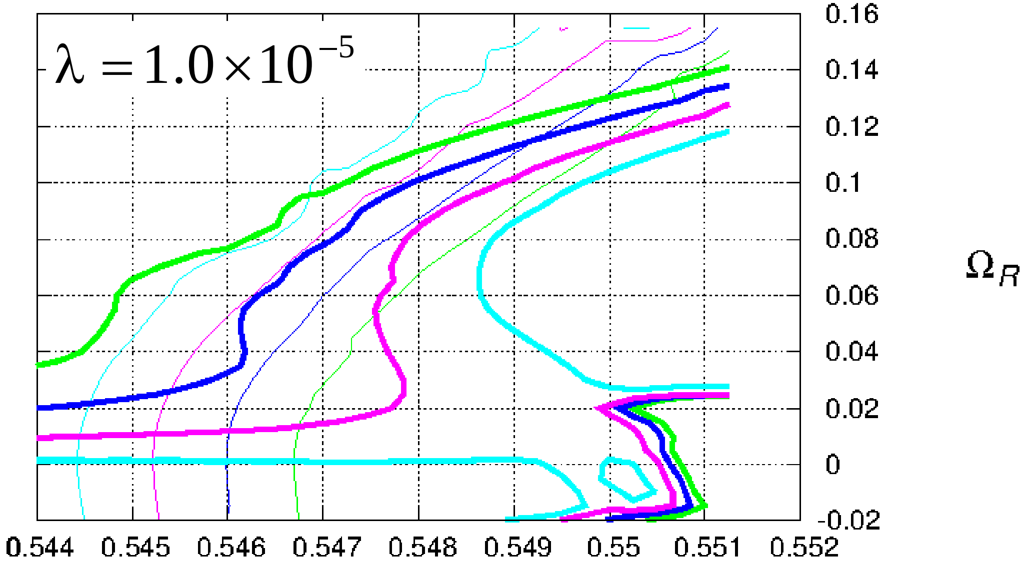


Critical surface at finite density

- Contour plot of the derivatives of $V_{\rm eff}(P,\Omega_R)$.

$$\boxed{\lambda \equiv K^{N_t}\sinh(\mu/T)}$$





- Blue line is the $dV_{\rm eff}/d\Omega_R$ around the critical point.
- Effect from the complex phase is small on the critical line, $\lambda < 2\times 10^{-5}$.

$$\frac{dV_{\rm eff}(\beta,K)}{dP} = \frac{dV_{\rm eff}(\beta_0,0)}{dP} - \left(6(\beta-\beta_0) + 288 N_{\rm f} K^4\right) N_{\rm site} + \underline{\frac{\left(3\times 2^{N_t+2} N_{\rm f} N_s^3 \lambda\right)^2}{2} \frac{\partial \langle \Omega_I^2 \rangle_c}{\partial P}}$$

$$\frac{dV_{\rm eff}(\beta,K)}{d\Omega_R} = \frac{dV_{\rm eff}(\beta_0,0)}{d\Omega_R} - 12\times 2^{N_t} N_{\rm f} N_s^3 K^{N_t} \cosh\left(\frac{\mu}{T}\right) + \underline{\frac{\left(3\times 2^{N_t+2} N_{\rm f} N_s^3 \lambda\right)^2}{2} \frac{\partial \langle \Omega_I^2 \rangle_c}{\partial \Omega_R}}$$

