Strong coupling study of Aoki phase in Staggered-Wilson fermion

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PoS (LAT2011), 108 (2011) [arXiv:1110.1231].

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- **Introduction**
  - Aoki phase in Wilson fermion
- **Aoki phase in Wilson fermion**
  - Strong coupling study
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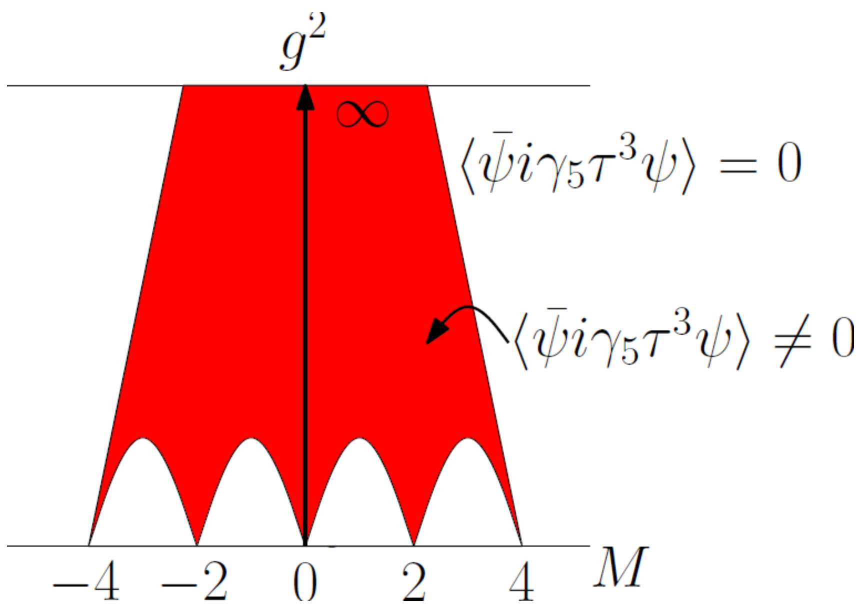
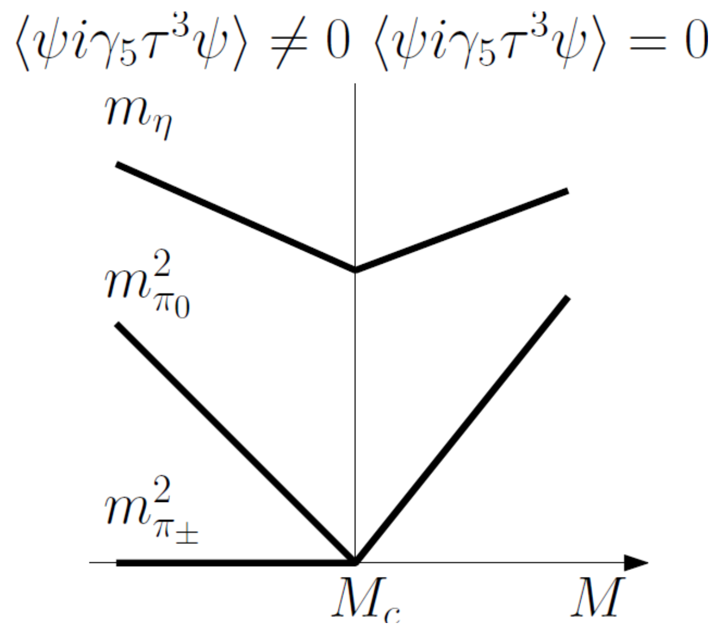
  - Strong coupling study
- **Summary & Future works**

Phase structure in Wilson Fermion

- Aoki Phase
  - Parity-flavor symmetry is spontaneously broken in some quark mass region. Aoki (1984)
- Quark mass tuning ➔ Chiral limit





Analysis for Aoki Phase

- Strong coupling analysis, Gross-Neveu Model
  - indicate $\langle\bar{\psi}i\gamma_5\tau^3\psi\rangle \neq 0$
    - Gross-Neveu Model : Aoki (1984)
    - Strong coupling (expansion) : Aoki : (1986); (1989)
- Numerical calculation
  - Vacuum :$\langle\bar{\psi}i\gamma_5\tau^3\psi\rangle \neq 0 \text{ for } M < M_c$ Aoki (1987); (1989)
  - High T : $\langle\bar{\psi}i\gamma_5\tau^3\psi\rangle = 0 \text{ for all } M$ Aoki, Gocksch (1990), Aoki, Ukawa, Umemura (1996), Aoki, Kaneda, Ukawa (1997), Aoki (1996)
- Chiral perturbation theory
  - Sharpe, Singleton (1998), Sharpe (2009)

Staggered-Wilson Fermion

- Flavored Mass : Generalized Wilson terms
  - Adams : $N_f = 2$
  - Hoelbling : $N_f = 1$

Adams (2010, 2011), Hoelbling (2011).
Creutz, Kimura, Misumi (2010)

- Aoki phase in Staggered-Wilson fermion ?

Purpose

- Applicability of Staggered-Wilson (Overlap) fermion
  - Chiral limit

- Previous study
  - Lattice Gross-Neveu model with flavored mass [Kimura's talk]
    Creutz, Kimura, Misumi (2011)

Purpose

- We study Aoki phase of Staggered-Wilson fermion in strong coupling lattice QCD.

