Diagrammatic Monte Carlo for Strong Coupling LQCD

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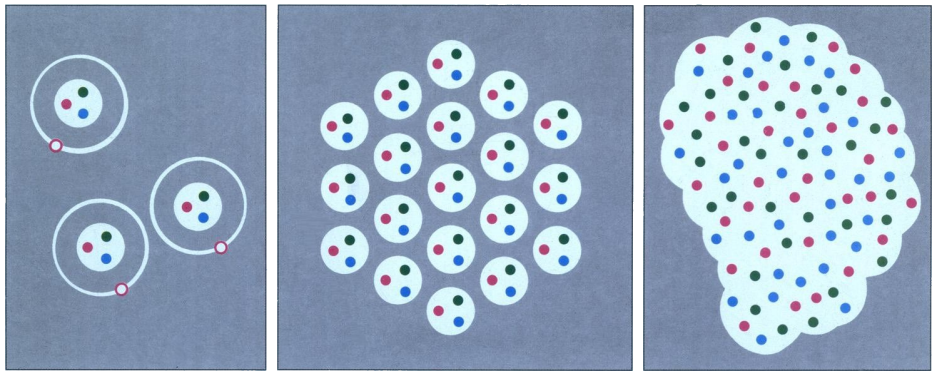

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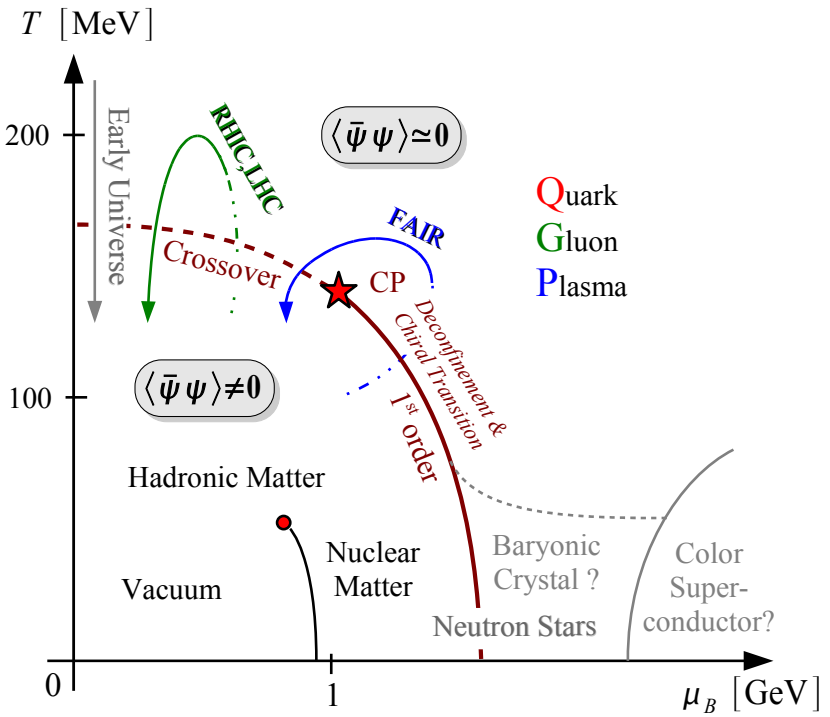
The QCD (μ, T) **phase diagram**:



- rich phase structure conjectured
- chiral and deconfinement transition
- QGP at high temperatures
- exotic matter at high density

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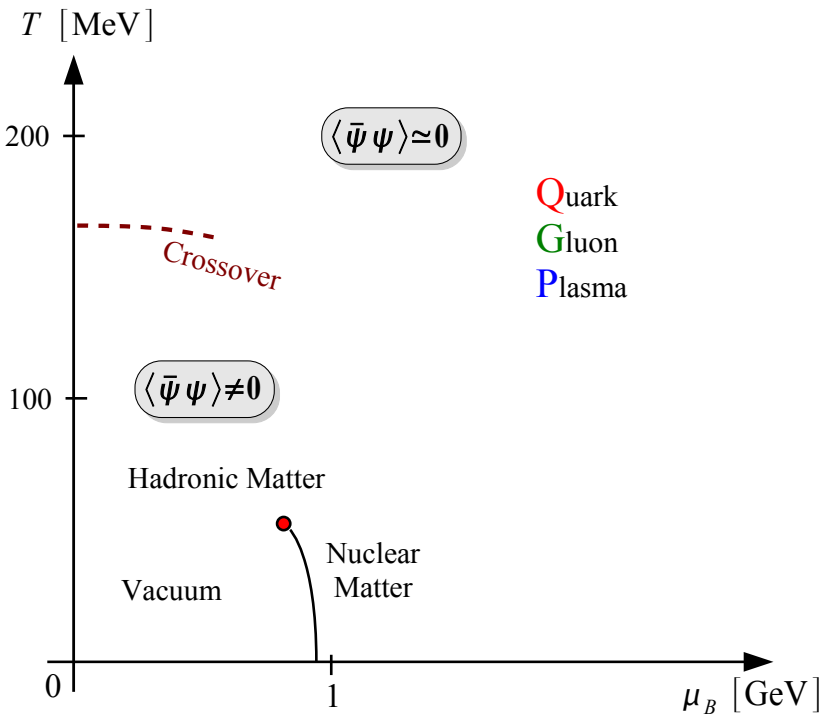