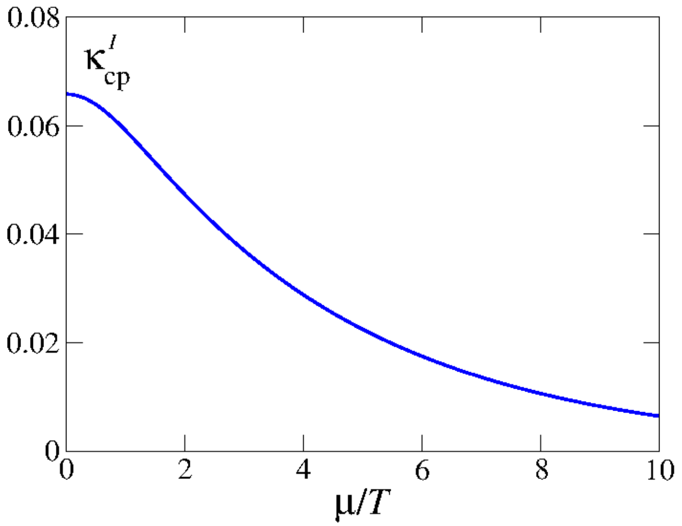
Critical line in finite density QCD

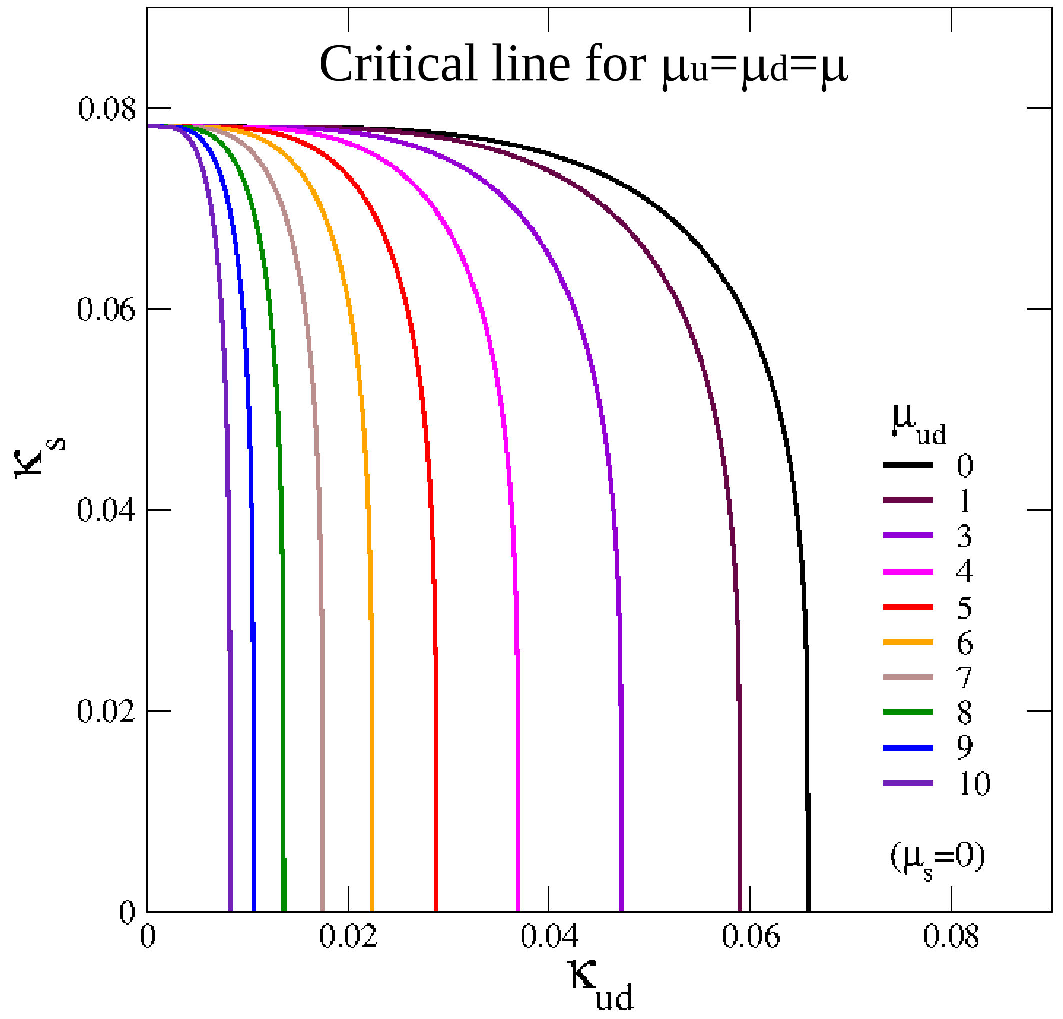
(WHOT-QCD Collab., in preparation, 11)

$$\theta = 12 \cdot 2^{N_t} i N_s^3 N_f K^{N_t} \sinh(\mu/T) \Omega_I$$

- The effect from the complex phase is very small for the determination of $K_{\rm cp}$ because $K_{\rm cp}^{N_t}(0) = K_{\rm cp}^{N_t}(\mu)\cosh(\mu/T)$ is small.
- On the critical line,

$$\theta = 12 \cdot 2^{N_t} N_s^3 N_f K_{\rm cp}^{N_t}(0) \underbrace{\tanh(\mu/T)}_{<1} \Omega_I$$