- Method
  - Hopping parameter expansion
  - Effective potential analysis

- Staggered-Wilson Fermion
  - Adams type , Hoelbling type

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- **Introduction**
  - Aoki phase in Wilson fermion
- **Aoki phase in Wilson fermion**
  - Strong coupling study (Condensate, Pion mass)
    - Hopping Parameter Expansion
    - Effective Potential Analysis
- **Aoki phase in Staggered-Wilson fermion**
  - Strong coupling study
- **Summary & Future works**

Hopping Parameter Expansion (HPE)

- Strong coupling study $(N_f = 1)$ Aoki (1984) $S_G \propto \cancel{1/g^2}$
  - Hopping Parameter Expansion (HPE)

$$S_F = M \sum_x \bar{\psi}_x \psi_x - \frac{1}{2} \sum_{x,\mu} \left[\bar{\psi}_x (r - \gamma_\mu) U_{\mu,x} \psi_{x+\hat{\mu}} - \bar{\psi}_x (r + \gamma_\mu) U^\dagger_{\mu,x-\hat{\mu}} \psi_{x-\hat{\mu}} \right]$$

$$M = m_0 + 4r$$

$$\psi \to \sqrt{2K}\psi$$

$$K = 1/\left[2(m_0 + 4r)\right]$$

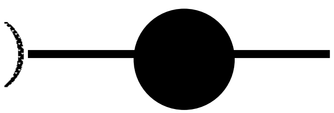
$$S_F = \sum_x \bar{\psi}_x \psi_x - K \sum_{x,\mu} \left[\bar{\psi}_x (r - \gamma_\mu) U_{\mu,x} \psi_{x+\hat{\mu}} + \bar{\psi}_x (r + \gamma_\mu) U^\dagger_{\mu,x-\hat{\mu}} \psi_{x-\hat{\mu}} \right]$$

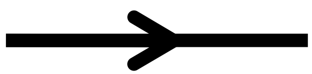
Hopping Parameter

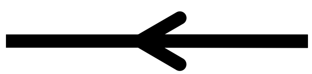
$$r = 1$$

Hopping Parameter Expansion (HPE)

- Feynman rule a : color, α : spinor

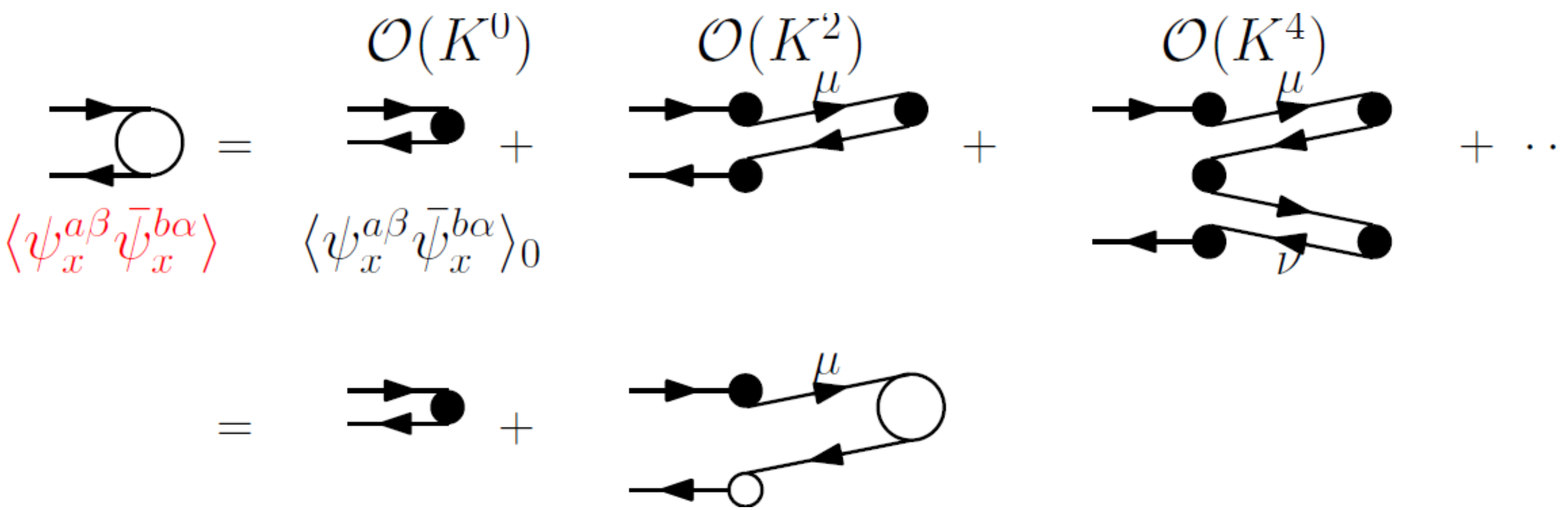
(x, a, α)  (y, b, β) $\langle \psi^{a}_{x,\alpha} \bar{\psi}^{b}_{y,\beta} \rangle_0 = -\delta_{xy}\delta_{\alpha\beta}\delta^{ab}$

(x, a, α)  $(x+\hat{\mu}, b, \beta) = -K(1-\gamma_\mu)_{\alpha\beta}U^{ab}_{\mu,x}$

(x, b, β)  $(x+\hat{\mu}, a, \alpha) = -K(1+\gamma_\mu)_{\alpha\beta}U^{\dagger ab}_{\mu,x}$

1 pt. Function

- Diagrams (1 pt. function) ➔ $\langle \bar{\psi}_x^{\alpha} \psi_x^{\beta} \rangle$
  - Expansion for hopping parameter (K)
  - Link variables appear as a pair of $\left(U, U^{\dagger}\right)$.
  - Summation for K. (K^2 , K^4 ,…)



1 pt. Function

- Self-Consistent Equation for $\langle \bar{\psi}_x^{\alpha} \psi_x^{\beta} \rangle$

$$
\begin{aligned}
\langle \psi_x^{a\beta} \bar{\psi}_x^{b\alpha} \rangle &= \langle \psi_x^{a\beta} \bar{\psi}_x^{b\alpha} \rangle_0 \\
&+ K^2 \sum_{\mu} \langle \psi_x^{\beta} \bar{\psi}_x \rangle_0 (r \mp \gamma_\mu) U_{\mu,x} \langle \psi_{x+\hat{\mu}} \bar{\psi}_{x+\hat{\mu}} \rangle_0 (r \pm \gamma_\mu) U_{\mu,x}^{\dagger} \langle \psi_x \bar{\psi}_x^{\alpha} \rangle_0 \\
&+ \cdots
\end{aligned}
$$

$$
\begin{aligned}
\langle \psi_x^{a\beta} \bar{\psi}_x^{b\alpha} \rangle &= \langle \psi_x^{a\beta} \bar{\psi}_x^{b\alpha} \rangle_0 \\
&+ K^2 \sum_{\mu} \langle \psi_x^{\beta} \bar{\psi}_x \rangle_0 (r \mp \gamma_\mu) U_{\mu,x} \langle \psi_{x+\hat{\mu}} \bar{\psi}_{x+\hat{\mu}} \rangle (r \pm \gamma_\mu) U_{\mu,x}^{\dagger} \langle \psi_x \bar{\psi}_x^{\alpha} \rangle
\end{aligned}
$$

