# ***The QCD phase diagram: universality and continuity***

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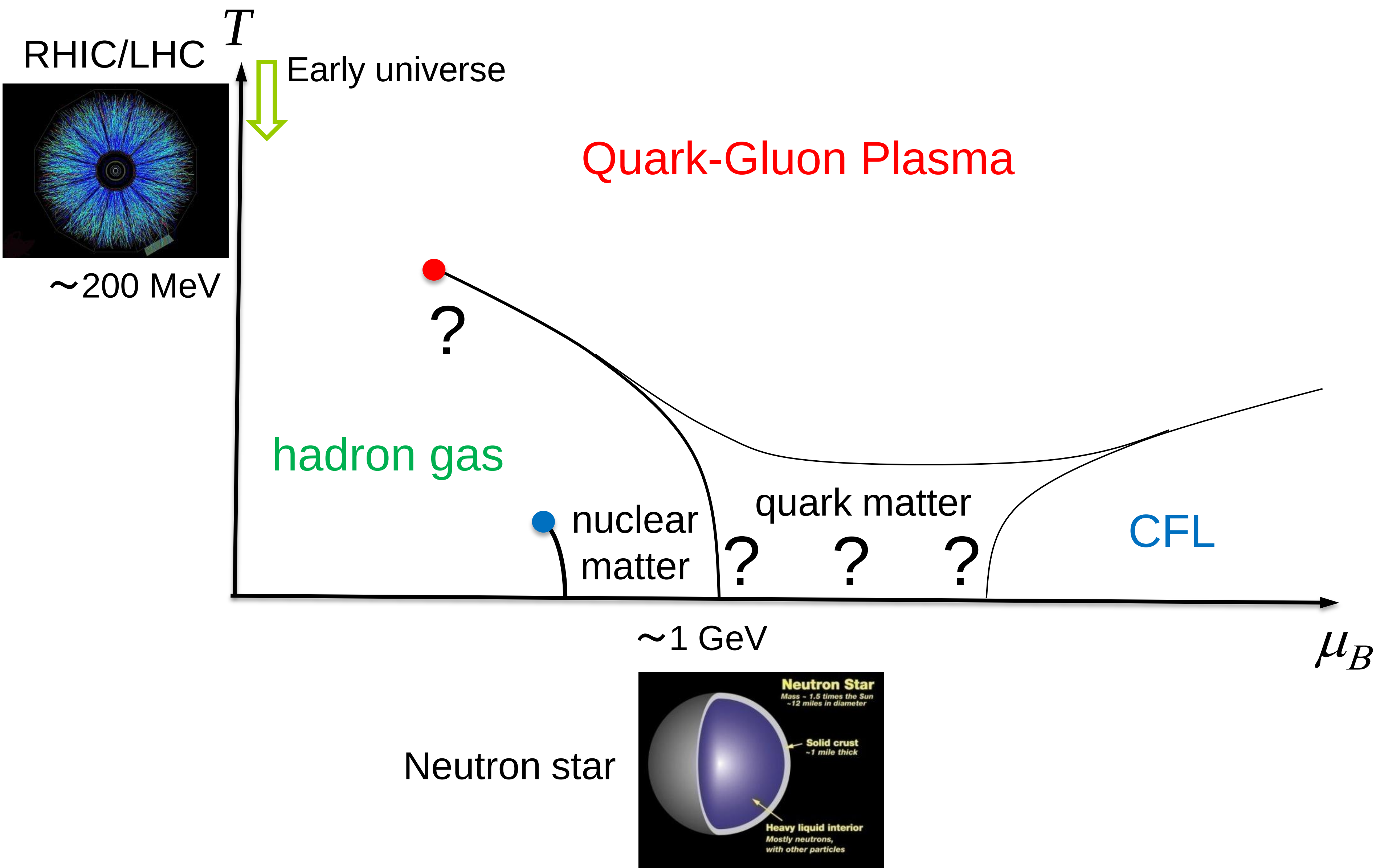
# ***Outline***

- Introduction of QCD phase diagram
- Universality of phases in QCD and QCD-like theories
- A no-go theorem for critical phenomena
- Quark-hadron continuity/duality

Refs: M. Hanada and NY, arXiv:1103.5480 (to appear in JHEP).

Y. Hidaka and NY, arXiv:1110.3044 (to appear in PRL).

# A conjectured QCD phase diagram



# ***Finite μ phase transition (T=0)***

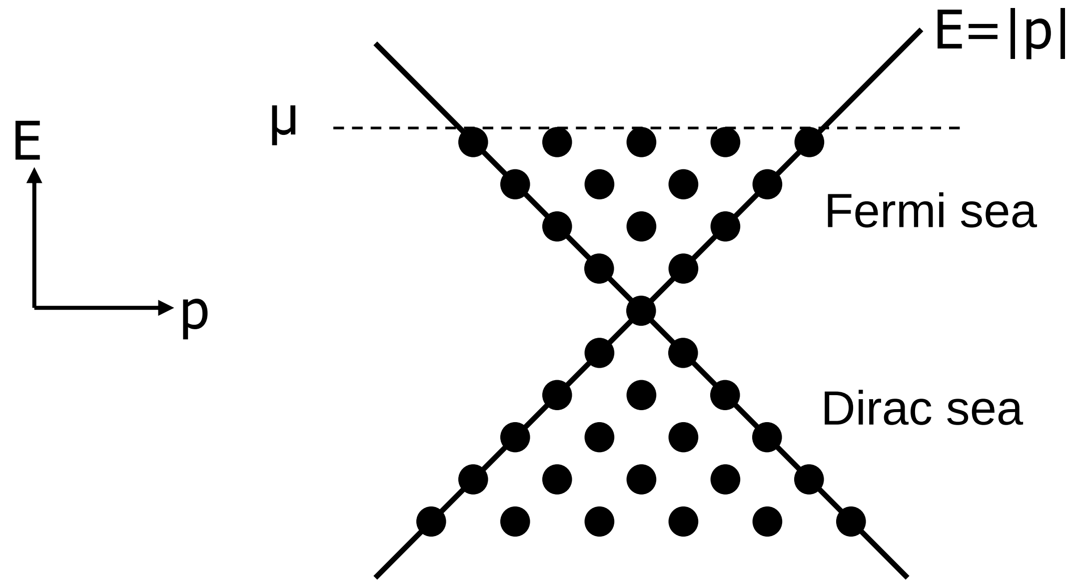
Consider $T$=0.

- A baryon with excitation energy $m_B - \mu_B$ can be created for $\mu_B > m_B$;

  Baryon (=fermion) forms a Fermi surface due to the Pauli principle.
- Phase transition at $\mu_B \approx m_B$ distinguishing between $n_B$=0 and $n_B$>0.

# Large μ: color superconductivity (T=0)

- At high density, nuclear matter → quark matter (quark Fermi surface): weak-coupling due to asymptotic freedom



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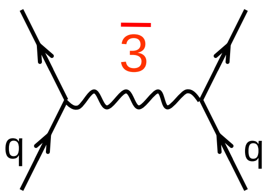
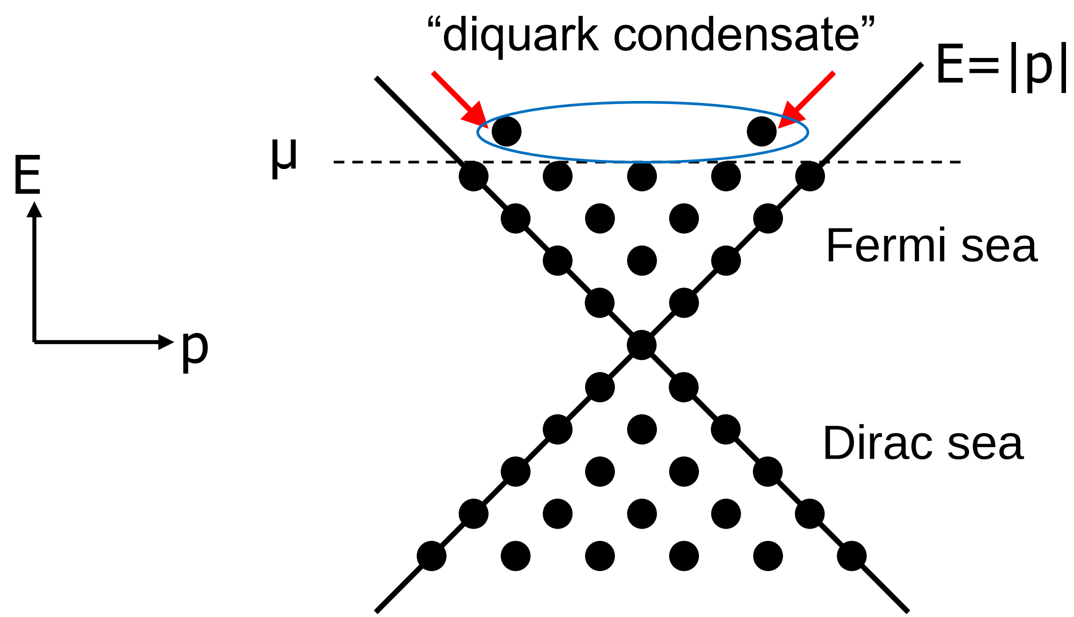
- Attractive interaction → BCS pairing




$$3 \times 3 = 6_S + \bar{3}_A$$

$$(\tau_a)_{ij}(\tau_a)_{kl} = \frac{2}{3}(\tau_S)_{ik}(\tau_S)_{lj} - \frac{4}{3}(\tau_A)_{ik}(\tau_A)_{lj}$$



