Aoki phases in the lattice Gross-Neveu model with staggered Wilson fermion

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Workshop on *New-type of Fermions on the Lattice*

Collaboration with M. Creutz (BNL) and T. Misumi (YITP)
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Lattice fermions

Naive

Lattice fermions

Naive $\longrightarrow$ Wilson

- Wilson [Wilson]

Lattice fermions

Naive $\longrightarrow$ Wilson $\longrightarrow$ Overlap (DW)

- Wilson [Wilson]
- Domain-wall & Overlap [Kaplan] [Furman-Shamir] [Neuberger]

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- Domain-wall & Overlap [Kaplan] [Furman-Shamir] [Neuberger]
- Staggered [Susskind]

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- $\#$ of doublers reduced $16 \to 1$
- Chiral symmetry broken due to "mass term"

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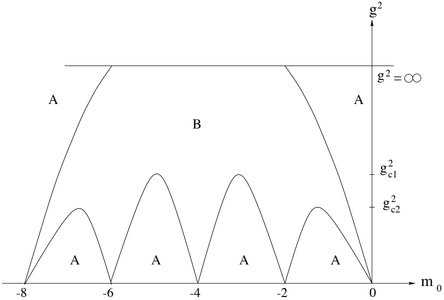

$$\langle \bar{\psi}\gamma_5\psi \rangle \begin{cases} = 0 & \text{(A phase)} \\ \neq 0 & \text{(B phase)} \end{cases}$$

Parity broken phase
(Aoki phase)

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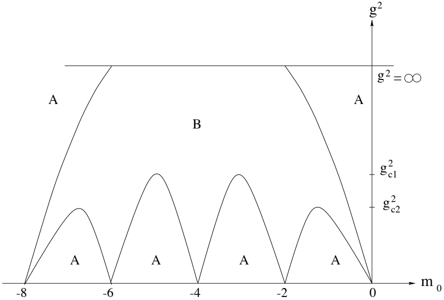

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Parity broken phase
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Chiral limit at 2nd order phase boundary

Staggered Wilson fermion

- Staggered fermion with Wilson term (doubler sensitive mass)
- # of doublers reduced
  - $4 \to 2$ **Adams' type**
  - $4 \to 1$ **Hoelbling's type**
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Chiral limit?

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$\downarrow$

Phase structure

- Phase structure with staggered Wilson

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- Asymptotic free
- Spontaneous chiral symmetry breaking
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- Phase structure of $4d$ QCD with staggered Wilson

[Nakano et al.]

- Phase structure with staggered Wilson

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Toy model for 4d QCD

- Phase structure of $4d$ QCD with staggered Wilson
  [Nakano et al.]

- GN model with Wilson fermion [Aoki-Higashijima]
  - Chiral limit [Izubuchi-Noaki-Ukawa] cf. [Izubuchi-Nagai]

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Gross-Neveu model

$$\mathcal{L} = \bar{\psi}_a (i\partial\!\!\!/ - m)\psi_a - \frac{g^2}{2N}(\bar{\psi}_a \psi_a)^2$$

- N-flavor system $\longrightarrow$ Global $\mathrm{U}(N)$ symmetry
- $2d$ version of NJL model

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- $2d$ version of NJL model

- Asymptotic free
- Chiral $\mathbb{Z}_4$ ($\mathbb{Z}_2$) symmetry: $\psi \to e^{i\frac{\pi}{2}\gamma_5}\psi$, $\bar{\psi} \to \bar{\psi}e^{i\frac{\pi}{2}\gamma_5}$
- Exactly solvable in $2d$ at large N

“Chiral” Gross-Neveu model

$$\mathcal{L} = \bar{\psi}_a (i\partial\!\!\!/ - m)\psi_a - \frac{g^2}{2N}\left[(\bar{\psi}_a \psi_a)^2 + (\bar{\psi}_a i\gamma_5 \psi_a)^2\right]$$

- Chiral $\mathrm{U}(1)$ symmetry at $m = 0$:

$$\psi \longrightarrow e^{i\theta\gamma_5}\psi, \qquad \bar{\psi} \longrightarrow \bar{\psi} e^{i\theta\gamma_5}$$

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- Spontaneous chiral $\mathrm{U}(1)$ breaking $\longrightarrow$ NG mode
  - Remark: IR behavior

- GN model on the lattice?