- because of the sign problem: very little is known
- only recently: agreement on crossover temperature T_c at zero baryon chemical potential

QCD has a severe **sign problem** for finite chemical potential $\mu = \frac{1}{3}\mu_B$:

- fermions anti-commute:

$$\gamma_5(i\not{p}+m+\mu\gamma_0)\gamma_5 = (i\not{p}+m+\mu\gamma_0)^\dagger$$

- the **fermion determinant** $\det M(\mu)$ becomes **complex!**

$$e^{-S_f} = \det M(\mu) = \overline{\det M(-\bar{\mu})}$$

- little hope that it can be circumvented:
  - Taylor expansion,
  - imaginary μ with analytic continuation,
  - reweighting method

  are all **limited to small** $\mu/T \lesssim 1$

What is Strong Coupling Lattice QCD?

Look at QCD in a regime where the **sign problem** can be made mild:
This is obtained by changing the nature of integration variables:

— no sampling of gauge fields $\{U\}$!

— no fermion determinant (no HMC)!

Staggered QCD in the strong coupling limit:

- start from the "1-flavor" staggered QCD Lagrangian in Euclidean time: $\mathcal{L}_{\rm QCD} = \sum_\nu \frac{1}{2}\eta_\nu(x)\left(e^{\mu\delta_{\nu 0}}\bar{\chi}(x)U_\nu(x)\chi(x+\hat{\nu}) - e^{-\mu\delta_{\nu 0}}\bar{\chi}(x+\hat{\nu})U_\nu^\dagger(x)\chi(x)\right) + am_q\bar{\chi}\chi$
$-\frac{\beta}{2N_c}\sum_P\left(\mathrm{tr}\,U_P + \mathrm{tr}\,U_P^\dagger\right) + \mathcal{O}(a^2)$

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- take limit of **infinite gauge coupling:** $g \to \infty,\ \beta = \frac{2N_c}{g^2} \to 0$
- allows to integrate out the gauge fields completely, as **link integration factorizes**

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Drawback: strong coupling limit is converse to asymptotic freedom:

- degrees of freedom in SC-QCD live on a **crystal**
- fermions have **no spin** (taste splitting maximal)

Why Strong Coupling Lattice QCD?

1-flavor strong coupling QCD might appear crude, but

- exhibits **confinement**, i.e. only color singlet degrees of freedom survive:
  - **mesons** (rep. by monomers and dimers)
  - **baryons** (rep. by oriented self-avoiding loops)
- and **chiral symmetry breaking/restoration:**

$$\mathrm{U}_A(1): \qquad \chi(x) \mapsto e^{i\eta_5(x)\theta_A}\chi(x),\ \bar{\chi}(x) \mapsto \bar{\chi}(x)e^{i\eta_5(x)\theta_A}, \qquad \eta_5(x) = (-1)^{\sum_{\nu=0}^{d} x_\nu}$$

  is spontaneously broken below T_c,

hence its phase diagram might be **similar to the QCD phase diagram**

SC-LQCD as effective theory for **nuclear matter**:

- derive nuclear interactions between hadrons from (Lattice) QCD
- transition at high density is the **nuclear transition**

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SC-LQCD as effective theory for **nuclear matter**:

- derive nuclear interactions between hadrons from (Lattice) QCD
- transition at high density is the **nuclear transition**

Just another effective model for QCD?

- yes and no: think of SC-LQCD rather as a **1-parameter deformation of QCD**

SC-QCD Partition Function (1)

Partition function for SU(N_c):

$$\mathcal{Z} = \int \prod_x d\bar{\chi} d\chi e^{2am_q M(x)} \prod_\mu \Big\{ \sum_{k_\mu(x)=0}^{N_c} \frac{(N_c - k_\mu(x))!}{N_c! k_\mu(x)!} \left(M(x)M(x+\hat{\mu})\right)^{k_\mu(x)}$$

$$+ \kappa \left(\rho(x,y)^{N_c} \bar{B}(x) B(x+\hat{\mu}) + (-\rho(y,x))^{N_c} \bar{B}(x+\hat{\mu}) B(x) \right) \Big\}$$

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Only confined, colorless degrees of freedom remain after link integration ($N_c = 3$):

- **Mesons** $M(x) = \bar{\chi}(x)\chi(x)$, represented by **monomers** and **dimers** (non-oriented meson hoppings):
- **Baryons** $B(x) = \frac{1}{3!}\epsilon_{i_1 \dots i_3} \chi_{i_1}(x) \dots \chi_{i_3}(x)$ form self-avoiding **oriented loops**:

$$\rho(x,y) = \eta_{\hat{\mu}}(x) \exp(\pm a_\tau \mu) \delta_{\hat{\mu}\hat{0}}$$

$n(x) \in \{0, 1, \dots, 3\}$

$k_\mu(x) \in \{0, 1, \dots, 3\}$

$\bar{B}(x)B(y) \in \{0, 1\}$

- Note: baryons transform under gauge trafo $\Omega \in U(3)$ as $B(x) \to B(x) \det \Omega$
  $\Rightarrow$ **not gauge invariant**, only SU(3) contains baryons ($\kappa = 1$), U(3) purely mesonic ($\kappa = 0$)