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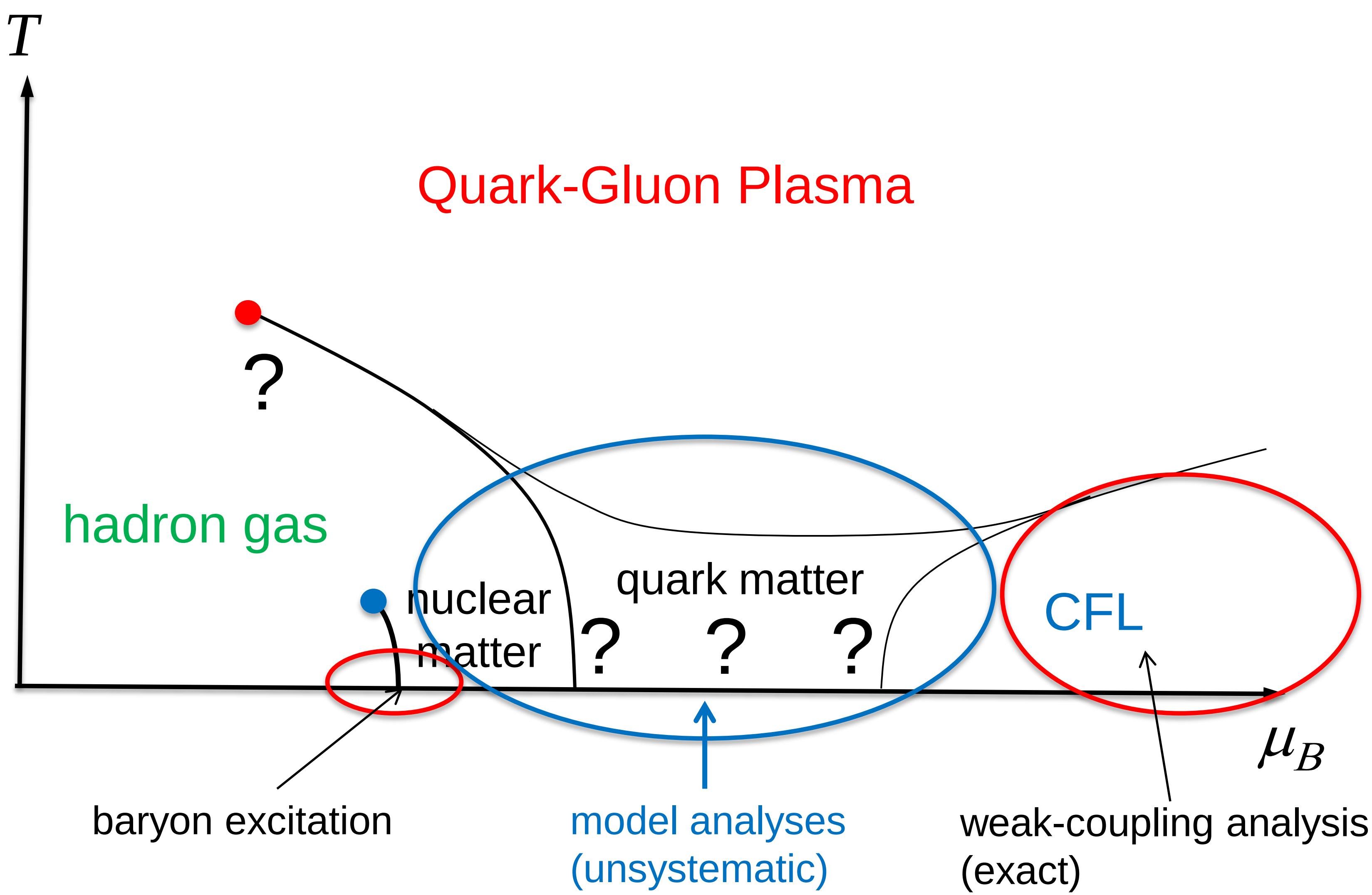

# ***Phase diagrams of QCD-like theories***

- QCD at $\mu_I=2\mu>0$: chemical potentials $\mu$ & $-\mu$ for $u$ & $d$ quark (= $\pi^+$)

  → Dirac eigenvalues for $u$ & $d$ are complex conjugate.

  → Positive fermion determinant (no sign problem)

- Different gauge groups at $\mu_B>0$ (instead of SU($N_c$)):

  ▪ SO($N_c$) with any flavors: real

  (similar to QCD with adjoint fermions)

  ▪ Sp($N_c$) with even degenerate flavors: pseudo-real

  (similar to 2-color QCD; $N_c$ "color" version of SU(2)=Sp(2) )

  → Positive fermion determinant (no sign problem)

# ***QCD at finite isospin density***

Consider $T$=0. Son-Stephanov ('01)

- At small $\mu_I$:
  - $\pi^+$ with excitation energy $m_\pi - \mu_I$ can be created for $\mu_I > m_\pi$.
  - $\pi^+$ (=boson) forms a Bose-Einstein condensation (BEC) at $\mu_I = m_\pi$.

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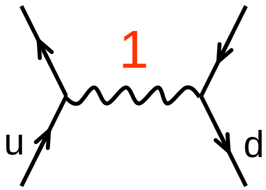
- At large $\mu_I$: Fermi surface of $u$ & $\bar{d}$ quarks
  - attractive interaction in color singlet channel: $3 \times \bar{3} = 1 + 8$

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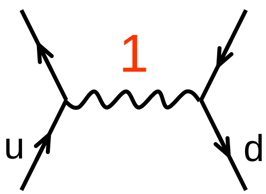


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- Same quantum numbers of condensates/ symmetry breaking pattern:

BEC-BCS crossover (continuity from hadronic to quark matter)