- GN model on the lattice? [Aoki-Higashijima]

Gross-Neveu model with Wilson fermion

$$S_{\mathrm{WGN}} = \frac{1}{2}\sum_{n,\mu}\bar{\psi}_n\gamma_\mu\left(\psi_{n+\hat{\mu}} - \psi_{n-\hat{\mu}}\right) + \sum_n \bar{\psi}_n M \psi_n + S_{\mathrm{W}}$$
$$-\frac{g^2}{2N}\sum_n\left[\left(\bar{\psi}_n\psi_n\right)^2 + \left(\bar{\psi}_n i\gamma_5\psi_n\right)^2\right]$$

- Wilson term

$$S_{\mathrm{W}} = -\frac{r}{2}\sum_{n,\mu}\bar{\psi}_n\left(\psi_{n+\hat{\mu}} - 2\psi_n + \psi_{n-\hat{\mu}}\right)$$

- Staggered fermion

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  - spin-diagonalization: $\psi \longrightarrow \chi$
  - chirality: $\gamma_5 \longrightarrow \epsilon_n = (-1)^n \simeq \gamma_5 \otimes \xi_5$

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Summation over spinor components $\longrightarrow$ sublattice

$$\left(\bar{\psi}_n i\gamma_5 \psi_n\right)^2 \quad \longrightarrow \quad \left(\sum_A i\epsilon_A \bar{\chi}_A \chi_A\right)^2$$

  - Sublattice structure: $n = 2\mathcal{N} + A$

- Wilson term for staggered fermion [Adams] [Hoelbling]
  cf. [Goltermann-Smit] [Goltermann]

$$S_{\mathrm{stW}} \simeq -i\epsilon_n \eta_1 \eta_2 \cos p_1 \cos p_2 \longrightarrow \mathbb{1} \otimes \tau_3$$

taste-splitting mass

Gross-Neveu model with staggered Wilson fermion

$$\begin{aligned} S_{\mathrm{stWGN}} &= \frac{1}{2}\sum_{n,\mu}\eta_\mu\bar{\chi}_n(\chi_{n+\hat{\mu}}-\chi_{n-\hat{\mu}})+\sum_n\bar{\chi}_n M\chi_n+S_{\mathrm{stW}} \\ &\quad -\frac{g^2}{2N}\sum_{\mathcal{N}}\left[\left(\sum_A\bar{\chi}_{2\mathcal{N}+A}\chi_{2\mathcal{N}+A}\right)^2 \right. \\ &\qquad \left. +\left(\sum_A i\epsilon_A\bar{\chi}_{2\mathcal{N}+A}\chi_{2\mathcal{N}+A}\right)^2\right] \end{aligned}$$

Gross-Neveu model with staggered Wilson fermion

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&\qquad \left. + \left(\sum_A i\epsilon_A\bar{\chi}_{2\mathcal{N}+A}\chi_{2\mathcal{N}+A}\right)^2\right]
\end{aligned}
$$

- Analysis with auxiliary fields to eliminate the 4pt interactions

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How to study?

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1. Introduce auxiliary fields (mesons) to eliminate the 4pt interactions

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2. Integrate out the fermionic fields, and then obtain the effective action for mesons
3. Solve the saddle point equation, giving the gap equation

Bosonization

$$Z = \int \mathcal{D}\bar{\chi}\mathcal{D}\chi \, e^{-S_{\mathrm{stWGN}}} = \int \mathcal{D}\sigma\mathcal{D}\pi \, e^{-S_{\mathrm{eff}}(\sigma,\pi)}$$

- Auxiliary "mesonic" fields: $\bar{\chi}_n\chi_n \to \sigma, \quad \epsilon_n\bar{\chi}_n\chi_n \to \pi$
- Effective action for mesons

$$S_{\mathrm{eff}} = \frac{1}{2g^2}\sum_{\mathcal{N}}\left(\sigma_{\mathcal{N}}^2 + \pi_{\mathcal{N}}^2\right) - \mathrm{Tr}\,\log D$$

$$D_{n,m} = (\sigma_{\mathcal{N}} + i\epsilon_n\pi_{\mathcal{N}})\delta_{n,m} + \frac{1}{2}\eta_\mu\left(\delta_{n+\hat{\mu},m} - \delta_{n-\hat{\mu},m}\right) + (S_{\mathrm{stW}})_{n,m}$$

- Saddle point equation:

$$\frac{\delta S_{\text{eff}}}{\delta \sigma} = \frac{\delta S_{\text{eff}}}{\delta \pi} = 0$$