$U^{\dagger} U = 1$

1 pt. Function

- Mean-field treatment
  - Scalar $\langle \bar{\psi}\psi \rangle$
  - Pesudo-scalar $\langle \bar{\psi} i\gamma_5 \psi \rangle$

- Two solution
  - Parity conserved

$$\frac{1}{4N_c}\langle \bar{\psi}\psi \rangle = \frac{1}{2K}, \quad \frac{1}{4N_c}\langle \bar{\psi} i\gamma_5 \psi \rangle = 0$$

  - Parity broken

$$\frac{1}{4N_c}\langle \bar{\psi}\psi \rangle = \frac{1}{8K}, \quad \frac{1}{4N_c}\langle \bar{\psi} i\gamma_5 \psi \rangle = \pm\frac{\sqrt{16K^2-1}}{8K}$$

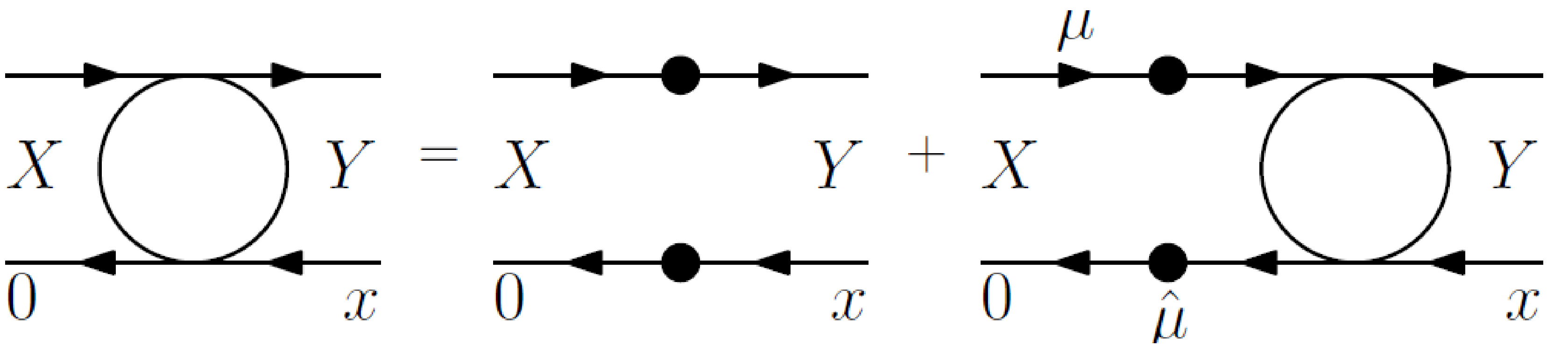
⇒ For $|K| > \frac{1}{4}$, Parity-broken?

2 pt. Function

- Diagrams (2 pt. function) ➔ m_π

$$S_{XY}(0,x) \equiv \langle M_X(0) M^Y(x) \rangle = \langle \bar{\psi}_0^a \Gamma_X \psi_0^a \bar{\psi}_x^b \Gamma^Y \psi_x^b \rangle$$

$$\Gamma_X = \mathbf{1}, \gamma_5, \gamma_\mu, \gamma_5\gamma_\mu, \gamma_\mu\gamma_\nu$$



- Self-consistent equation

$$S_{XY}(0,x) = -\delta_{0x}\delta_{XY}4N_c + K^2 \sum_{Z,\mu} \left[T^{\mu}_{XZ} S_{ZY}(\mu,x) + T^{-\mu}_{XZ} S_{ZY}(-\mu,x) \right]$$

$$T^{\pm\mu}_{AC} = \frac{1}{4} \mathrm{tr} \Gamma_A (1 \mp \gamma_\mu) \Gamma^C (1 \pm \gamma_\mu)$$

2 pt. Function

- Pion mass
  - F.T. $S_{XY}(0,x) \to S_{XY}(p)$
  - Insert $p = (p_0, \mathbf{p}) = (im_\pi, \vec{0})$

$$\cosh m_\pi = 1 + \frac{1 - 16K^2}{8K^2(1 - 6K^2)}(1 - 4K^2)$$

For $|K| > \frac{1}{4}$, $m_\pi^2 < 0$?

Tachyon : instability of ground state

➔Another lower energy state

Effective Potential (Condensate)

- Effective Potential for Meson (large N_c)
  ➔We can identify the Parity-broken phase

$$Z = \int \mathcal{D}[\psi, \bar{\psi}, U] \exp[S_F]$$
$$\sim \exp\left[S_{\text{eff}}(M^{\alpha\beta} = \bar{\psi}^\alpha \psi^\beta)\right]$$

	$\frac{1}{4N_c}\langle\bar{\psi}\psi\rangle$	$\frac{1}{4N_c}\langle\bar{\psi}i\gamma_5\psi\rangle$
$M^2 > 4\ (\lvert K \rvert < 1/4)$	$\frac{1}{M}$	0
$M^2 < 4\ (\lvert K \rvert > 1/4)$	$\frac{3M}{16-M^2}$	$\pm 2\frac{\sqrt{3(4-M^2)}}{16-M^2}$

$$\frac{\partial V_{\text{eff}}}{\partial \sigma} = \frac{\partial V_{\text{eff}}}{\partial \pi} = 0$$

For $\lvert K \rvert > \frac{1}{4}$, $\langle\bar{\psi}i\gamma_5\psi\rangle \neq 0$