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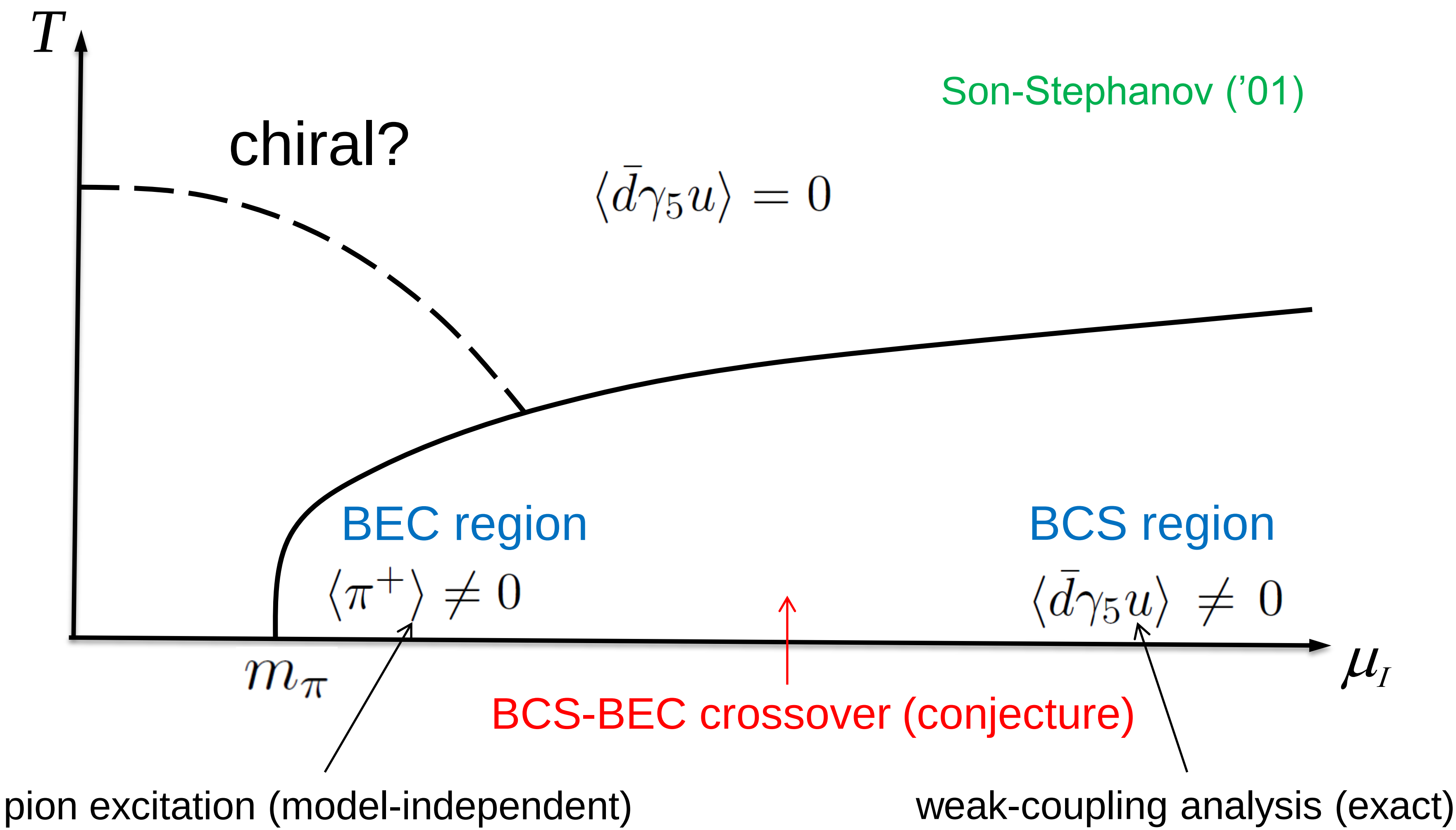
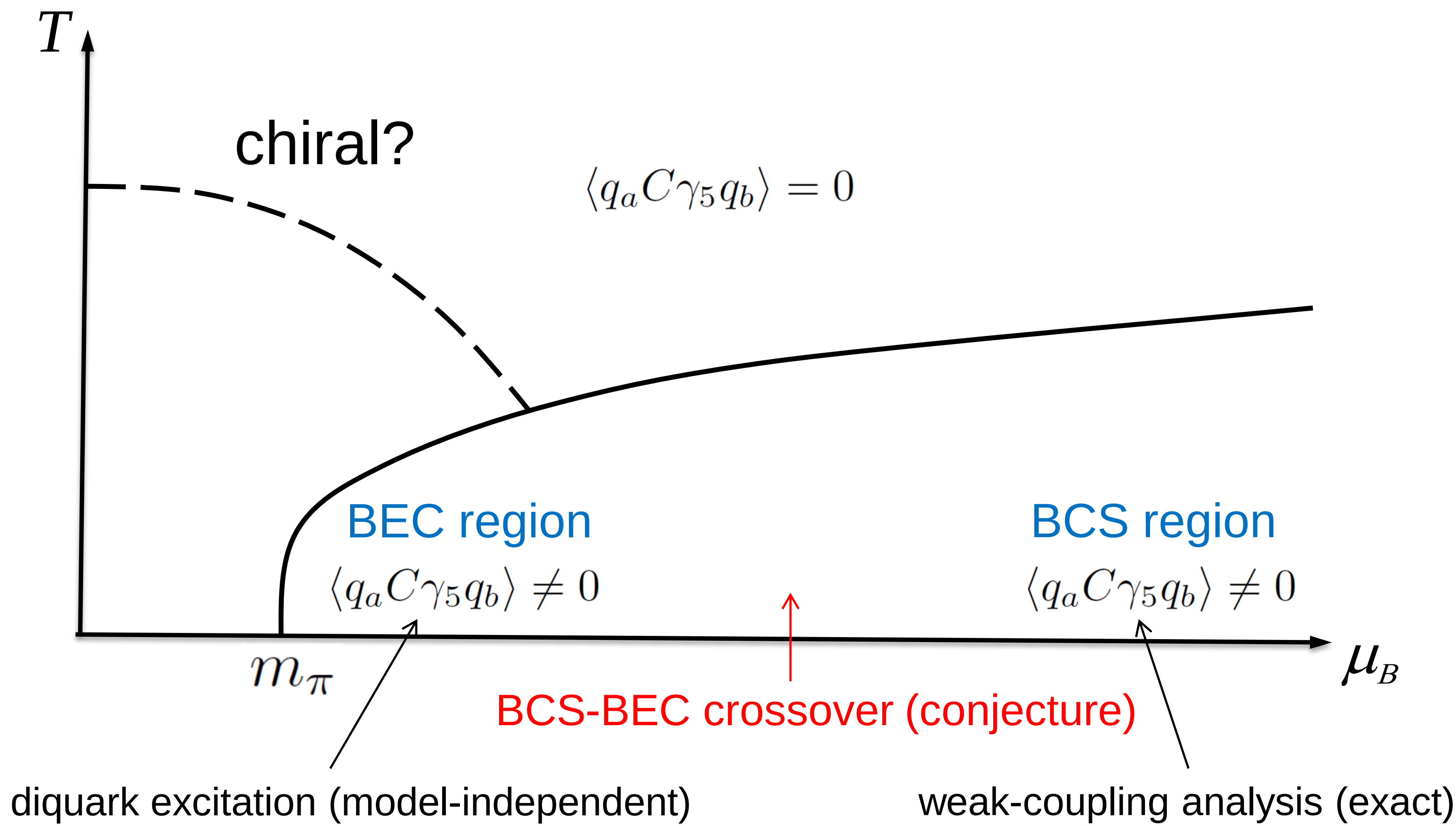


# SO & Sp gauge theories at $\mu_B$>0




2-color QCD and adjoint QCD also have similar phase diagrams.

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- Main results: Hanada-NY ('11); Hidaka-NY ('11)

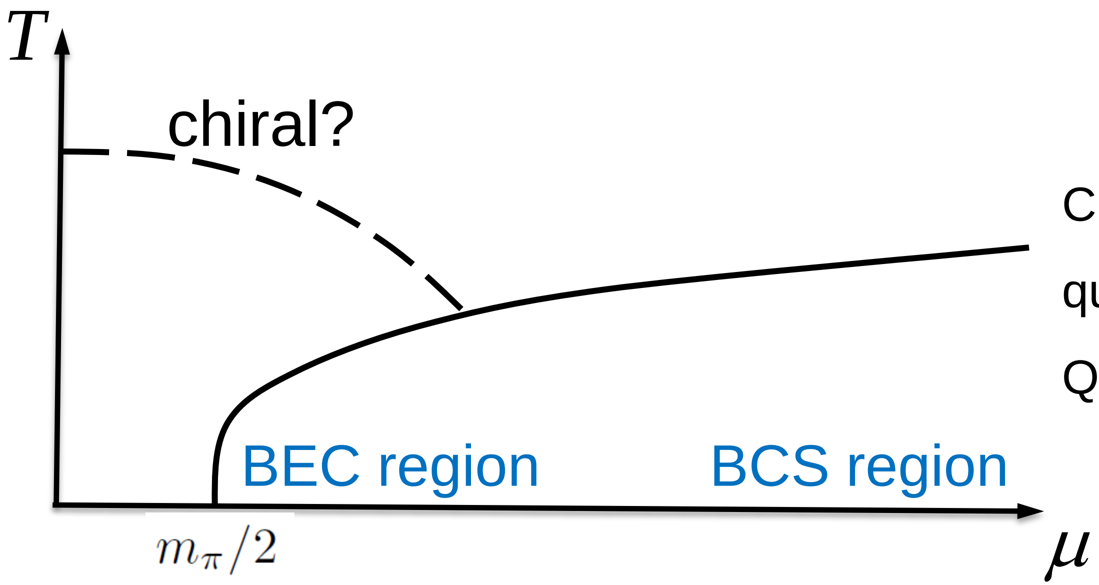
1. Whole phase diagrams are quantitatively *universal* in the large-$N_c$ limit between QCD at $\mu_I>0$ & SO/Sp gauge theories at $\mu_B>0$.

2. Universality holds for QCD at $\mu_B>0$ outside the BCS/BEC region.

3. **No-go theorem**: QCD critical point is ruled out in QCD at $\mu_B>0$ outside the BCS/BEC region.

**Practically this makes it possible to evade the sign problem for a class of observables at least large $N_c$ and hopefully $N_c$=3.**

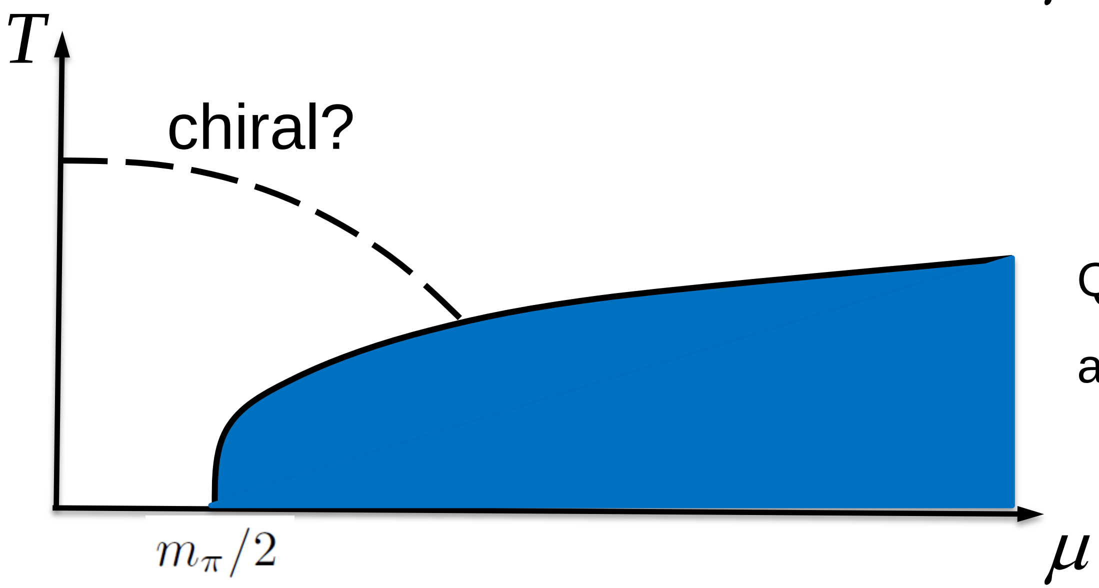
# Universality of phase diagrams




Chiral, deconfinement, BEC/BCS, ...

quantitatively *universal* between

QCD at $\mu_I$>0, SO/Sp theories at $\mu_B$>0.