  - We now find a spatially constant solution σ_0, π_0

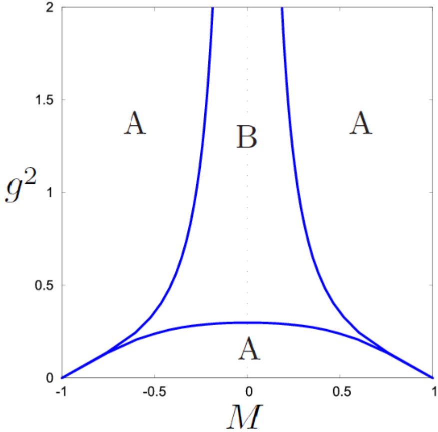
- Impose the critical condition: $\pi = 0$

$$\frac{M_c}{g^2} = 4 \int \frac{d^2 k}{(2\pi)^2} \frac{2 c_1^2 c_2^2 \sigma_0}{((\sigma_0 + c_1 c_2)^2 + \pi_0^2 + s^2)((\sigma_0 - c_1 c_2)^2 + \pi_0^2 + s^2)}$$

$$\frac{1}{g^2} = 4 \int \frac{d^2 k}{(2\pi)^2} \frac{\sigma_0^2 + s^2 + c_1^2 c_2^2}{((\sigma_0 + c_1 c_2)^2 + \pi_0^2 + s^2)((\sigma_0 - c_1 c_2)^2 + \pi_0^2 + s^2)}$$

- $c_\mu = \cos \frac{k_\mu}{2}$, $s^2 = \sin^2 \frac{k_1}{2} + \sin^2 \frac{k_2}{2}$

- Phase diagram



$$\pi \begin{cases} = 0 & \text{(A phase)} \\ \neq 0 & \text{(B phase)} \end{cases}$$

Parity broken phase
(Aoki phase)

- Pion mass

$$m_\pi^2 \propto \left\langle \frac{\delta^2 S_{\text{eff}}}{\delta \pi_n \delta \pi_m} \right\rangle \Bigg|_{M=M_c} = 0 \quad \longrightarrow$$ **Critical at phase boundary**

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- Critical phase boundary suggests the chiral limit for the staggered Wilson fermion

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- but, even through renormalization?
  - chiral symmetry broken due to the (staggered) Wilson term

Renormalization effect

"Generic" Gross-Neveu model with staggered Wilson fermion

$$
\begin{aligned}
S_{\rm stWGN} \;=\; & \frac{1}{2}\sum_{n,\mu}\eta_\mu\bar{\chi}_n(\chi_{n+\hat{\mu}}-\chi_{n-\hat{\mu}}) + \sum_n \bar{\chi}_n M\chi_n + S_{\rm stW} \\
& -\frac{g_\sigma^2}{2N}\sum_{\mathcal{N}}\left(\sum_A \bar{\chi}_{2\mathcal{N}+A}\chi_{2\mathcal{N}+A}\right)^2 \\
& -\frac{g_\pi^2}{2N}\sum_{\mathcal{N}}\left(\sum_A i\epsilon_A\bar{\chi}_{2\mathcal{N}+A}\chi_{2\mathcal{N}+A}\right)^2
\end{aligned}
$$

- Two kinds of coupling constants: (g_σ, g_π)

Chiral symmetry is preserved only when $g_\sigma = g_\pi$ at the classical level, but broken due to the Wilson term.

- Effective action

$$
\begin{aligned}
\tilde{S}_{\text{eff}} &= -\left(\frac{M+1/a}{g_\sigma^2}+\frac{2}{a}C_1\right)\sigma_0+\left(\frac{1}{2g_\pi^2}-\tilde{C}_0+\frac{1}{\pi}\log 4a^2\right)\pi_0^2 \\
&\quad +\left(\frac{1}{2g_\sigma^2}-\tilde{C}_0+2C_2+\frac{1}{\pi}\log 4a^2\right)\sigma_0^2 \\
&\quad +\frac{1}{\pi}(\sigma_0^2+\pi_0^2)\log\frac{\sigma_0^2+\pi_0^2}{e}+\mathcal{O}(a)
\end{aligned}
$$

- Effective action

$$\begin{aligned}\tilde{S}_{\text{eff}} &= -\left(\frac{M+1/a}{g_\sigma^2}+\frac{2}{a}C_1\right)\sigma_0+\left(\frac{1}{2g_\pi^2}-\tilde{C}_0+\frac{1}{\pi}\log 4a^2\right)\pi_0^2 \\ &\quad +\left(\frac{1}{2g_\sigma^2}-\tilde{C}_0+2C_2+\frac{1}{\pi}\log 4a^2\right)\sigma_0^2 \\ &\quad +\frac{1}{\pi}(\sigma_0^2+\pi_0^2)\log\frac{\sigma_0^2+\pi_0^2}{e}+\mathcal{O}(a) \\ &\to \frac{1}{\pi}(\sigma_0^2+\pi_0^2)\log\frac{\sigma_0^2+\pi_0^2}{e\Lambda^2}\end{aligned}$$