SC-QCD Partition Function (2)

Exact rewriting of SC-QCD partition function (no approximation!):

$$\mathcal{Z}(m_q, \mu, N_\tau) = \sum_{\{k,n,l\}} \prod_{b=(x,\hat{\mu})} \frac{(3-k_b)!}{3!k_b!} \prod_x \frac{3!}{n_x!} (2am_q)^{n_x} \prod_l w(\ell, \mu)$$

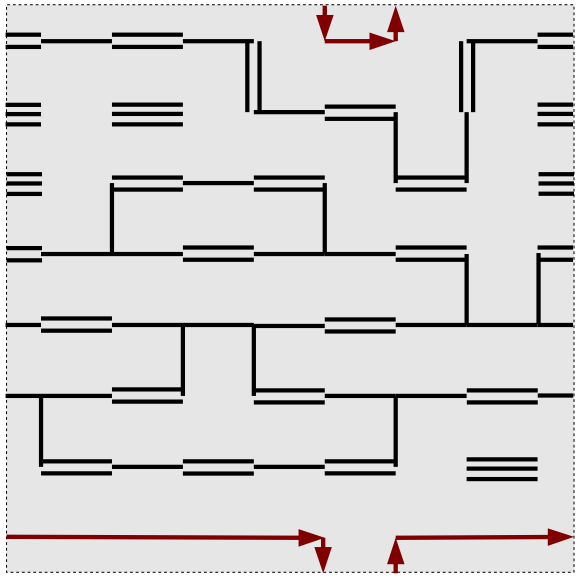
- **Grassmann constraint:**
  color neutral states at each site

$$n_x + \sum_{\hat{\mu}=\pm\hat{0},\ldots\pm\hat{d}} k_{\hat{\mu}} = 3 \qquad \forall x \in {N_\sigma}^3 \times N_\tau$$

- weight for baryon loop l (sign $\sigma(\ell)$):

$$w(\ell, \mu) = \frac{1}{\prod_{x\in\ell} 3!} \sigma(\ell) e^{3N_\tau r_l a_\tau \mu}$$

- in the following: restrict to **chiral limit**,
  where monomer density $\langle n \rangle = 0$



SC-LQCD at finite temperature

How to vary the temperature?

- $aT = 1/N_\tau$ is discrete with N_τ even
- $aT_c \simeq 1.5 \quad \Rightarrow \quad$ we cannot address the phase transition!

Solution: introduce an **anisotropy** γ in the Dirac couplings:

$$\mathcal{L}_{\mathrm{QCD}} = \sum_\mu \frac{\gamma^{\delta_{\mu 0}}}{2} \eta_\nu(x) \left(e^{\mu \delta_{\mu 0}} \bar{\chi}(x) U_\nu(x) \chi(x+\hat{\mu}) - e^{-\mu \delta_{\mu 0}} \bar{\chi}(x+\hat{\mu}) U_\mu^\dagger(x) \chi(x) \right)$$

Should we expect $a/a_\tau = \gamma$, as suggested at weak coupling?

- **No**: meanfield predicts $a/a_\tau = \gamma^2$, since $\gamma_c^2 = N_\tau \frac{(d-1)(N_c+1)(N_c+2)}{6(N_c+3)}$

  $\Rightarrow$ sensible, N_τ-independent definition of the temperature: $aT \simeq \frac{\gamma^2}{N_\tau}$

- Moreover: SC-QCD partition function is a function of γ^2:

$$\mathcal{Z}(m_q, \mu, \gamma, N_\tau) = \sum_{\{k,n,l\}} \prod_{b=(x,\mu)} \frac{(3-k_b)!}{3! k_b!} \gamma^{2 k_b \delta_{\mu 0}} \prod_x \frac{3!}{n_x!} (2am_q)^{n_x} \prod_l w(\ell, \mu)$$

However: **precise correspondence between a/a_τ and γ^2 not known**

SC-LQCD at finite Temperature and Continuous Time:

Strategy for **unambiguous** answer: the **continuous Euclidean time limit** (CT-limit):

$$N_\tau \to \infty, \qquad \gamma \to \infty, \qquad \gamma^2/N_\tau \quad \text{fixed}$$

- same as in analytic studies: $a_\tau = 0$, $aT = \beta^{-1} \in \mathbb{R}$

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Several **advantages** of continuous Euclidean time approach:

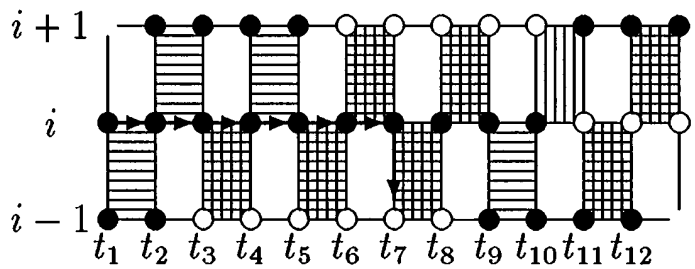
- ambiguities arising from the functional dependence of observables on the anisotropy parameter will be circumvented, **only one parameter** setting the temperature
- no need to perform the continuum extrapolation $N_\tau \to \infty$
- allows to estimate critical temperatures more precisely, with a faster algorithm (about 10 times faster than $N_t = 16$ at T_c)
- baryons become static in the CT-limit, the **sign problem is completely absent!**