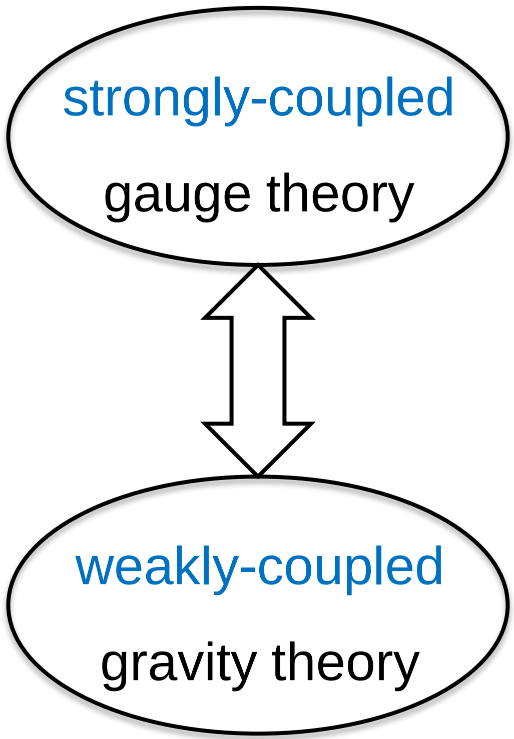

QCD at $\mu_B$>0 also equivalent to

above outside BEC/BCS.

# ***Large-Nc dualities/equivalences***

**gauge/gravity duality**

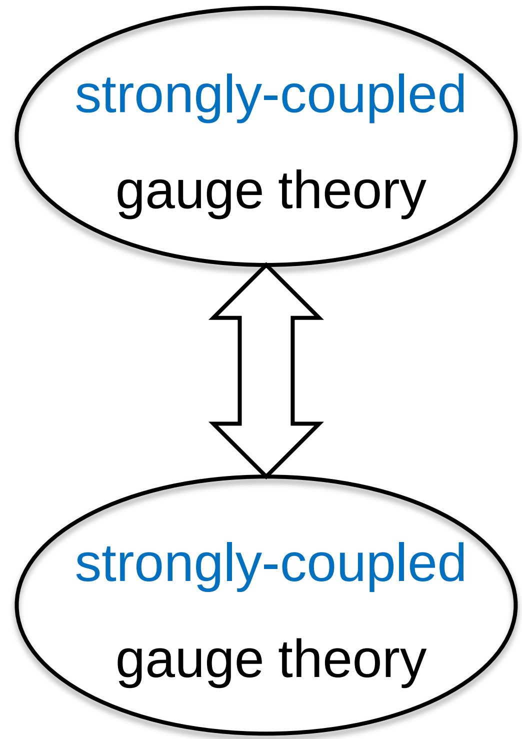
Maldacena ('98), ...




AdS/QGP,
AdS/QCD,
AdS/CMT, ...

**orbifold equivalence**

Kachru-Silverstein ('98), ...




Cherman-Hanada-Robles-Llana;
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# Large-Nc dualities/equivalences

**gauge/gravity duality**

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strongly-coupled
gauge theory

⇕

weakly-coupled
gravity theory

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gauge theory
w/ sign problem

an example ⇕

gauge theory
w/o sign problem

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# ***Large-Nc orbifold equivalence***

**[How to use]**

1. Identify a discrete global symmetry of the original theory (parent).

Refs: Bershadsky-Johansen ('98), Kovtun-Ünsal-Yaffe ('03, '05, '06)

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**[Caution]**

Valid as long as the discrete symmetry is not broken spontaneously.

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# ***Example: from SO(2$N_c$) to SU($N_c$)***

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$$A_\mu^{\mathrm{proj}} = \begin{pmatrix} (A_\mu^{\mathrm{U}})^C & 0 \\ 0 & A_\mu^{\mathrm{U}} \end{pmatrix}, \quad \psi^{\mathrm{proj}} = \begin{pmatrix} 0 \\ \psi^{\mathrm{U}} \end{pmatrix}$$

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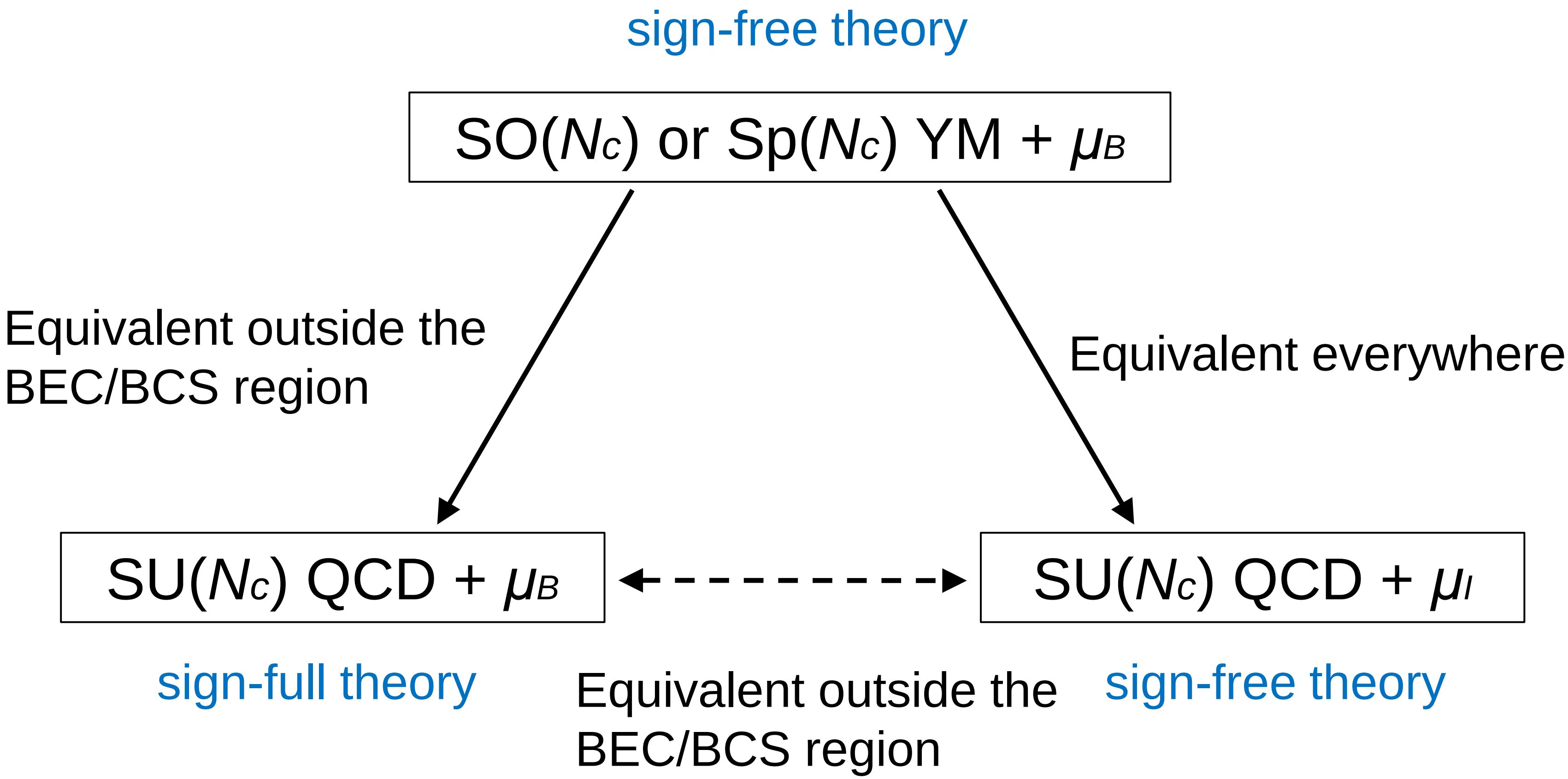
**[Caution]**

- $Z_4 \in \mathrm{U}(1)_B$ broken in the BCS-BEC crossover.
- $Z_4 \in \mathrm{SU}(2)_{iso}$ unbroken from SO(2$N_c$) to QCD at $\mu_I$>0 everywhere.