Effective Potential (Pion Mass)

- Critical Mass (M_c) corresponds to the chiral limit.

$$m_\pi \leftarrow \frac{\partial^2 S_{\text{eff}}(M)}{\partial M_x \partial M_y}$$

$$\cosh m_\pi = \begin{cases} 1 + \dfrac{(M^2-4)(M^2-1)}{2M^2-3} & \text{for } M^2 > 4 \\ 1 + \dfrac{2(4-M^2)(16-M^2)(8+M^2)}{15M^4-64M^2+256} & \text{for } M^2 < 4 \end{cases}$$

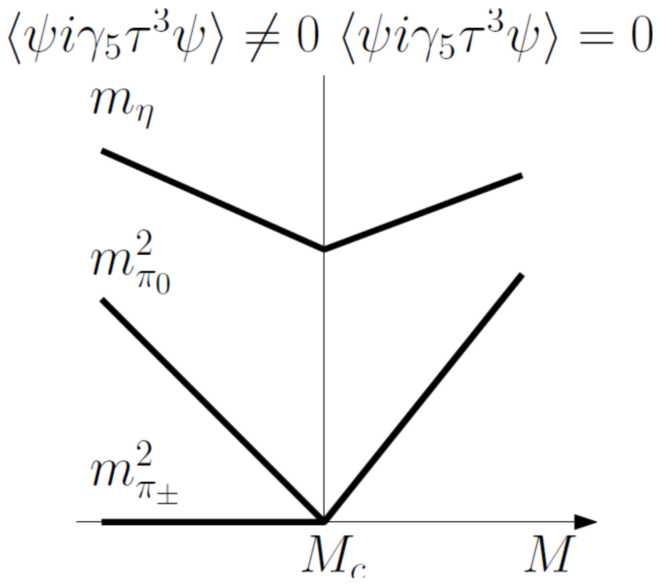
2 Flavor Case

- Possibility
  - (1)Parity symmetry breaking

    $$\langle \bar{\psi} i \gamma_5 \psi \rangle \neq 0, \langle \bar{\psi} i \gamma_5 \tau^3 \psi \rangle = 0$$

  - (2)Parity-flavor symmetry breaking

    $$\langle \bar{\psi} i \gamma_5 \psi \rangle = 0, \langle \bar{\psi} i \gamma_5 \tau^3 \psi \rangle \neq 0$$



- Parity-flavor symmetry breaking
  - Vafa-Witten's theorem ➔ $\langle \bar{\psi} i \gamma_5 \psi \rangle = 0$
  - $\pi_\pm$: NG boson for spontaneous breaking of parity-flavor symmetry

Aoki Phase in Wilson Fermion

- Hopping Parameter Expansion

	$\frac{1}{4N_c}\langle\bar{\psi}\psi\rangle$	$\frac{1}{4N_c}\langle\bar{\psi}i\gamma_5\psi\rangle$	m_π^2
Parity Conserved	$\frac{1}{2K}$	0	$+$
Parity Broken	$\frac{1}{8K}$	$\pm\frac{\sqrt{16K^2-1}}{8K}$	$-$

- Effective Potential Analysis

	$\frac{1}{4N_c}\langle\bar{\psi}\psi\rangle$	$\frac{1}{4N_c}\langle\bar{\psi}i\gamma_5\psi\rangle$	m_π^2
$M^2 > 4 \ (\mid K \mid < 1/4)$	$\frac{1}{M}$	0	$+$
$M^2 < 4 \ (\mid K \mid > 1/4)$	$\frac{3M}{16-M^2}$	$\pm 2\frac{\sqrt{3(4-M^2)}}{16-M^2}$	$+$

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  - Strong coupling study
    - Hoelbling type
    - Adams type
  - Discussion
- **Summary & Future works**

Staggered-Wilson Fermion

- Lattice action (Hoelbling type) C. Hoelbling (2011)

$$S_{\mathrm{F}} = \sum_{xy} \bar{\chi}_x \left[D_s\right]_{xy} \chi_y$$

$$D_s = D_{st} + r(2 + M_s) + m_0$$

- Flavored mass : 2 hopping

$$M_s = \sum_{\mu \neq \nu} \frac{1}{2\sqrt{3}} \epsilon_{\mu\nu} \eta_\mu \eta_\nu \left(C_\mu C_\nu + C_\mu C_\nu\right)$$

$$C_\mu = (V_\mu + V_\mu^\dagger)/2$$

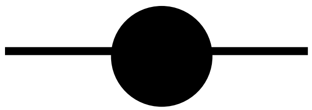
$$(V_\mu)_{xy} = U_{\mu,x} \delta_{y,x+\hat{\mu}} \ ,$$

$$\epsilon_{\mu\nu} = (-1)^{x_\mu + x_\nu} \ ,$$

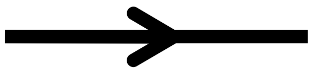
$$\eta_\mu = (-1)^{x_1 + \cdots + x_{\mu-1}}$$

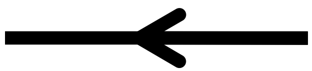
Hopping Parameter Expansion (HPE)

- Feynman rule $K = 1/\left[2(m_0 + 2r)\right]$ $\quad r = 2\sqrt{2}$

(x, a)  (y, b) $\quad \langle \chi_x^a \bar{\psi}_y^b \rangle_0 = -\delta_{xy}\delta^{ab}$

1 hopping

(x, a)  $(x + \hat{\mu}, b)$ $\quad = K U_{\mu,x}^{ab}$

(x, b)  $(x + \hat{\mu}, a)$ $\quad = -K U_{\mu,x}^{\dagger ab}$

2 hopping

(x, a)  $(x + \alpha\hat{\mu} + \beta\hat{\nu}, b)$ $\quad = \dfrac{i}{4\sqrt{3}} K r \eta_{\mu\nu} U_{2,\mu\nu}^{ab}$