Origin of Continuous Time / Diagrammatic Monte Carlo

Continuous time (CT) methods were introduced in **Quantum Monte Carlo**:
Suzuki-Trotter decomposition $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$ allows mapping onto classical spin system and sampling of diagrammatic expansion in continuous time

First proposed for the Heisenberg quantum antiferromagnet (Beard & Wiese, 1996):

1. persue a world-line formulation
   (i. e. rewrite the partition function in terms of transition/decay probabilities)



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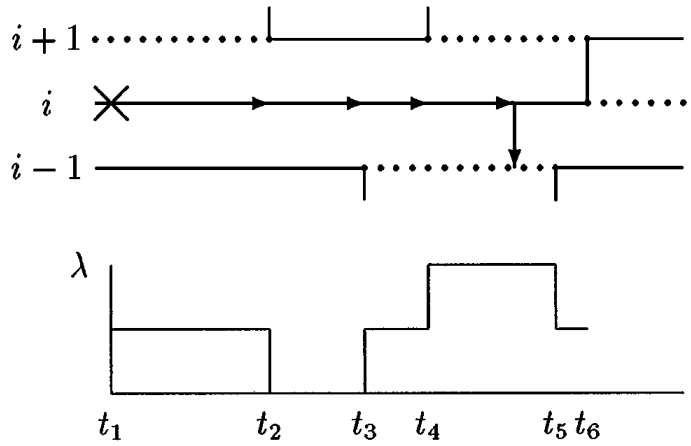

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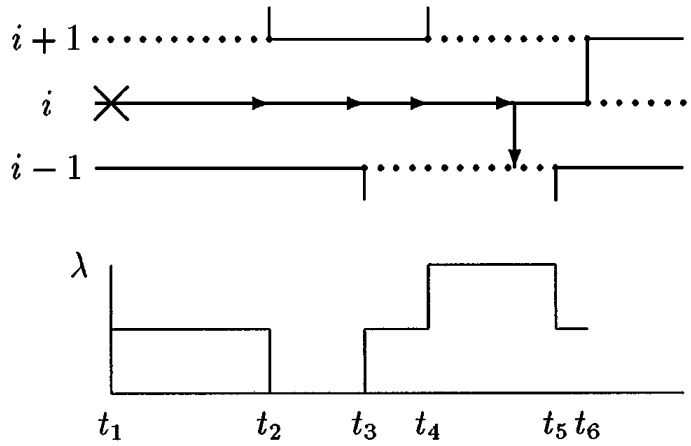

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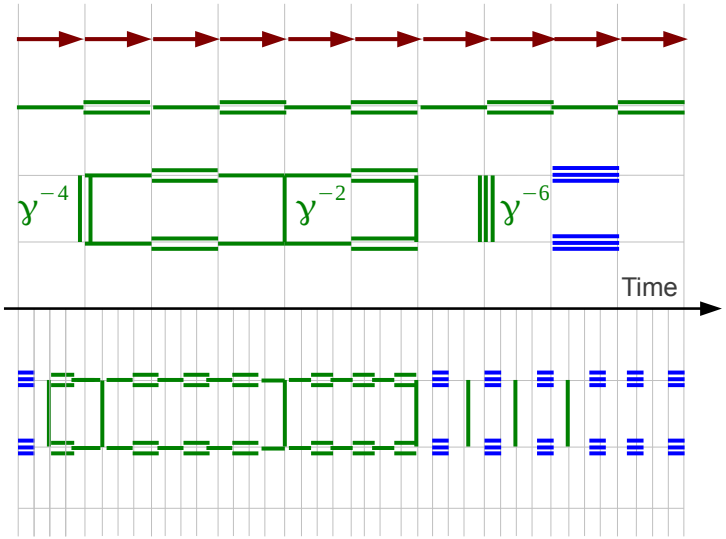
Situation today:

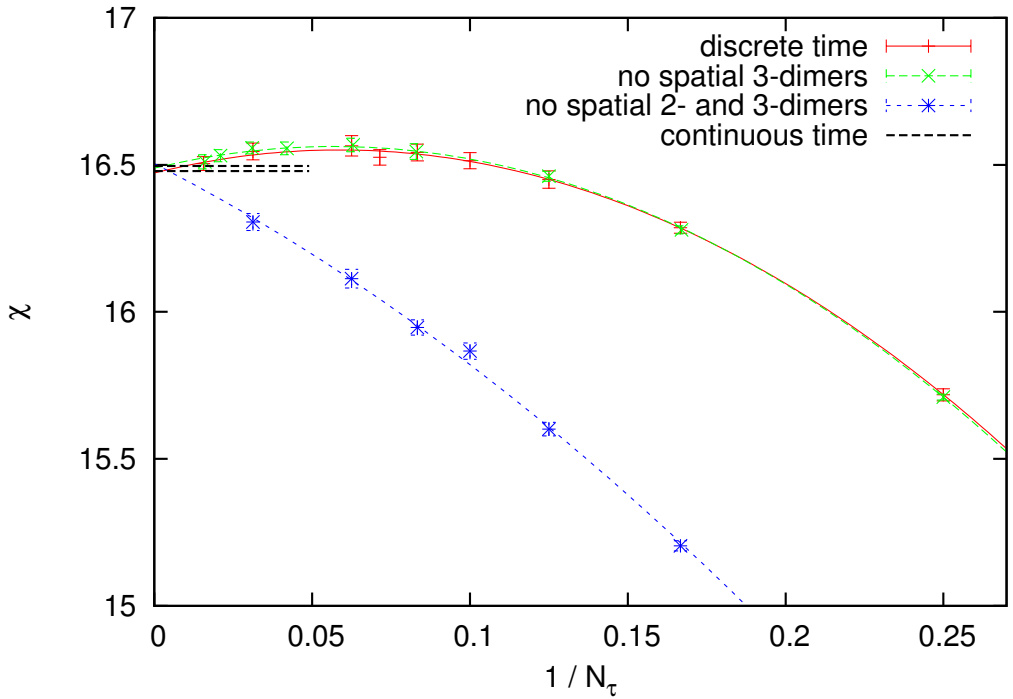
- rich literature for quantum impurity systems (see review by Gull *et al* 2010) based on diagrammatic Monte Carlo
- now: continuous time method applied to **gauge theories**

Continuous Time Partition Function (1)

In the following: restrict to chiral limit $m_q = 0$, where monomers are absent.
Key observation: **multiple spatial dimers are suppressed** with powers of γ^{-2}:

- double/triple spatial dimers become resolved into single dimers as $a_\tau \simeq a/\gamma^2 \to 0$
- single spat. dimers survive (constant density)
- baryonic spat. hops are suppressed with γ^{-1}

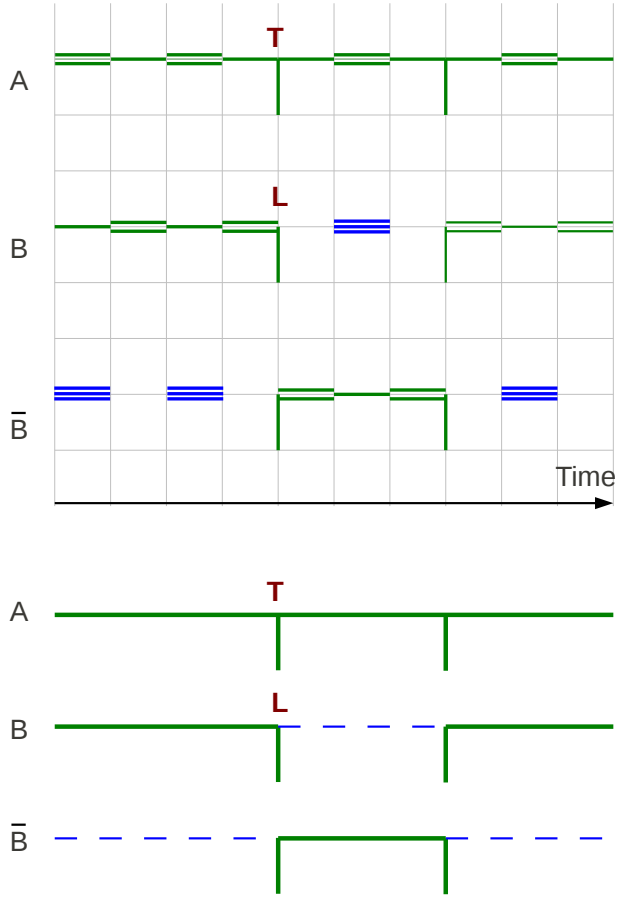




Continuous Time Partition Function (1)

The partition function can be decomposed into spatial and temporal parts, i. e.:

- in spatial dimers (meson **hoppings**) and two types of temporal intervals in between: **dashed** (3-0-3-0-...) and **solid intervals** (2-1-2-1-...)
- the weight of a configuration **only depends on** the kind and number of **vertices** at which spatial hoppings are attached to solid/dashed lines:
  - "L"-vertices of weight $v_L = 1$, and
  - "T"-vertices of weight $v_T = 2/\sqrt{3}$, where a solid line emits a spatial dimer.
- $n_L(x)$ and $n_T(x)$ denote the number of T-vertices and L-vertices at spatial position x.