# *“Family Tree” of QCD and QCD-like theories*

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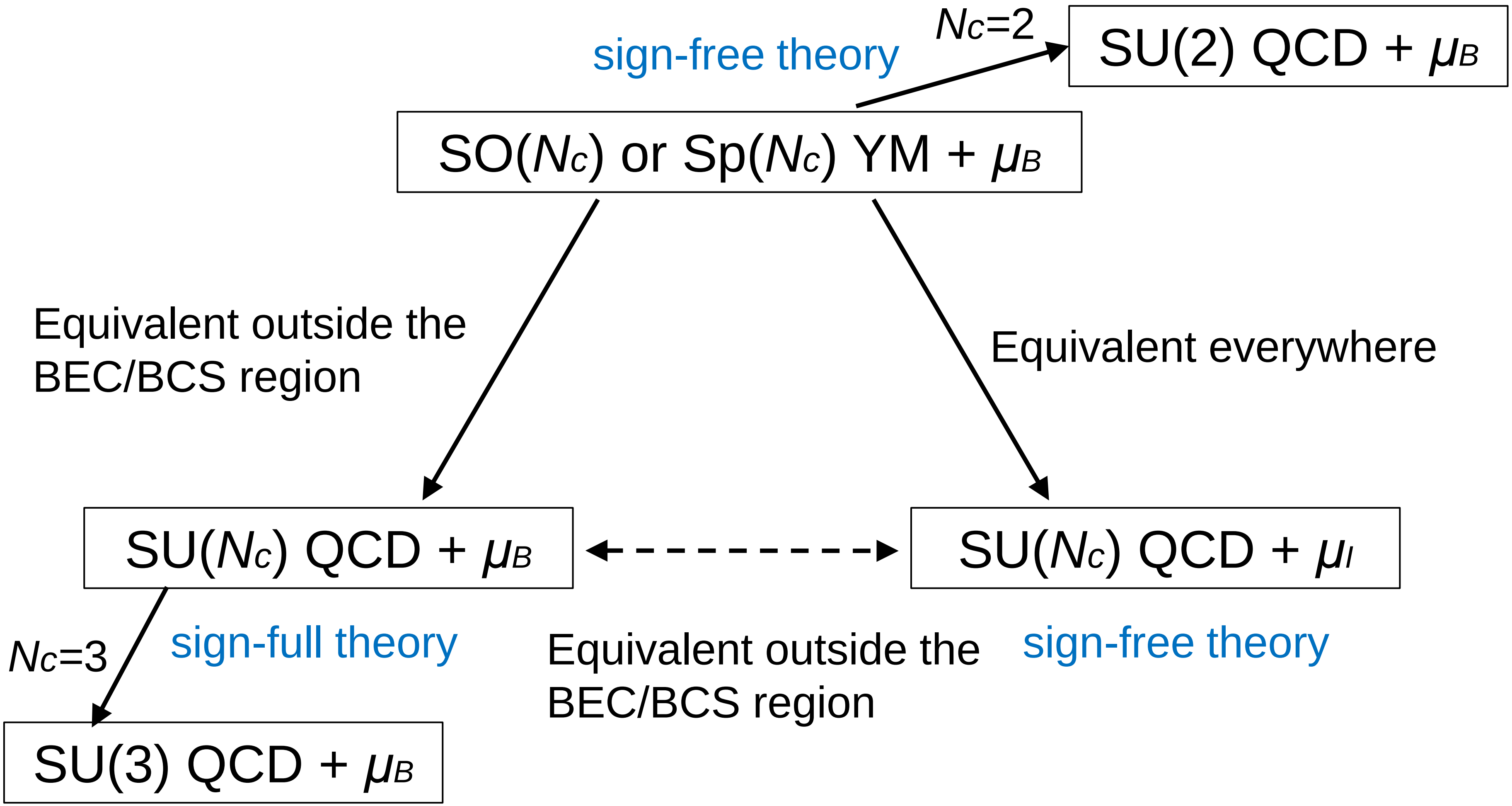


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- Not applicable to the quantity $\langle \pi^{\pm} \rangle$ (where *u* & *d* are mixed).
  - $\longrightarrow$ equivalence for a class of observables
- Not applicable if $\langle \pi^{\pm} \rangle \neq 0$ in the ground state (where *u* & *d* are mixed).
  - $\longrightarrow$ equivalence outside BEC/BCS region

# ***A recent lattice result***

- Critical temperature of chiral transition at small $\mu_I$ and $\mu_B$.

$$\frac{T_c(\mu)}{T_c(0)} = 1 + a_1 \left(\frac{\mu}{\pi T}\right)^2$$

Cea-Cosmai-D'Elia-Papa-Sanfilippo
arXiv:1110.3910

$$a_1 = -0.470(13) \quad \text{isospin chemical potential}$$

$$a_1 = -0.522(10) \quad \text{quark chemical potential}\,.$$

(staggered fermion, *ma*=0.05, $16^3 \times 4$ lattice)

Universality (or phase-quenching) looks approximately valid up to difference ~10% in three-color QCD.

# Reminder: QCD Dirac operator

- QCD Dirac operator at $\mu$=0:

$$\mathcal{D} = D + m \text{ with } D = \gamma_\mu(\partial_\mu + igA_\mu)$$

  - anti-Hermiticity: $D^\dagger = -D$
  - chiral symmetry: $\gamma_5 D \gamma_5 = -D$

$\longrightarrow$ $\gamma_5$ Hermiticity: $\gamma_5 \mathcal{D} \gamma_5 = \mathcal{D}^\dagger$

This ensures the positivity $\det \mathcal{D} \geq 0$ (which is not true at $\mu_B$ >0).

- QCD at $\mu_I$>0, $\mathcal{D}(\mu_I) = D + \dfrac{\mu_I}{2}\gamma_0\tau_3 + m$
  - property: $\tau_1\gamma_5\mathcal{D}\gamma_5\tau_1 = \mathcal{D}^\dagger$ $\rightarrow$ positivity at any $\mu_I$>0

# QCD inequalities (μ=0)

- Consider correlator of $M_\Gamma = \bar{\psi}\Gamma\psi$ Weingarten; Witten; Nussinov ('83)

$$\begin{aligned}\langle M_\Gamma(x) M_\Gamma^\dagger(y)\rangle_{\psi,A} &= -\langle \mathrm{tr}[S_A(x,y)\Gamma \underline{S_A(y,x)}\bar{\Gamma}]\rangle_A \\ &= \langle \mathrm{tr}[S_A(x,y)\Gamma \underline{i\gamma_5 S_A^\dagger(x,y) i\gamma_5}\bar{\Gamma}]\rangle_A \\ &\leq \langle \mathrm{tr}[S_A(x,y) S_A^\dagger(x,y)]\rangle_A .\end{aligned}$$

$\bar{\Gamma} \equiv \gamma_0 \Gamma^\dagger \gamma_0$

(∵ $\gamma_5$ Hermiticity)

(∵ Cauchy-Schwarz ineq.)

becomes maximum when $\Gamma = i\gamma_5 \tau_A$ (meson is $\pi^{0,\pm}$)

- Correlators behave $\sim e^{-m_\Gamma |x-y|}$ at large separation.
- Thus we have $m_\Gamma \geqq m_\pi$ (pion is the lightest meson).

This argument is not applicable to the flavor-singlet.

Reason: contributions of flavor-disconnected diagrams

# *A no-go theorem at large Nc (μ=0)*

- Disconnected diagram may be suppressed phenomenologically.

  e.g.) Okubo-Zweig-Iizuka rule: suppression of annihilation diagrams
- It can be justified in the large-$N_c$ limit. Witten ('79)


- Then QCD inequalities are applicable to the flavor singlet: $m_\sigma \geqq m_\pi$

  Turning on $m_q$>0, we have $m_\sigma \geqq m_\pi > 0$.
- 2nd-order chiral transition implies $\xi=\infty$ or $m_\sigma=0$ at some $T=T_c$, which is prohibited for *any* $m_q$>0 from above constraint.

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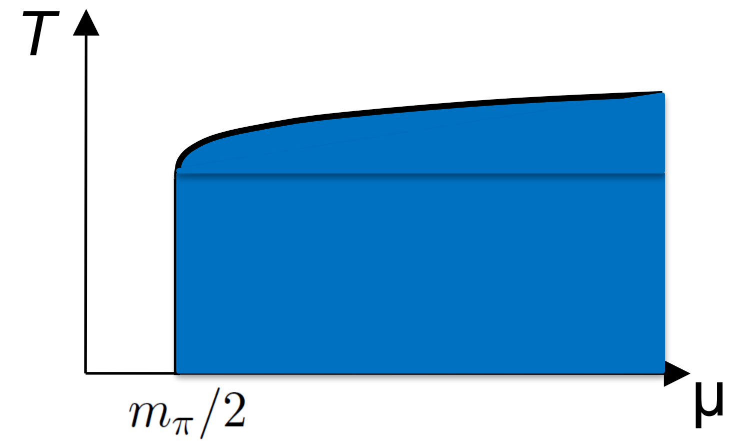
  inequality saturated when $\Gamma = i\gamma_5\tau_{1,2}$ (meson is $\pi^\pm$)

  $\rightarrow$ $m_\sigma \geqq m_{\pi^\pm} > 0$ for $m_q$>0 $\rightarrow$ no 2nd-order chiral transition for $m_q$>0

- Via our universality, this result can be mapped to QCD at $\mu_B$>0 outside BEC/BCS region.