Chiral limit

- Effective action

$$
\begin{aligned}
\tilde{S}_{\text{eff}} &= -\left(\frac{M+1/a}{g_\sigma^2} + \frac{2}{a}C_1\right)\sigma_0 + \left(\frac{1}{2g_\pi^2} - \tilde{C}_0 + \frac{1}{\pi}\log 4a^2\right)\pi_0^2 \\
&\quad + \left(\frac{1}{2g_\sigma^2} - \tilde{C}_0 + 2C_2 + \frac{1}{\pi}\log 4a^2\right)\sigma_0^2 \\
&\quad + \frac{1}{\pi}(\sigma_0^2 + \pi_0^2)\log\frac{\sigma_0^2+\pi_0^2}{e} + \mathcal{O}(a) \\
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\end{aligned}
$$

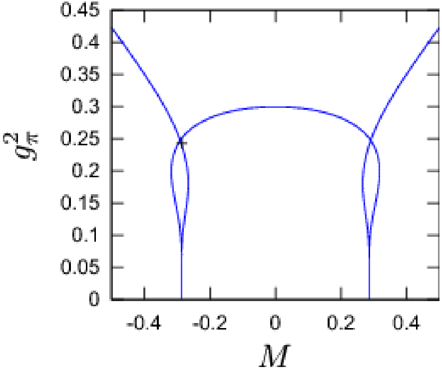
Chiral limit

- tuning couplings & Λ-parameter

$$
\frac{1}{2g_\sigma^2} = \tilde{C}_0 - 2C_2 + \frac{1}{\pi}\log\left(\frac{1}{4\Lambda^2 a^2}\right), \quad \frac{1}{2g_\pi^2} = \tilde{C}_0 + \frac{1}{\pi}\log\left(\frac{1}{4\Lambda^2 a^2}\right)
$$

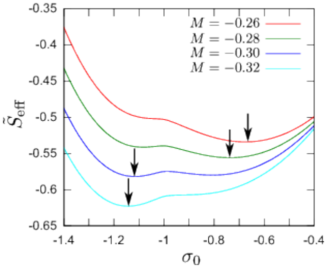
$$
M = -\frac{2g_\sigma^2}{a}C_1 - 1, \qquad 2a\Lambda = \exp\left[\frac{\pi}{2}\tilde{C}_0 - \pi C_2 - \frac{\pi}{4g_\sigma^2}\right]
$$

- Gap equation: $\pi_0 = 0$ with $g_\sigma^2 = 0.4$



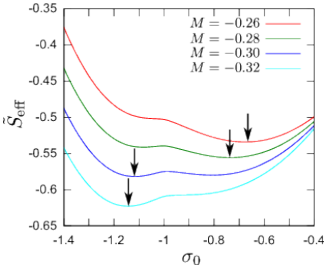
"2nd order" phase boundary $\longrightarrow$ other possibilities?

- Potential analysis for σ_0



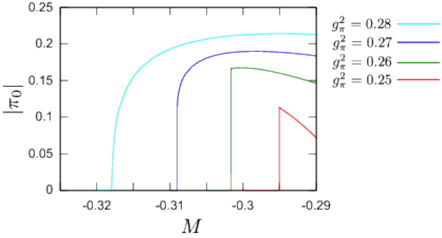
1st order σ-transition

- Potential analysis for σ_0



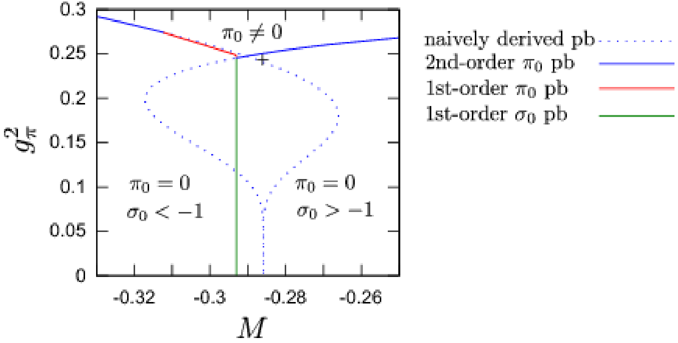
1st order σ-transition

- Order of π-transition



1st & 2nd order transition

- Phase structure & Chiral limit



We can safely take chiral & continuum limit

- cf. Chial perturbation approach [Sharpe-Singleton]

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Summary

- Staggered Wilson Gross-Neveu model as a toy model for QCD
- Phase structure of staggered Wilson GN model
  $\longrightarrow$ Parity broken phase (Aoki phase)
- Chiral limit of the model
  $\longrightarrow$ Renormalization effect, safety of chiral limit
- Phase structure in QCD [Nakano et al.]