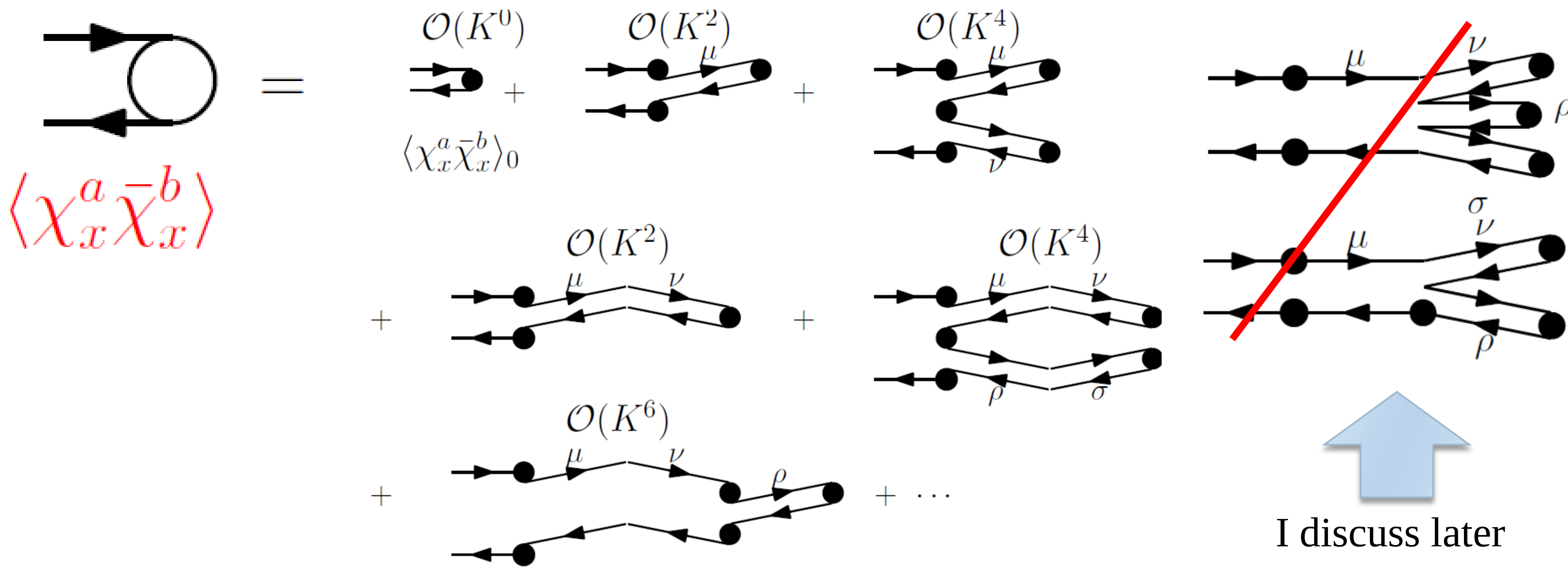
(x, b)  $(x + \alpha\hat{\mu} + \beta\hat{\nu}, a)$ $\quad = \dfrac{-i}{4\sqrt{3}} K r \eta_{\mu\nu} U_{2,\mu\nu}^{\dagger ab}$

$\alpha, \beta = \pm$

$$U_{2,\mu\nu} \equiv \left(U_{\mu,x} U_{\nu,x+\hat{\mu}}, U_{\mu,x} U_{\nu,x+\hat{\mu}-\hat{\nu}}^{\dagger}, U_{\mu,x-\hat{\mu}}^{\dagger} U_{\nu,x-\hat{\mu}} \right)$$

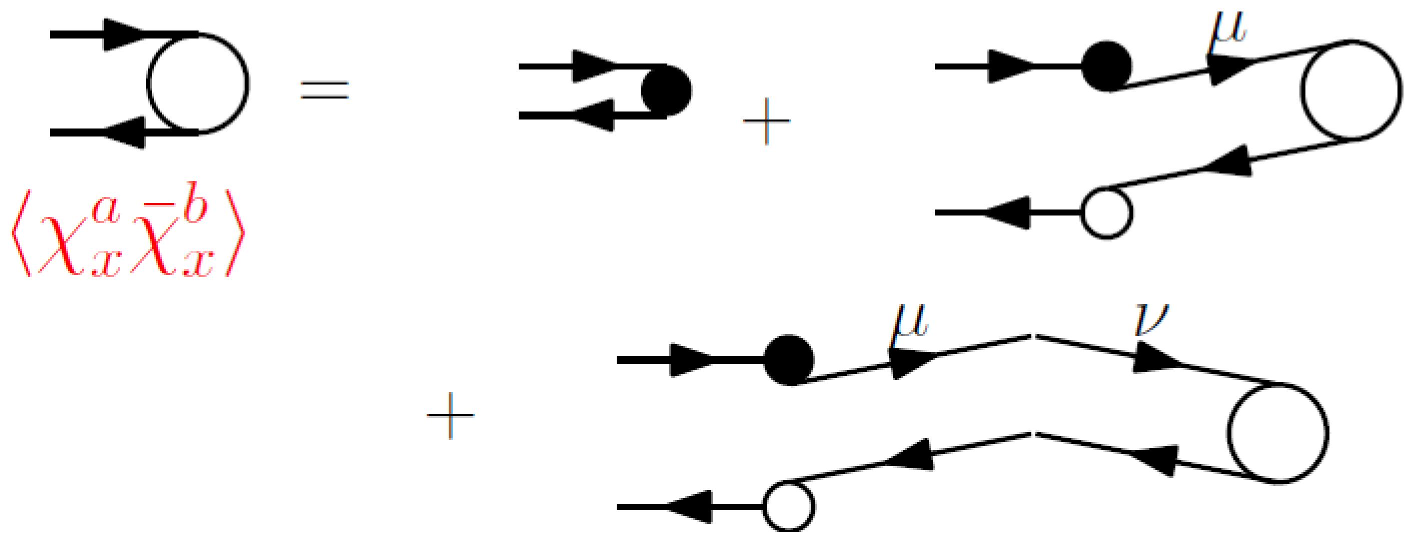
1 pt. Function

- Diagrams (1 pt. function) ➔ $\langle \bar{\chi}_x \chi_x \rangle$
  - Expansion for K
  - Link variables appear as $\left(U, U^\dagger\right), \left(U_{2,\mu\nu}, U^\dagger_{2,\mu\nu}\right)$.
  - Summation for K. (K^2 , K^4 ,…)