Continuous Time Partition Function (1)

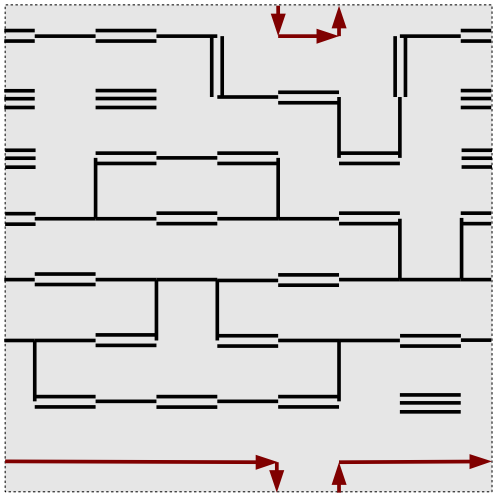
Partition function in the limit of large γ^2, N_τ with $T = \gamma^2/N_\tau$:

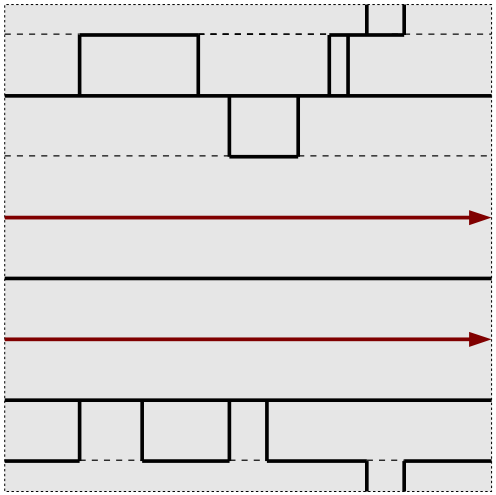
$$\mathcal{Z}(\gamma, N_\tau) \simeq \gamma^{3V} \sum_{\{k, n_B(x)\}} \prod_{x \in V_B} e^{3 n_B(x) \mu / T} \prod_{x \in V_M} \prod_{b=(x,\hat{i})} \frac{1}{3} \gamma^{-2k_b} \prod_{b_0=(x,\hat{0})} \frac{(3-k_{b_0})!}{3! k_{b_0}!}$$

$$= \sum_{\{k, n_B(x)\}} e^{3 n_B \mu / T} \prod_{x \in V_M} \left(\frac{v_L}{\gamma}\right)^{n_L(x)} \left(\frac{v_T}{\gamma}\right)^{n_T(x)}, \quad n_B(x) \in \{-1, 0, 1\}$$

with $V_M \dot{\cup} V_B = {N_\sigma}^3$ the mesonic/baryonic subvolumes and $n_B = \sum_x n_B(x)$ the baryon number

- typical 2-dimensional configurations in discrete and continuous time at the same temperature
- no multiple dimers, only **static baryons** in continuous time





Continuous Time Partition Function (3)

Final step: partition function in $\beta = 1/aT$:

- expansion in powers of γ^{-2},
  i. e. in total number of spatial hoppings: $\kappa = \frac{1}{2}\sum_{x\in V_M}(n_L(x) + n_T(x))$
- sum over all spatial hopping positions $\sim N_\tau/2$
  $\Rightarrow$ expansion in inverse temperature $\beta \simeq N_\tau/\gamma^2$:

$$\mathcal{Z}(\beta) = \sum_{\kappa\in 2\mathbb{N}} \frac{(\beta/2)^\kappa}{\kappa!} \sum_{\mathcal{C}\in\Gamma_\kappa} v_T^{n_T(\mathcal{C})} e^{\beta 3\mu B(\mathcal{C})}, \quad n_T = \sum_x n_T(x)$$

- Γ_κ is the set of equivalence classes of configurations with κ spatial hoppings
- $\mathcal{C} \in \Gamma_\kappa$ is a representative of all configurations obtained from shifts of spatial hoppings which leave their time ordering unchanged
- prefactor is weight for equivalence class of time ordered graph $\mathcal{C} \in \Gamma_\kappa$
  (weight does not depend on specific time coordinates):

$$\frac{1}{2^\kappa}\int_0^\beta dk_1 \int_{k_1}^\beta dk_2 \ldots \int_{k_{\kappa-1}}^\beta dk_\kappa v_T^{n_T(\mathcal{C})} = \frac{1}{\kappa!}\left(\frac{\beta}{2}\right)^\kappa v_T^{n_T(\mathcal{C})}$$

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Combinatorics governed by constraints concerning assignments of emission/absorption event to vertices:

- state vector χ characterizing time slice, $\chi(t) = (\chi_1, \ldots \chi_V)$, $\chi_x \in \{|0\rangle, \ldots |N_c\rangle\}$
- spatial dimers acts at time t_i on $\chi(t)$: $D_{<x,y>} = (J_x^{\pi(x)} J_t^{\pi(y)})$ with (for $N_c = 3$)
  $J^+ = \begin{pmatrix} 0 & & & & \\ v_L & 0 & & & \\ & v_T & 0 & & \\ & & v_L & 0 & \end{pmatrix}$, $J^- = (J^+)^T$ and $\pi(x) = \pm 1$ the parity of site x
- peridicity implies: $M\chi(0) = \chi(\beta) \equiv \chi(0)$, i. e. $M = \prod_{i=1}^{\kappa} D_{<x,y>t_i} \equiv \mathbb{1}$
- and the "Grassmann constraint" is: $J^+|N_c\rangle = 0$, $J^-|0\rangle = 0$ (forbidden)