Distribution function in the light quark region

WHOT-QCD Collaboration, in preparation,
(Nakagawa et al., arXiv:1111.2116)

- We perform phase quenched simulations
- The effect of the complex phase is added by the reweighting.
- We calculate the probability distribution function.

- Goal
  - The critical point
  - The equation of state

    Pressure, Energy density, Quark number density, Quark number susceptibility, Speed of sound, etc.

Probability distribution function by phase quenched simulation

- We perform phase quenched simulations with the weight:

$$\left|\det M(m,\mu)\right|^{N_\mathrm{f}} e^{-S_g}$$

$$\boxed{\begin{aligned} W(P',F',\beta,m,\mu) &= \int DU\ \delta(P-P')\delta(F-F')\left(\det M(m,\mu)\right)^{N_\mathrm{f}} e^{-S_g} \\ &= \int DU\ \delta(P-P')\delta(F-F')\, e^{i\theta}\left|\det M(m,\mu)\right|^{N_\mathrm{f}} e^{-S_g} \\ &= \underline{\left\langle e^{i\theta}\right\rangle_{P',F'}} \times \underline{W_0(P',F',\beta,m,\mu)} \end{aligned}}$$

expectation value with fixed *P,F* histogram

P: plaquette $F(\mu) = N_\mathrm{f} \ln\left|\dfrac{\det M(\mu)}{\det M(0)}\right|$ $\theta \equiv N_\mathrm{f}\ \mathrm{Im} \ln \det M$

Distribution function of the phase quenched.

$$W_0(P',F') = \int DU\ \delta(P-P')\delta(F-F')\left|\det M\right|^{N_\mathrm{f}} e^{6N_\mathrm{site}\beta P}$$

Phase quenched simulation

$$\boxed{W(P,F,\beta,m,\mu)=\underline{\left\langle e^{i\theta}\right\rangle_{P,F}}\times W_0(P,F,\beta,m,\mu)}$$

$$\det M(K,-\mu)=\left[\det M(K,\mu)\right]^*, \qquad \left|\det M(K,\mu)\right|^2=\det M(K,\mu)\det M(K,-\mu)$$

- When $\mu_u=-\mu_d$, pion condensation occurs.
- $\left\langle e^{i\theta}\right\rangle = 0$ is suggested in the pion condensed phase by phenomenological studies. [Han-Stephanov '08, Sakai et al. '10]

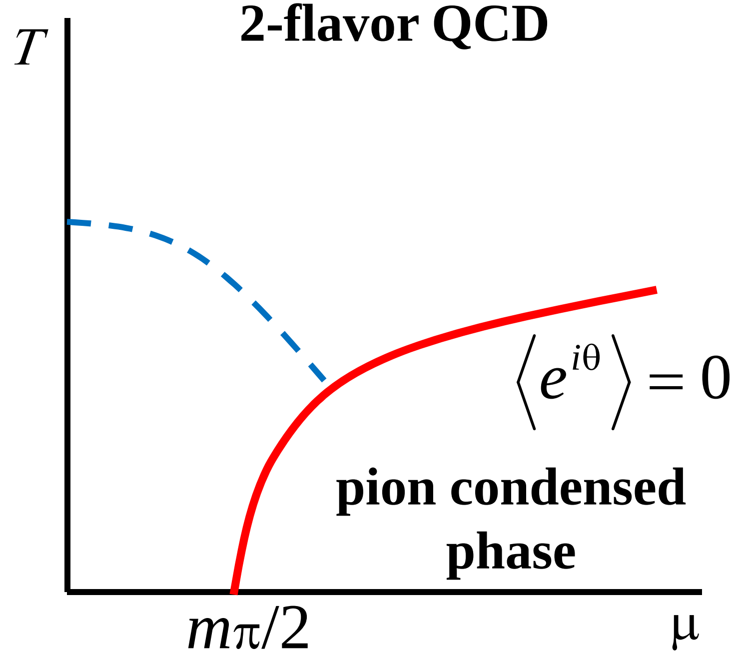
➡ No overlap between $W(\mu)$ and $W_0(\mu)$.