1 pt. Function

- Diagrams (1 pt. function) ➔ $\langle \bar{\chi}_x \chi_x \rangle$
  - Expansion for K
  - Link variables appear as $\left(U, U^\dagger\right), \left(U_{2,\mu\nu}, U^\dagger_{2,\mu\nu}\right)$.
  - Summation for K. (K^2 , K^4 ,…)



1 pt. Function

- Mean-field treatment
  - Scalar $\langle \bar{\chi}\chi \rangle$
  - Pesudo-scalar $\langle \bar{\chi} i\epsilon_x \chi \rangle$

$$\epsilon_x \equiv (-1)^{x_1+\cdots x_4} \sim \gamma_5 \otimes \xi_5$$

- Two solution
  - Parity conserved

$$\frac{1}{N_c}\langle \bar{\chi}\chi \rangle = \frac{1}{2K}, \ \frac{1}{N_c}\langle \bar{\chi} i\epsilon_x \chi \rangle = 0$$

  - Parity broken

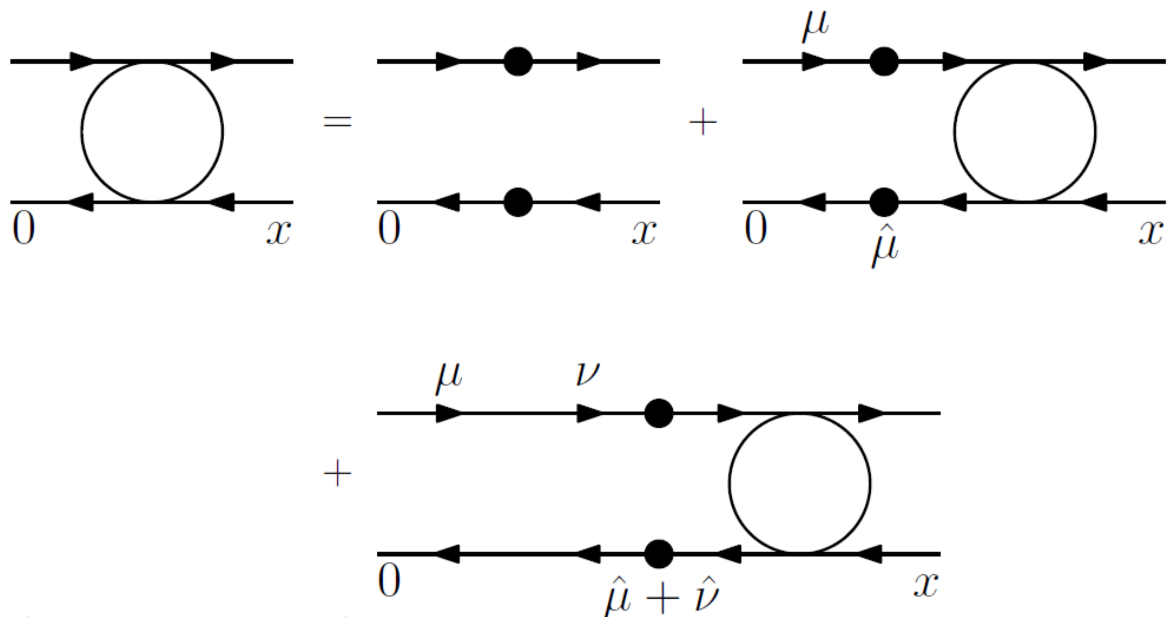
$$\frac{1}{N_c}\langle \bar{\chi}\chi \rangle = \frac{1}{8K}, \ \frac{1}{N_c}\langle \bar{\chi} i\epsilon_x \chi \rangle = \pm\frac{\sqrt{16K^2-1}}{8K}$$

➡ For $|K| > \frac{1}{4}$, Parity-broken?

2 pt. Function

- Diagrams (2 pt. function) ➔ m_π

$$S(0,x) \equiv \langle M_0 M_x \rangle = \langle \bar{\chi}_0^a \chi_0^a \bar{\chi}_x^b \chi_x^b \rangle$$



- Self-consistent equation

$$S(0,x) = -\delta_{0x} N_c + K^2 \sum_{\pm\mu} S(\mu, x) + \left(\frac{i}{4\sqrt{3}} K r\right)^2 \sum_{\pm\mu \neq \pm\nu} S(\mu+\nu, x)$$

2 pt. Function

- Pion mass
  - F.T. $S(0,x) \to S(p)$
  - Insert $p = (p_0, \mathbf{p}) = (im_\pi + \pi, \vec{\pi})$
    - Another method
      - Calculate $\langle \bar{\chi}_0^a i\epsilon_0 \chi_0^a \bar{\chi}_x^b i\epsilon_x \chi_x^b \rangle$, Insert $p = (im_\pi, 0)$

$$\cosh m_\pi = 1 + \frac{1 - 16K^2}{6K^2}$$

For $| K | > \frac{1}{4}$, $m_\pi^2 < 0$?