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- Still it is possible in QCD-like theories at $\mu$>0 preserving positivity.

  e.g., QCD at any $\mu_I$, $\langle M_\Gamma(x) M_\Gamma^\dagger(y)\rangle_{\psi,A} \le \langle \mathrm{tr}[S_A(x,y)S_A^\dagger(x,y)]\rangle_A$

  inequality saturated when $\Gamma = i\gamma_5\tau_{1,2}$ (meson is $\pi^\pm$)

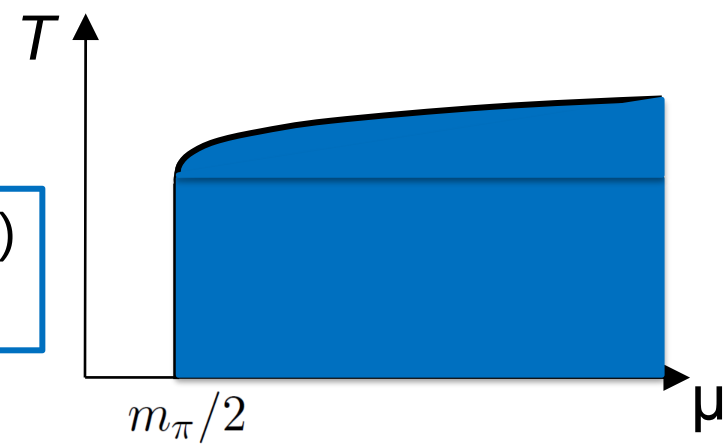
  → $m_\sigma \geqq m_{\pi^\pm} > 0$ for $m_q$>0 → no 2nd-order chiral transition for $m_q$>0

- Via our universality, this result can be mapped to QCD at $\mu_B$>0 outside BEC/BCS region.

- Thus we obtain an exact no-go theorem:

Critical phenomena (e.g. QCD critical point) are forbidden for $m_q$>0 outside BEC/BCS.

Hidaka-NY ('11)



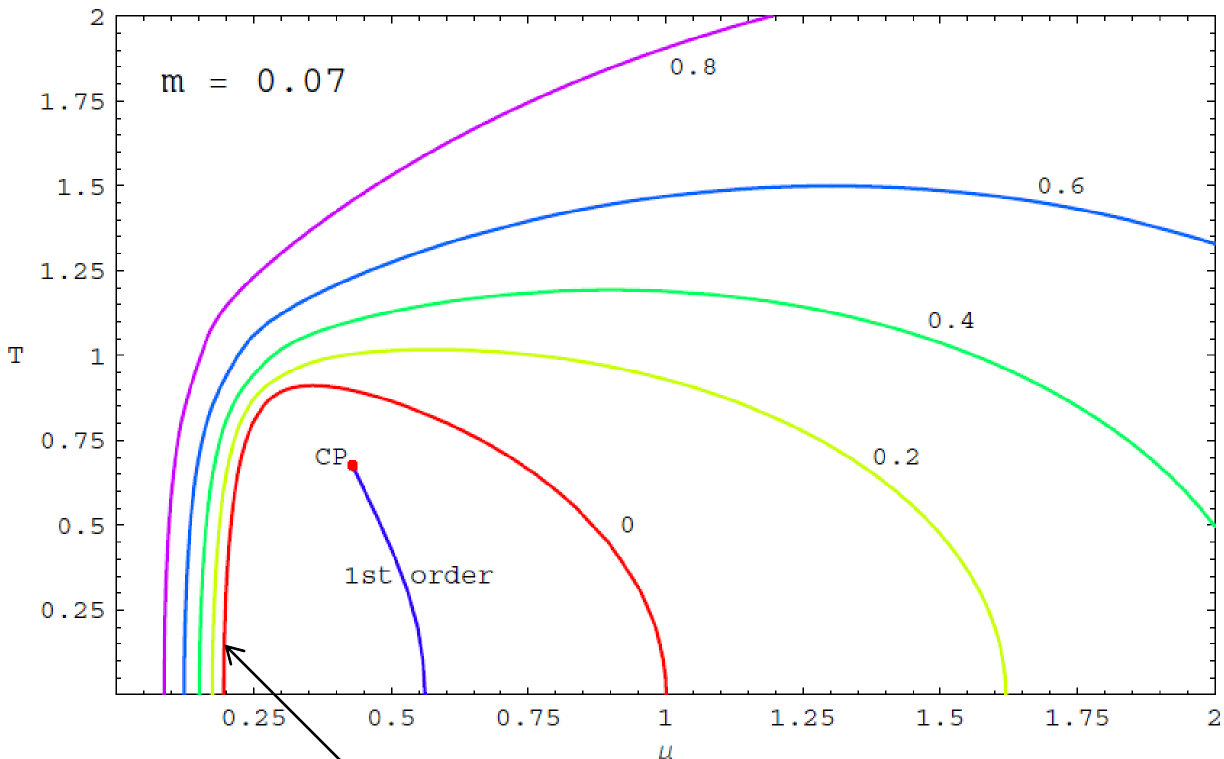
# ***Remarks***

- In this argument, we take large $N_c$ only to justify 2 empirical facts:
  - suppression of sea quark effects (quenched approximation)
  - suppression of disconnected diagrams (OZI rule).

- In other words, the no-go theorem would be violated if either of them is not satisfied in real QCD.

# Model results

- Critical point & average phase factor in random matrix model




Han-Stephanov ('08)

$$e^{2i\theta} = \frac{\det(D + \mu_q\gamma_0 + m)}{\det(D + \mu_q\gamma_0 + m)^*}$$

QCD at $\mu_B>0$

$$\boxed{\langle e^{2i\theta}\rangle = \frac{Z_{1+1}}{Z_{1+1^*}}}$$

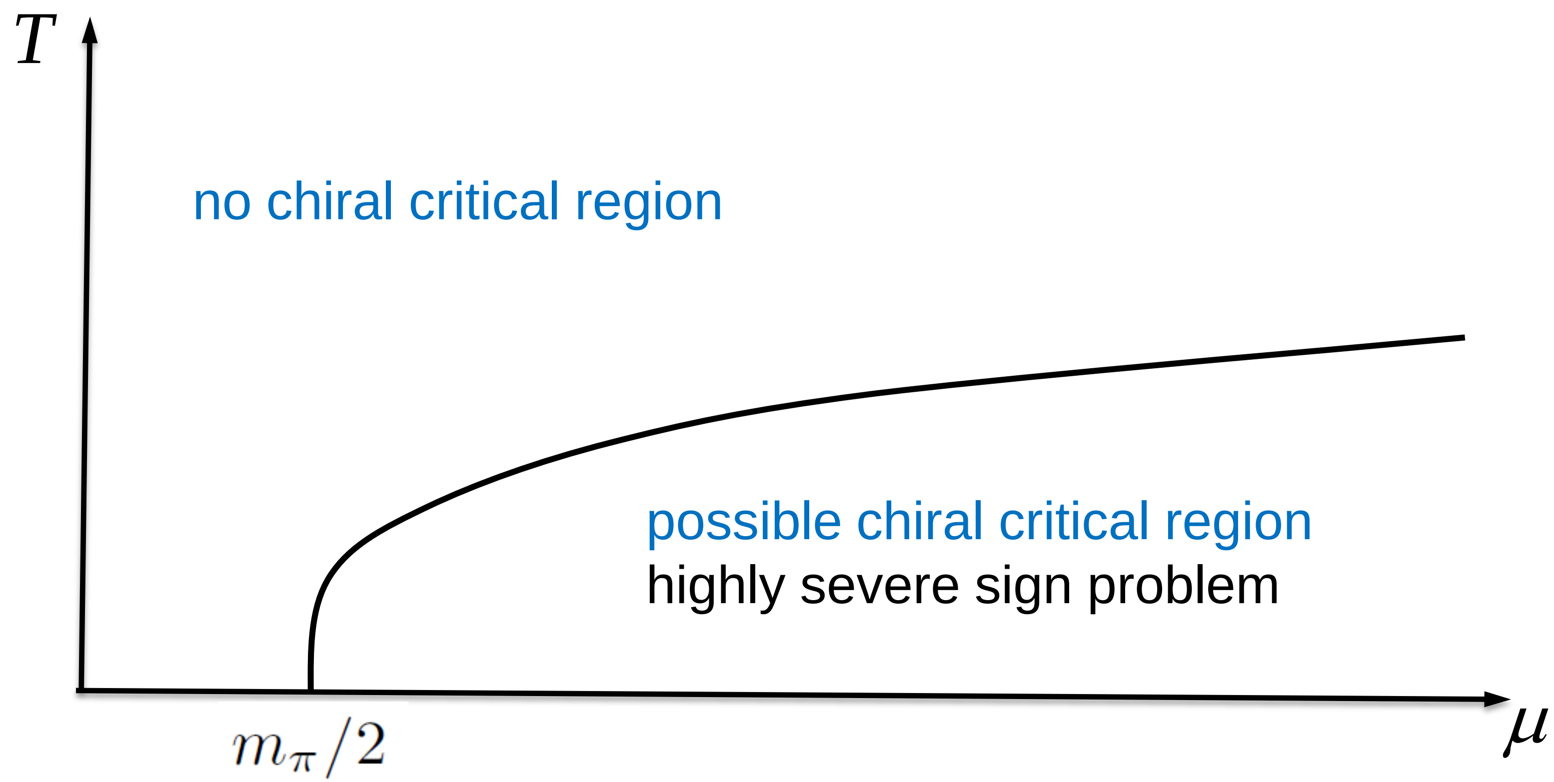
QCD at $\mu_I>0$

pion BEC phase (ends at large $\mu$ due to a model artifact)

- Similar results obtained in NJL & PNJL models
  Andersen- Kyllingstad- Splittorff; Sakai-Sasaki-Kouno-Yahiro ('10)

# ***Lesson***

- Nature somehow hides the QCD critical point and other interesting physics inside the region where the sign problem is maximally severe.