- Near the phase boundary,
  - large fluctuations in θ: expected.

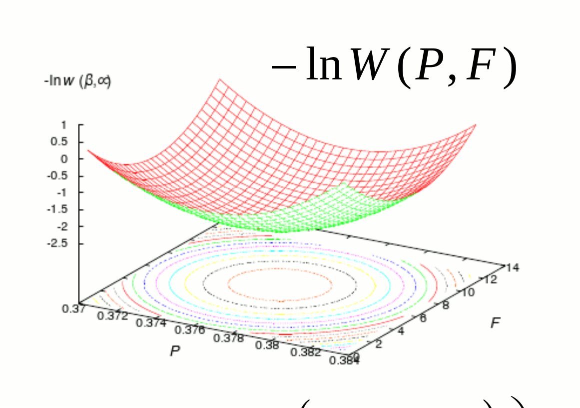
$$\left\langle e^{i\theta}\right\rangle_{P,F}\to 0 \qquad \left(\ln\left\langle e^{i\theta}\right\rangle_{P,F}\to -\infty\right)$$

  - $W(P,F)$ and $W_0(P,F)$ are completely different.

Phase structure of the phase quenched 2-flavor QCD



Peak position of $W(P,F)$



- The slopes are zero at the peak of $W(P,F)$. $\dfrac{\partial \ln W}{\partial P} = 0, \quad \dfrac{\partial \ln W}{\partial F} = 0$

$$\left(R(P,F,\mu,\mu_0) = \frac{W_0(P,F,\beta,\mu)}{W_0(P,F,\beta,\mu_0)} \right)$$

$$\frac{\partial \ln W}{\partial P}(P,F,\beta,\mu) = \frac{\partial \ln W_0}{\partial P}(P,F,\beta,\mu) + \frac{\partial \ln \left\langle e^{i\theta} \right\rangle_{P,F}}{\partial P}$$

$$= \frac{\partial \ln W_0}{\partial P}(P,F,\beta_0,\mu_0) + \underbrace{6N_{site}(\beta-\beta_0) + \frac{\partial \ln R}{\partial P}(P,F,\mu,\mu_0) + \frac{\partial \ln \left\langle e^{i\theta} \right\rangle_{P,F}}{\partial P}}_{=0}$$

$$\frac{\partial \ln W}{\partial F}(P,F,\beta,\mu) = \frac{\partial \ln W_0}{\partial F}(P,F,\beta,\mu) + \frac{\partial \ln \left\langle e^{i\theta} \right\rangle_{P,F}}{\partial F}$$

$$= \frac{\partial \ln W_0}{\partial F}(P,F,\beta_0,\mu_0) + \underbrace{\frac{\partial \ln R}{\partial F}(P,F,\mu,\mu_0) + \frac{\partial \ln \left\langle e^{i\theta} \right\rangle_{P,F}}{\partial F}}_{=0}$$

If these terms are canceled,

$$\boxed{W(P,F,\beta,\mu) \approx W_0(P,F,\beta_0,\mu_0) \times (\text{const.})}$$

- $W(\beta, \mu)$ can be computed by simulations around (β_0, μ_0).

Avoiding the sign problem

(SE, Phys.Rev.D77,014508(2008), WHOT-QCD, Phys.Rev.D82, 014508(2010))

θ: complex phase $\qquad \theta \equiv \mathrm{Im} \ln \det M$

- Sign problem: If $e^{i\theta}$ changes its sign,

$$\left\langle e^{i\theta} \right\rangle_{P,F \text{ fixed}} << \text{(statistical error)}$$

- Cumulant expansion $\qquad$ <..>$_{P,F}$: expectation values fixed F and P.