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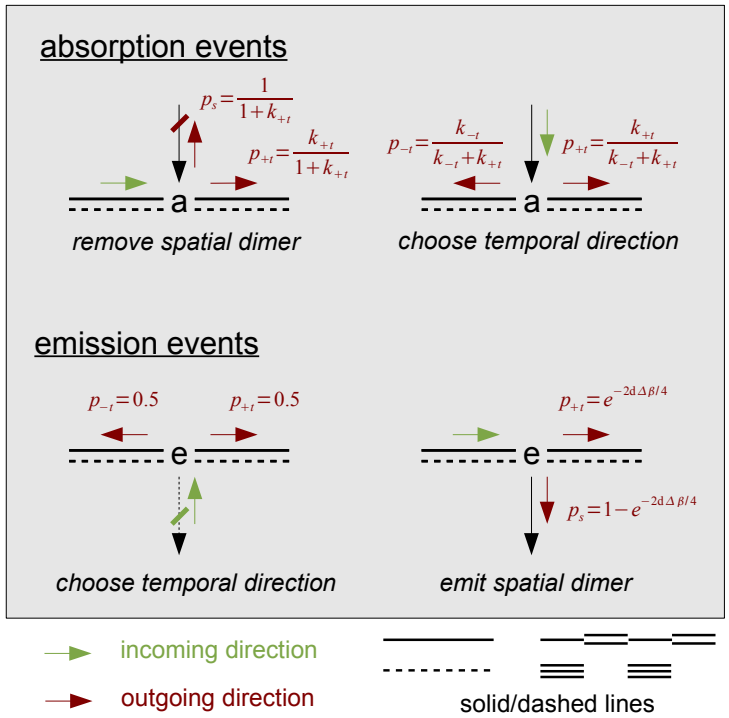
successfully used so far: **Worm algorithm in continuous time**
here: attempt **diagrammatic MC** based on the diagrammatic expansion in κ

Short Comment on CT-Worm Algorithm

CT Worm algorithm is derived from directed path Worm algorithm (Adams & Chandrasekharan, 2003):

- key step: hopping times are **uniformly distributed** and according to the statistics of a **Poisson process**:
- the dashed/solid interval lengths are **exponentially distributed:**

$$P(\Delta\beta) = e^{-\lambda\Delta\beta}, \quad \Delta\beta \in [0, \beta = 1/aT]$$



- λ is the "decay constant" for **spatial dimer emissions**:

$$\lambda = d_M(x)/4, \qquad d_M(x) = 2d - \sum_{\hat{\mu}=\pm i} |n_B(x+\hat{\mu})|$$

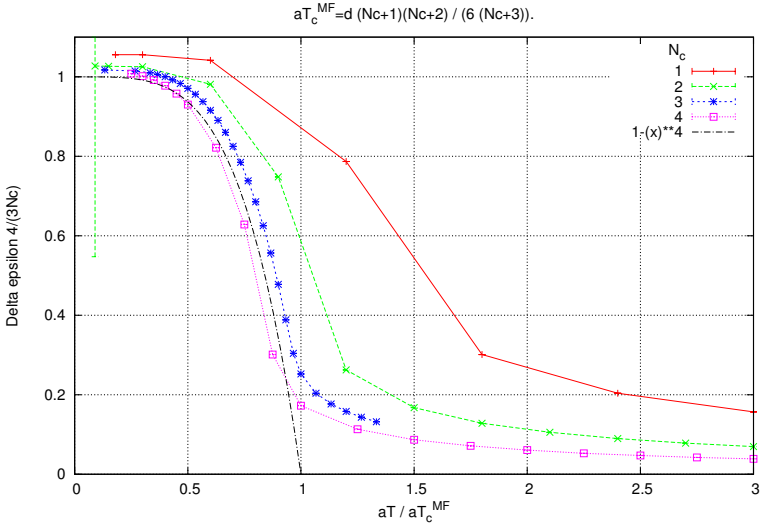
with $d_M(x)$ the number of mesonic neighbors at x

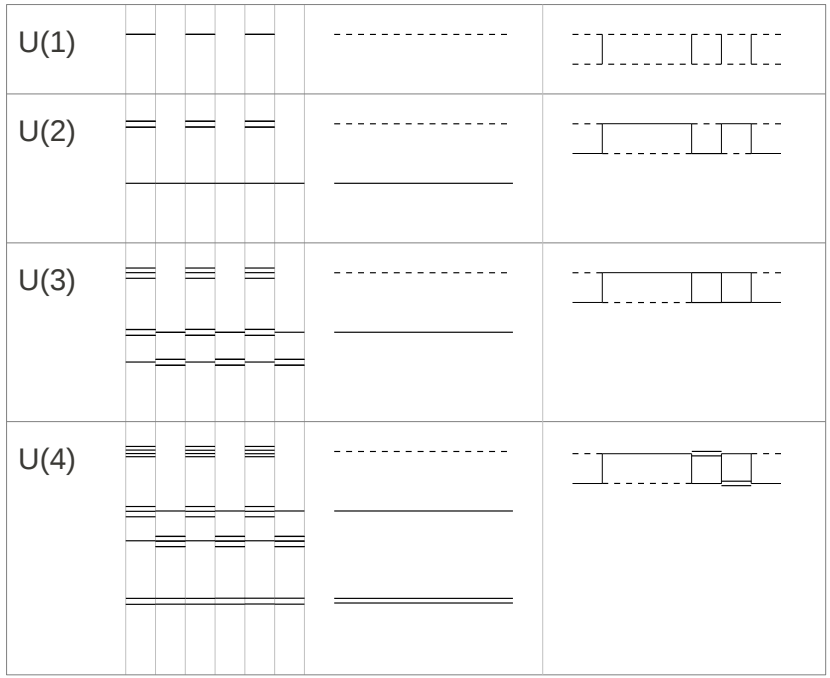
- idea of continuous time worm in SC-QCD: use **probability distribution** $p(\Delta\beta)$ for spatial dimer emission after time interval $\Delta\beta$