# ***Quark-hadron continuity***

- QCD phase diagram at $\mu_B$>0 where universality breaks down?

- Still there exists a similarity in $N_c$=$N_f$=3: quark-hadron continuity

  Schafer-Wilczek (‘99); Baym-Hatsuda-Tachibana-NY (’06, ’07, ‘08, ‘10);

# $N_f$=3 QCD at finite baryon density

Consider QCD with $m_u = m_d = m_s$ at $T=0$.

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Consider QCD with $m_u=m_d=m_s$ at $T$=0.

- At small $\mu_B$: recent lattice QCD results by HAL QCD & NPLQCD collaborations ('11)
  - H dibaryon has the smallest excit. energy per baryon since $m_H/2<m_B$.
  - For $\mu_B>m_H/2$, H dibaryon (= boson) forms BEC which breaks $U(1)_B$.

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- At large $\mu_B$: color flavor locking (CFL) Alford-Rajagopal-Wilczek ('99)
  - Diquark condensate: $X^{ia} = \epsilon^{ijk}\epsilon^{abc}\langle q_L^{bj} q_L^{ck}\rangle^*, \quad Y^{ia} = \epsilon^{ijk}\epsilon^{abc}\langle q_R^{bj} q_R^{ck}\rangle^*$

    u,d,s **r**,**g**,**b**
  - Gauge invariant order parameter: $\Sigma = XY^\dagger$

$\rightarrow$ Chiral & U(1)$_B$ symmetries are broken.

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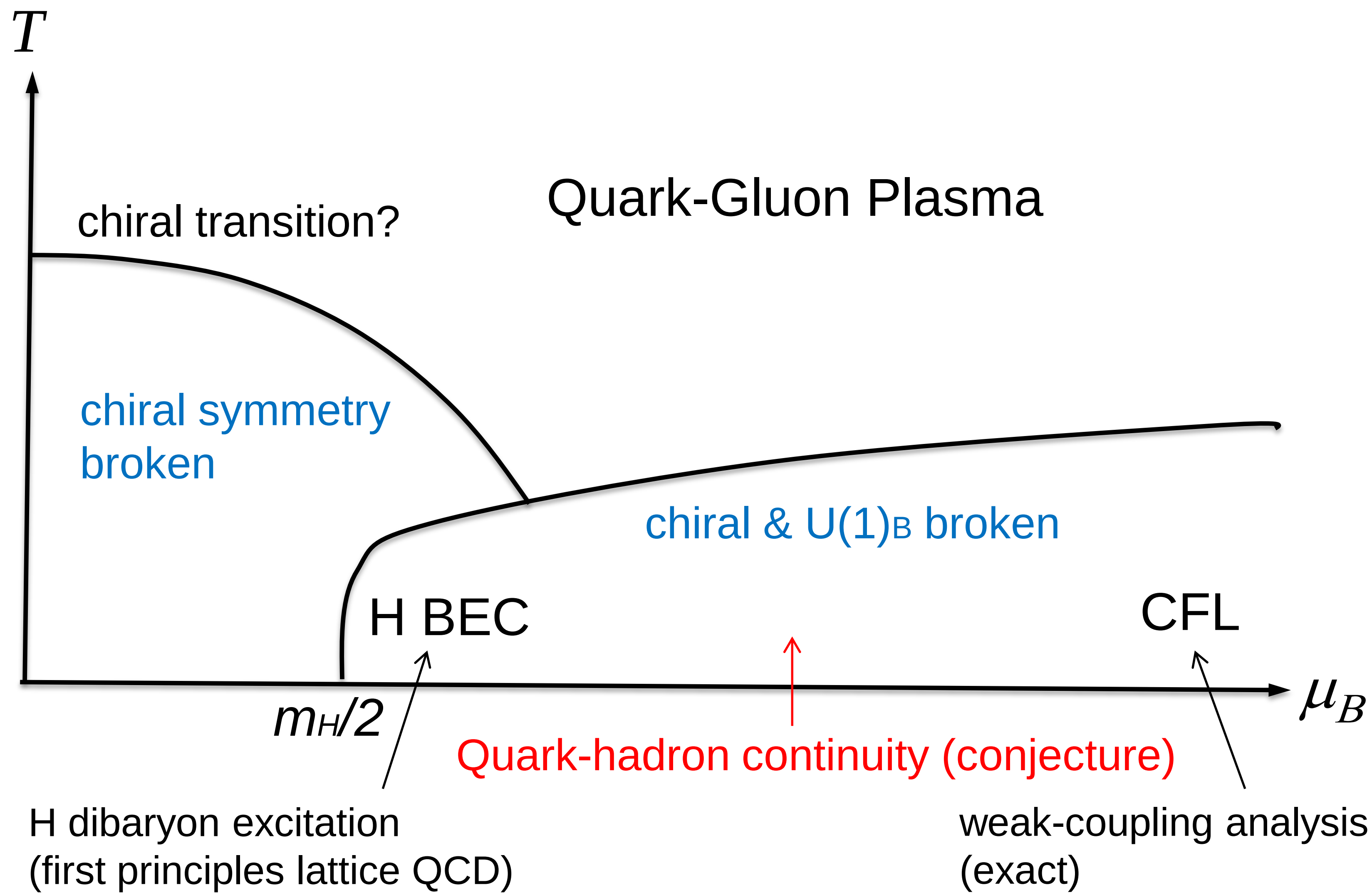
  → Chiral & $U(1)_B$ symmetries are broken.

- Same symmetry breaking pattern: quark-hadron continuity

# QCD phase diagram ($m_{u,d,s}$=0)



# ***Color-flavor locking (contd.)***

Elementary excitations

- Quarks: massive by the BCS pairing (Majorana mass).
- Gluons: massive by the Higgs mechanism ($SU(3)_C$ breaking).
- Light "pions": chiral & $U(1)_B$ symmetry breaking.

# ***Hadron/CFL correspondence ($N_c=N_f=3$)***

| Phase | Hadron (confined) | CFL (Higgs) |
|---|---|---|
| symmetry breaking | $SU(3)_L \times SU(3)_R \times U(1)_B$ $\rightarrow SU(3)_{L+R}$ | $SU(3)_L \times SU(3)_R \times SU(3)_C \times U(1)_B$ $\rightarrow SU(3)_{L+R+C}$ |
| (pseudo)scalars | 8 pseudo-scalars & 1 scalar | 8 pseudo-scalars & 1 scalar |
| vectors | vector mesons | gluons |
| fermions | baryons | quarks |
| coupling constant | strong-coupling | weak-coupling |
| quarks | confined | condensed |
| monopoles | ***(condensed?)*** | ***confined*** |

Existence of confined monopoles in CFL is recently shown.

Gorsky-Shifman-Yung; Eto-Nitta-NY ('11)

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| vectors | vector mesons | | gluons |
| fermions | baryons | one-to-one correspondence | quarks |
| coupling constant | strong-coupling | strong-weak dual? | weak-coupling |
| quarks | confined | | condensed |
| monopoles | ***(condensed?)*** | electromagnetic dual? | ***confined*** |

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# ***Summary & Outlook***

- Universality of phases in QCD and QCD-like theories at large $N_c$, which seems approximately valid even at $N_c$=3
  → future lattice QCD applications
- Universality can be shown in all effective & holographic models (PNJL, Sakai-Sugimoto models, ...): e.g., phase-quench for Polyakov loop up to one-meson-loop corrections. Hanada-Matsuo-NY, to appear.
- A no-go theorem for chiral critical phenomena.
- The quark-hadron continuity appears in the phase diagrams with flavor symmetry.
- Flavor symmetry breaking & finite $N_c$ effects may be crucial for the real dense matter.

*Back up slides*

# ***What are (aren't) equivalent?***

Not all the quantities are equivalent in the orbifold equivalence.