$$\left\langle e^{i\theta} \right\rangle_{P,F} = \exp\left[\underbrace{\cancel{i\langle\theta\rangle_C}}_{\to 0} - \frac{1}{2}\langle\theta^2\rangle_C - \underbrace{\cancel{\frac{i}{3!}\langle\theta^3\rangle_C}}_{\to 0} + \frac{1}{4!}\langle\theta^4\rangle_C + \cancel{\cdots} \right]$$

cumulants

$$\langle\theta\rangle_C = \langle\theta\rangle_{P,F}, \quad \langle\theta^2\rangle_C = \langle\theta^2\rangle_{P,F} - \langle\theta\rangle_{P,F}^2, \quad \langle\theta^3\rangle_C = \langle\theta^3\rangle_{P,F} - 3\langle\theta^2\rangle_{P,F}\langle\theta\rangle_{P,F} + 2\langle\theta\rangle_{P,F}^3, \quad \langle\theta^4\rangle_C = \cdots$$

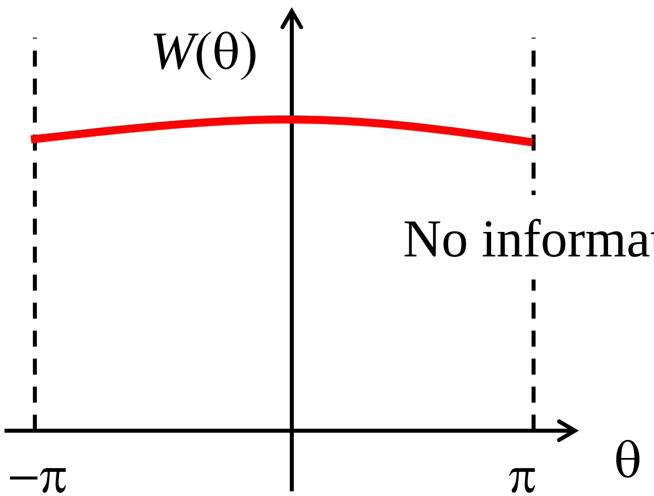
– Odd terms vanish from a symmetry under $\mu \leftrightarrow -\mu$ $(\theta \leftrightarrow -\theta)$
  Source of the complex phase

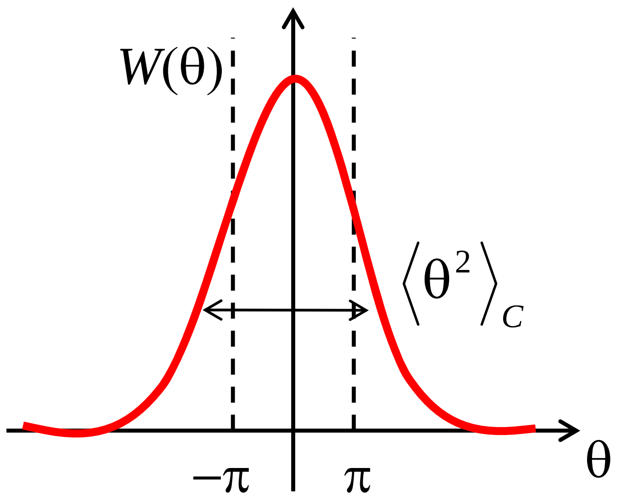
If the cumulant expansion converges, No sign problem.

Complex phase distribution

- We should not define the complex phase in the range from $-\pi$ to π.
- When the distribution of θ is perfectly Gaussian, the average of the complex phase is give by the second order (variance),

$$\left\langle e^{i\theta} \right\rangle_{P,F} = \exp\left[-\frac{1}{2} \left\langle \theta^2 \right\rangle_C \right]$$