This result implies Aoki phase

Effective Potential (Condensate)

- Effective Potential for Meson (large N_c)
  ➔We can identify the Parity-broken phase

$$Z = \int \mathcal{D}[\chi, \bar{\chi}, U] \exp[S_F]$$
$$\sim \exp[S_{\text{eff}}(M = \bar{\chi}\chi)]$$

	$\frac{1}{N_c}\langle\bar{\chi}\chi\rangle$	$\frac{1}{N_c}\langle\bar{\chi}i\epsilon_x\chi\rangle$
$M^2 > 4$ $(\mid K \mid < 1/4)$	$\frac{1}{M}$	0
$M^2 < 4$ $(\mid K \mid > 1/4)$	$\frac{M}{8-M^2}$	$\pm\frac{\sqrt{2(4-M^2)}}{8-M^2}$

$$\frac{\partial V_{\text{eff}}}{\partial \sigma} = \frac{\partial V_{\text{eff}}}{\partial \pi} = 0$$

➔ For $\mid K \mid > \frac{1}{4}$, $\langle\bar{\chi}i\epsilon_x\chi\rangle \neq 0$

Effective Potential (Pion Mass)

- Critical Mass (M_c) corresponds to the chiral limit.

$$m_\pi \leftarrow \frac{\partial^2 S_{\mathrm{eff}}(M)}{\partial M_x \partial M_y}$$

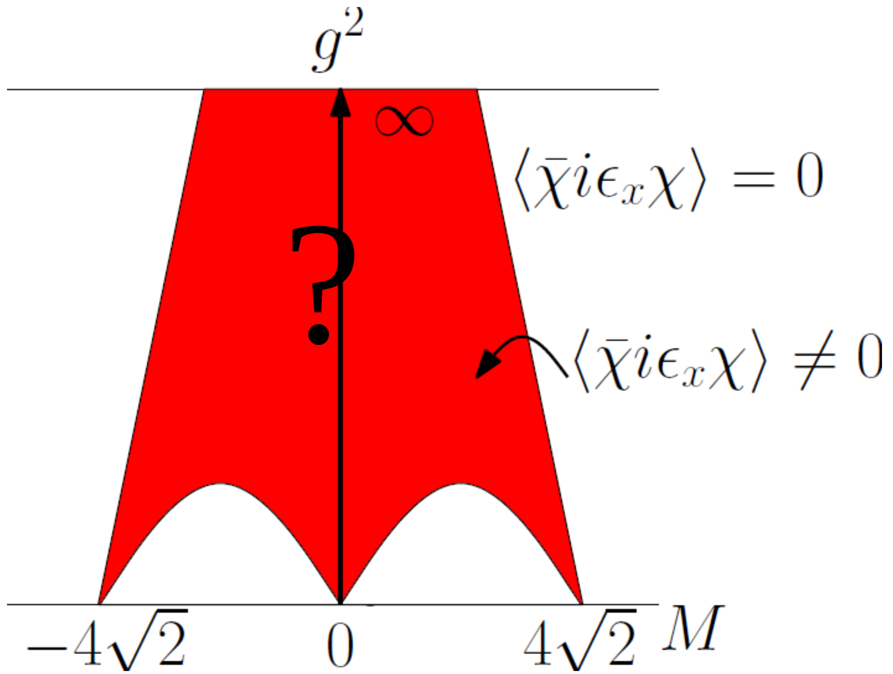
$$\cosh m_\pi = 1 + \frac{(2M^2 - 8)}{3} \quad \text{for } M^2 > 4$$

Aoki Phase in Staggered-Wilson Fermion

- Hoelbling type

$$\langle \bar{\chi} i \epsilon_x \chi \rangle \neq 0 \text{ for } |M| < M_c$$
$$m_\pi^2 = 0 \text{ for } |M| = M_c$$

- We can perform the lattice simulation with staggered -Wilson fermions by tuning mass parameter.



2 Flavor (Taste) Case

- Possibility
  - (1)Parity symmetry breaking

$$\langle \bar{\chi} i \epsilon_x \chi \rangle \neq 0, \langle \bar{\chi} i \epsilon_x \tau^3 \chi \rangle = 0$$

  - (2)Parity-flavor symmetry breaking

$$\langle \bar{\chi} i \epsilon_x \chi \rangle = 0, \langle \bar{\chi} i \epsilon_x \tau^3 \chi \rangle \neq 0$$

- Parity-flavor symmetry breaking
  - Vafa-Witten's theorem ➔ $\langle \bar{\chi} i \epsilon_x \chi \rangle = 0$

Staggered-Wilson Fermion

- Lattice action (Adams type) D. H. Adams (2011)

$$S_{\mathsf{F}} = \sum_{xy} \bar{\chi}_x \, [D_A]_{xy} \, \chi_y$$

$$D_A = D_{st} + r(1 + M_1) + m_0$$

- Flavored mass : 4 hopping

$$M_A = \epsilon \sum_{\text{sym.}} \eta_1 \eta_2 \eta_3 \eta_4 C_1 C_2 C_3 C_4$$

$$C_\mu = (V_\mu + V_\mu^\dagger)/2$$

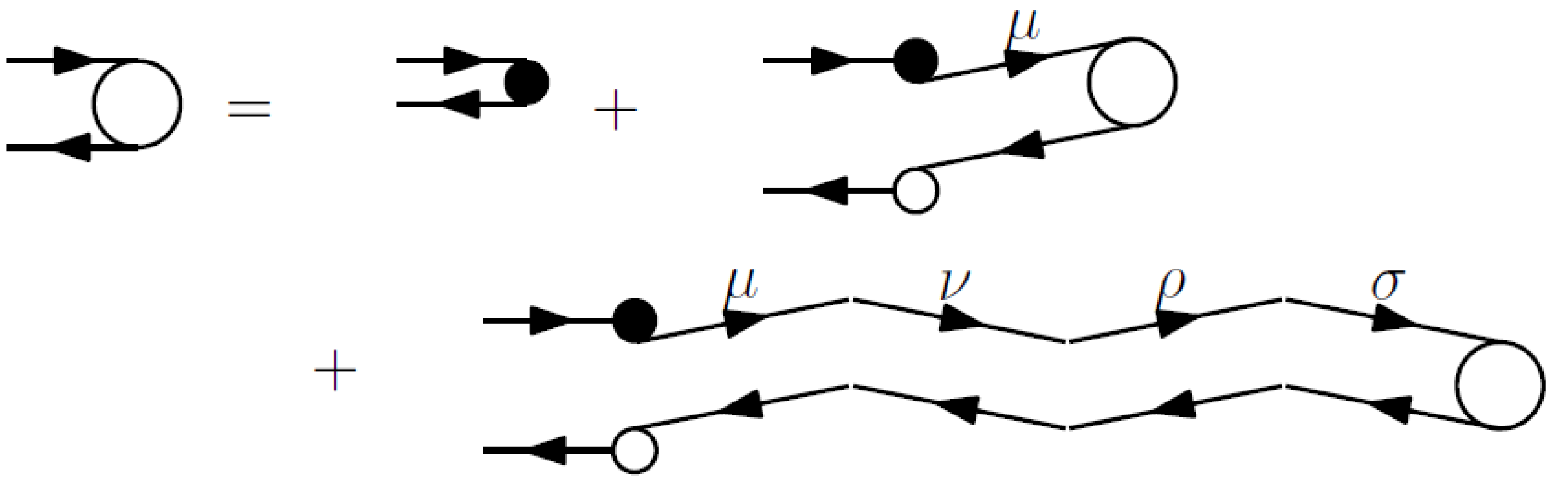
$$(V_\mu)_{xy} = U_{\mu,x} \delta_{y,x+\hat{\mu}} \ ,$$