- Projection symmetry must be unbroken.
- Observables must keep the projection symmetry (neutral).
- Symmetry breaking patterns, quantum numbers of the condensates can be different, but their magnitudes are the same.
- Example: BCS gap (inside the BEC-BCS crossover)

$$\Delta^{\mathrm{SU}}_{\mu_B} \sim \mu \exp\left(-\frac{\pi^2}{g}\sqrt{\frac{6N_c}{N_c+1}}\right) \longrightarrow 0$$

'$t$ Hooft limit (large $N_c$, $g^2N_c$ fixed)

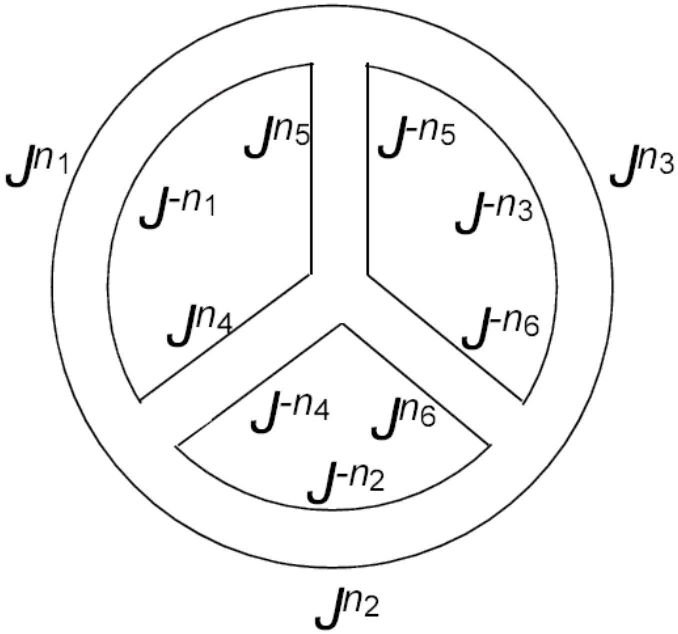
$$\Delta^{\mathrm{SU}}_{\mu_I} \sim \mu \exp\left(-\frac{\pi^2}{g}\sqrt{\frac{6N_c}{N_c^2-1}}\right)$$

$$\Delta^{\mathrm{SO}}_{\mu_B} \sim \mu \exp\left(-\frac{\pi^2}{g}\sqrt{\frac{12}{2N_c-1}}\right)$$

$$\Delta^{\mathrm{Sp}}_{\mu_B} \sim \mu \exp\left(-\frac{\pi^2}{g}\sqrt{\frac{12}{2N_c+1}}\right)$$

$$\longrightarrow \sim \mu \exp\left(-\pi^2\sqrt{\frac{6}{g^2N_c}}\right)$$

# ***Perturbative proof***




Bershadsky-Johansen ('98)

- Insert $\mathcal{P}(A_\mu^{\rm SO}) = \frac{1}{2}\left(A_\mu^{\rm SO} + J_c A_\mu^{\rm SO} J_c^{-1}\right)$ for each propagator.
- Take the same 't Hooft coupling.
- Difference comes from color factors.
- Condition: ${\rm tr}(J_c^n) = 0$, when $J_c^n \neq \pm \mathbf{1}_{2N_c}$

$$\sum_{n_i=0,1} \left(\frac{1}{2}\right)^{N_P} \cdot {\rm tr}(J^{-n_1}J^{n_4}J^{n_5}) \cdot {\rm tr}(J^{-n_2}J^{-n_4}J^{n_6}) \cdot {\rm tr}(J^{-n_3}J^{-n_5}J^{-n_6}) \cdot {\rm tr}(J^{n_1}J^{n_2}J^{n_3})$$

$$= 2^{-6} \cdot 2^{6-3} \cdot 2^4 = 2$$

Generally, $2^{-N_P} \cdot 2^{N_P-(N_L-1)} \cdot 2^{N_L} = 2$ for any planar diagrams.

# From SO(2$N_c$) to SU($N_c$) at finite $\mu_I$

Start with SO(2$N_c$) gauge theory at $\mu_B$>0.

1. Discrete symmetry: $J_c = -i\sigma_2 \otimes \mathbf{1}_{N_c} \in \mathrm{SO}(2N_c)$
$$J_i = -i\sigma_2 \otimes \mathbf{1}_{N_f/2} \in \mathrm{SU}(2)_{\mathrm{iso}}$$
2. Projection: $A_\mu^{\mathrm{SO}} = J_c A_\mu^{\mathrm{SO}} J_c^{-1}, \quad \psi^{\mathrm{SO}} = J_c \psi^{\mathrm{SO}} J_i^{-1}$
3. Daughter theory: U($N_c$) ≈ SU($N_c$) gauge theory at $\mu_I$>0.
$$A_\mu^{\mathrm{proj}} = \begin{pmatrix} (A_\mu^{\mathrm{U}})^C & 0 \\ 0 & A_\mu^{\mathrm{U}} \end{pmatrix}, \quad \psi^{\mathrm{proj}} = \begin{pmatrix} \psi_+^{\mathrm{U}} \\ \psi_-^{\mathrm{U}} \end{pmatrix}$$
4. Orbifold equivalence: $\langle\bar{\psi}\psi\rangle^{\mathrm{SO}} = \langle\bar{\psi}\psi\rangle^{\mathrm{SU}}$ etc.

**[Caution]**

- $Z_4 \in \mathrm{SU}(2)_{iso}$ unbroken from SO(2$N_c$) to QCD at $\mu_I$>0 everywhere.

# Approximate universality in real QCD

- One can check the universality (e.g. BCS gap) at sufficiently large $\mu$:

$$\Delta_{\mu_B}^{\mathrm{SO}} \sim \mu \exp\left(-\frac{\pi^2}{g}\sqrt{\frac{12}{2N_c-1}}\right)$$

$$\Delta_{\mu_I}^{\mathrm{SU}} \sim \mu \exp\left(-\frac{\pi^2}{g}\sqrt{\frac{6N_c}{N_c^2-1}}\right)$$

$$\Delta_{\mu_B}^{\mathrm{Sp}} \sim \mu \exp\left(-\frac{\pi^2}{g}\sqrt{\frac{12}{2N_c+1}}\right)$$

ratios

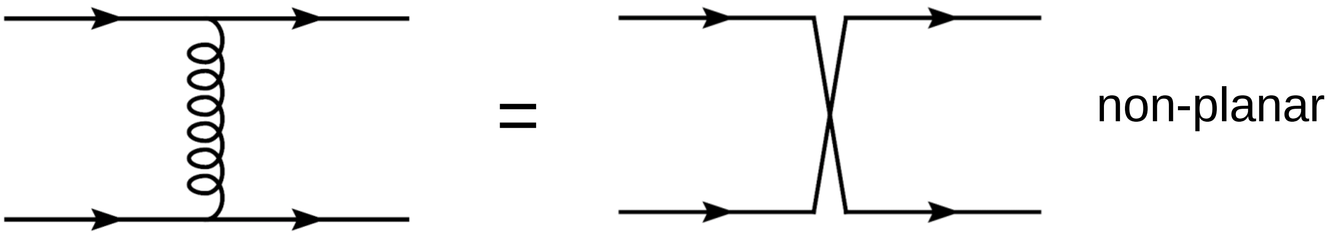
$$\sqrt{\frac{2(N_c^2-1)}{N_c(2N_c-1)}} = \begin{cases} 1.033 & (N_c = 3) \\ 1 & (N_c = \infty) \end{cases}$$

$$\sqrt{\frac{2(N_c^2-1)}{N_c(2N_c+1)}} = \begin{cases} 0.873 & (N_c = 3) \\ 1 & (N_c = \infty) \end{cases}$$

- Universality of phase diagrams can also be shown in effective models of QCD, e.g., chiral random matrix models.
  - chiral unitary matrix model — Klein-Toublan-Verbaarschot ('03)
  - generalized to all the universality classes — Hanada-NY ('11)

# ***Remarks***

- QCD phase diagram at $\mu_B$>0 (nuclear domain) qualitatively changes as a function of $N_c$ inside BEC/BCS region:
  - no nuclear liquid-gas transition at large $N_c$ Torrieri-Mishustin ('10).
  - no color superconductivity Deryagin-Grigoriev-Rubakov ('92); Shuster-Son ('00).




- QCD phase diagram at $\mu_B$>0 is similar between $N_c$=3 and large $N_c$ outside BEC/BCS region: chiral symmetry breaking/restoration.
- The whole phase diagrams of QCD-like theories (including BEC/BCS region) do not change qualitatively as a function of $N_c$.
- Universality is valid where strong $N_c$-dependence is absent .

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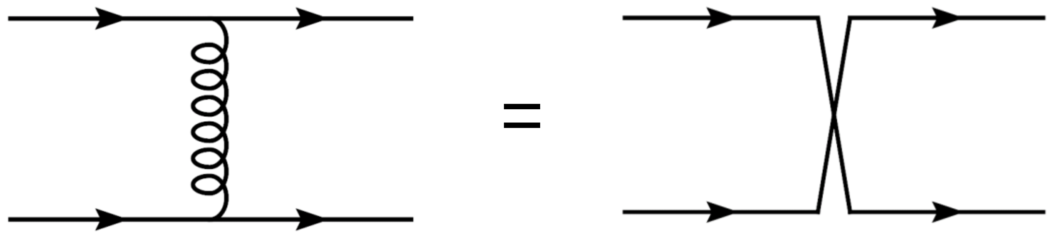


non-planar

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