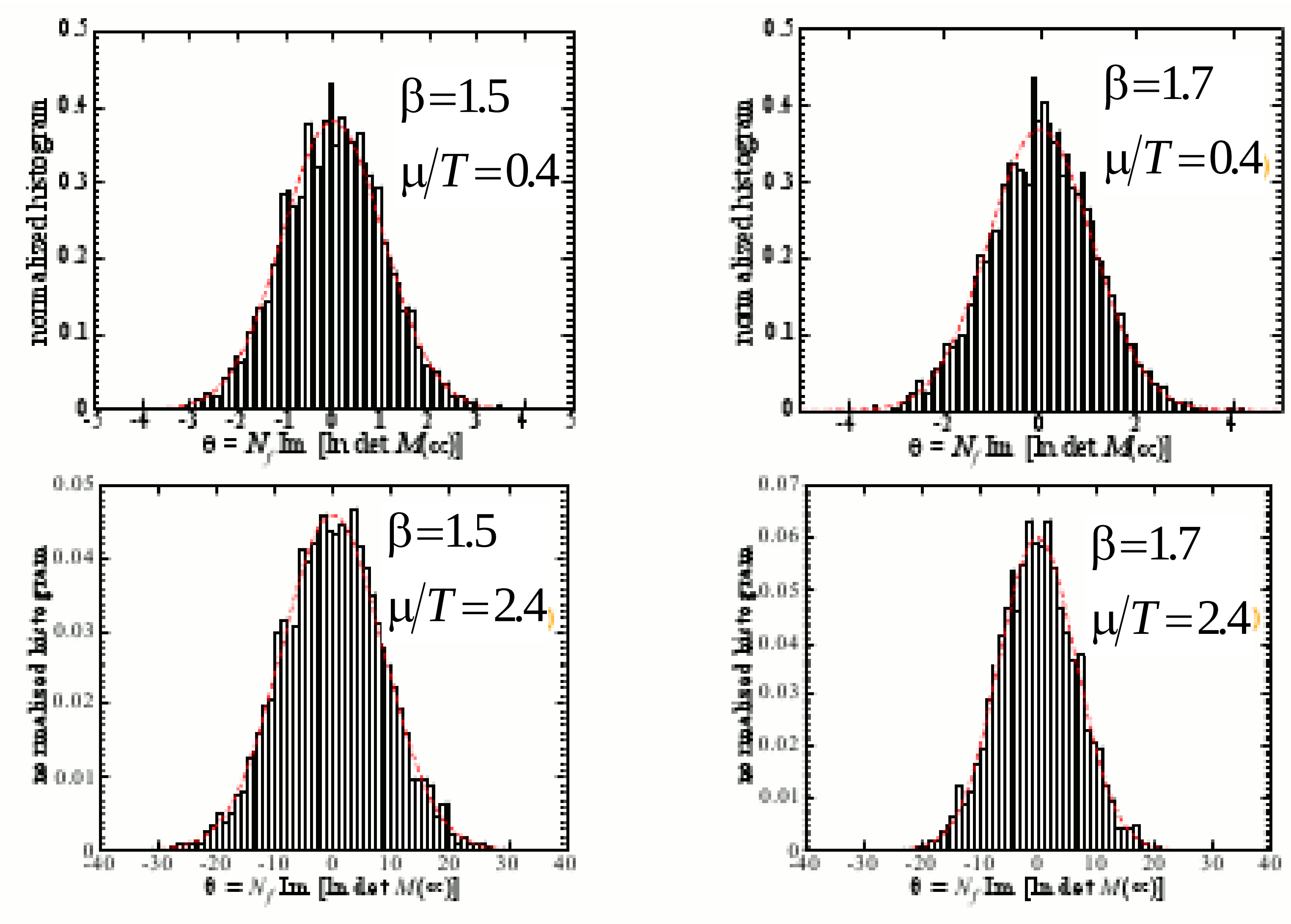


- Gaussian distribution $\rightarrow$ The cumulant expansion is good.
- We define the phase

$$\theta(\mu) = N_{\mathrm{f}}\ \mathrm{Im}\left(\ln \frac{\det M(\mu)}{\det M(0)} \right) = N_{\mathrm{f}} \int_0^{\mu/T} \mathrm{Im}\left[\frac{\partial \ln \det M}{\partial(\mu/T)} \right]_{\bar{\mu}} d\left(\frac{\bar{\mu}}{T} \right)$$

  – The range of θ is from $-\infty$ to ∞.