$$\epsilon_\mu = (-1)^{x_1+x_2+x_3+x_4} \ ,$$

$$\eta_\mu = (-1)^{x_1+\cdots+x_{\mu-1}}$$

Analysis in Adams type

- Similar way as Hoelbling type
  - e.g. HPE (1 pt. function)



$$\langle \bar{\chi} i \epsilon_x \chi \rangle \neq 0 \text{ for } |M| < M_c$$

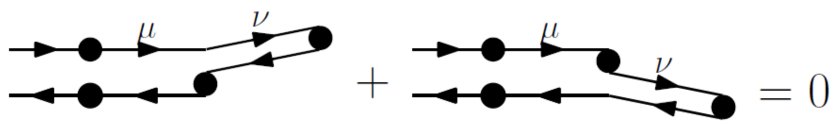
$$m_\pi^2 = 0 \text{ for } |M| = M_c$$

Discussion

- This HPE analysis is exact up to $O(K^4)$
- Other higher order terms in HPE

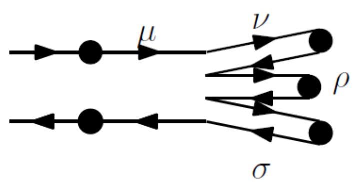
$\mathcal{O}(K^3)$

$(2 \text{ link}, 1 \text{ link}) = (1, 2)$

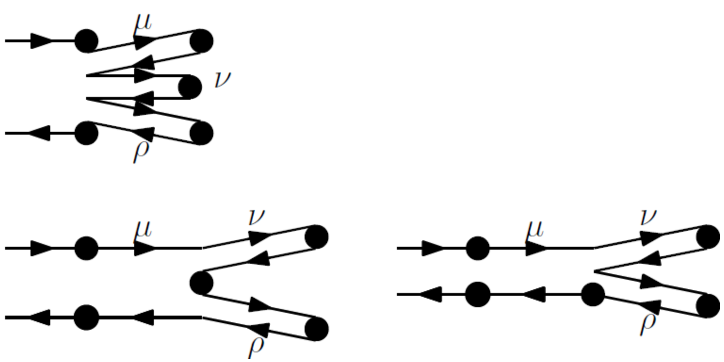


$\mathcal{O}(K^4)$

$(2 \text{ link}, 1 \text{ link}) = (4, 0)$



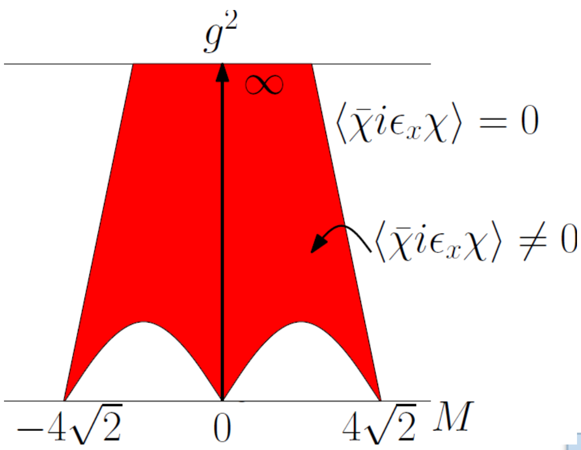
$(2 \text{ link}, 1 \text{ link}) = (2, 2)$



- Is Summation possible ?

Summary & Future Works

- We have studied Aoki phase in Staggerd-Wilson fermion in strong coupling limit.



$$\langle \bar{\chi} i \epsilon_x \chi \rangle \neq 0 \text{ for } |M| < M_c$$
$$m_\pi^2 = 0 \text{ for } |M| = M_c$$

- We can perform the lattice simulation with staggered-Wilson fermions by tuning mass parameter.
- Future Works
  - Evaluation of the summation for other higher diagrams

Back Up

2 pt. Function

- Self-consistent equation for $S_{XY}(0,x)$

$$
\begin{aligned}
S_{XY}(0,x) = &-\delta_{0x} N_c \delta_{XY} \\
&+ K^2 \langle \bar{\psi}_0^a \Gamma_X \psi_0^a \bar{\psi}_x^c \Big[(1-\gamma_\mu) U_{\mu,0}^{cd} \psi_{\hat{\mu}} \bar{\psi}_{\hat{\mu}} (1+\gamma_\mu) (U_{\mu,0}^\dagger)^{ef} \\
&+ (1+\gamma_\mu)(U_{\mu,-\hat{\mu}}^\dagger)^{cd} \psi_{-\hat{\mu}}^d \bar{\psi}_{-\hat{\mu}}^e (1-\gamma_\mu) U_{\mu,-\hat{\mu}}^{ef} \Big] \psi_0^f \bar{\psi}_x^b \Gamma^Y \chi_x^b \rangle
\end{aligned}
$$

Aoki Phase in Staggered-Wilson Fermion

- Adams type