Distribution of the complex phase



$8^3 \times 4$ lattice

$m_\pi / m_\rho = 0.8$

2-flavor QCD
Iwasaki gauge
+ clover Wilson
quark action

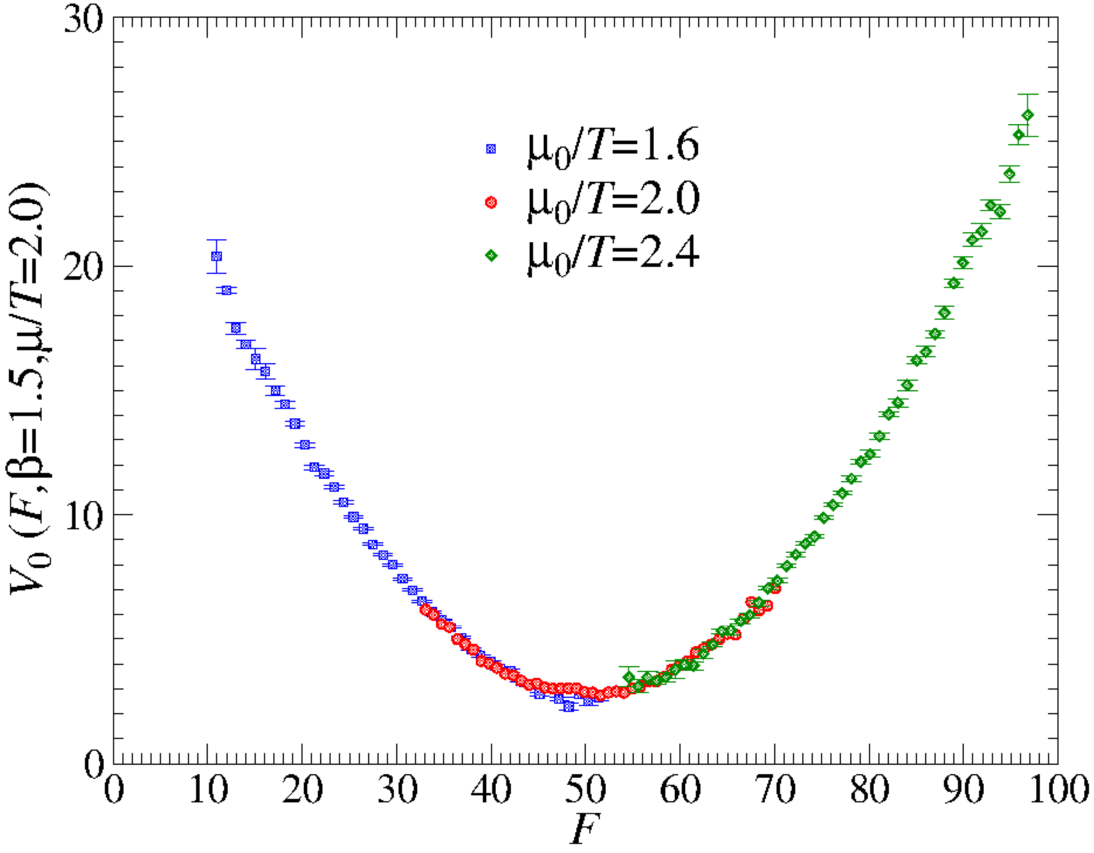
Random noise method is used.

- Well approximated by a Gaussian function.
- Convergence of the cumulant expansion: good.

Distribution in a wide range

Reweighting method W_0: distribution function in phase quenched simulations.

$$R(P,F)=\frac{W_0(P,F,\mu)}{W_0(P,F,\mu_0)}=\left\langle \left|\frac{\det M(\mu)}{\det M(\mu_0)}\right|^{N_f}\right\rangle_{P,F\ \text{fixed}}$$



- Perform phase quenched simulations at several points.
  - Range of F is different.
- Change μ by reweighting method.
- Combine the data.

Distribution in a wide range: obtained.

- The error of R is small because F is fixed.

$$-\ln W_0(P,F,\mu_0)-\ln R(P,F,\mu,\mu_0)=-\ln W_0(P,F,\mu)$$

Summary

- We studied the quark mass and chemical potential dependence of the nature of QCD phase transition.
- The shape of the probability distribution function changes as a function of the quark mass and chemical potential.
- To avoid the sign problem, the method based on the cumulant expansion of θ is useful.
- To find the critical point at finite density, further studies in light quark region are important applying this method.