Generalization to arbitrary SU(N_c)

Continuous time methods can be applied to any gauge group SU(N_c):

- baryons only become static for SU(N_c) with $N_c \geq 3$
- $N_c = 2$: diquark loops can have spatial hoppings, no T-vertices present
- large N_c feasible





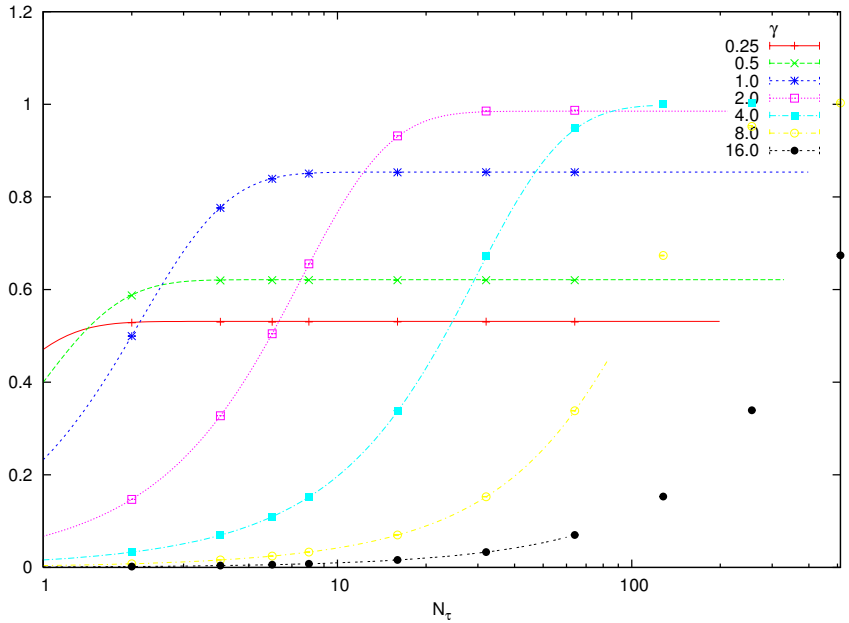
- list of line types:
  dashed, single, double, triple...
- for $N_c > 3$: generalization of T-vertices change line-type by one unit or its parity

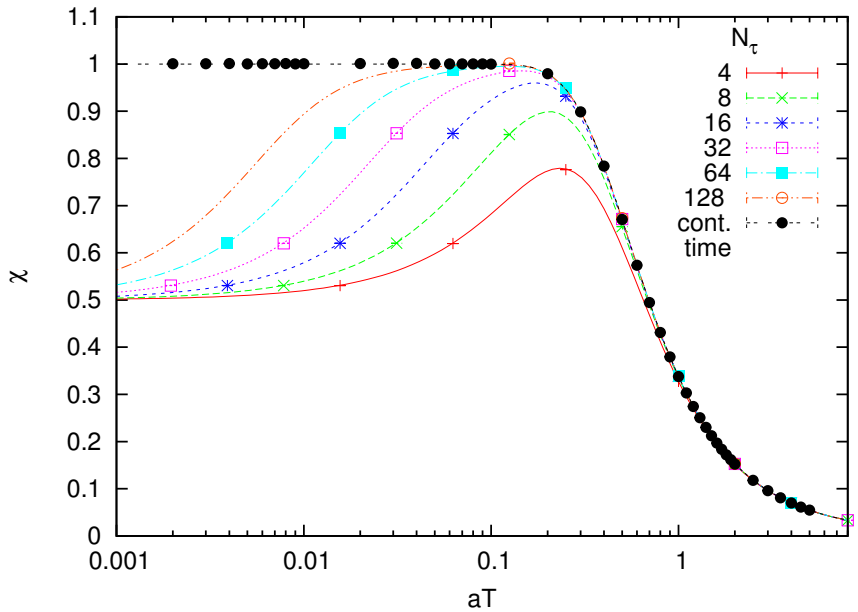
Crosscheck with analytic results for U(1)

Analytic solution of the U(1) system on $2xN_\tau$ latties:

$$\chi(N_\tau, \gamma^2) = \frac{1}{2}\tanh\left(\frac{N_\tau}{2}\mathrm{acsch}(\gamma^2)\right)\left((1+\gamma^{-4})^{-\frac{1}{2}} + \tanh\left(\frac{N_\tau}{2}\mathrm{acsch}(\gamma^2)\right)\right)$$

$$\chi(T) = \frac{1}{2}\tanh\left(1/2T\right)\left(1+\tanh\left(1/2T\right)\right)$$





→ Monte Carlo results on chiral susceptibility agrees well with analytic results

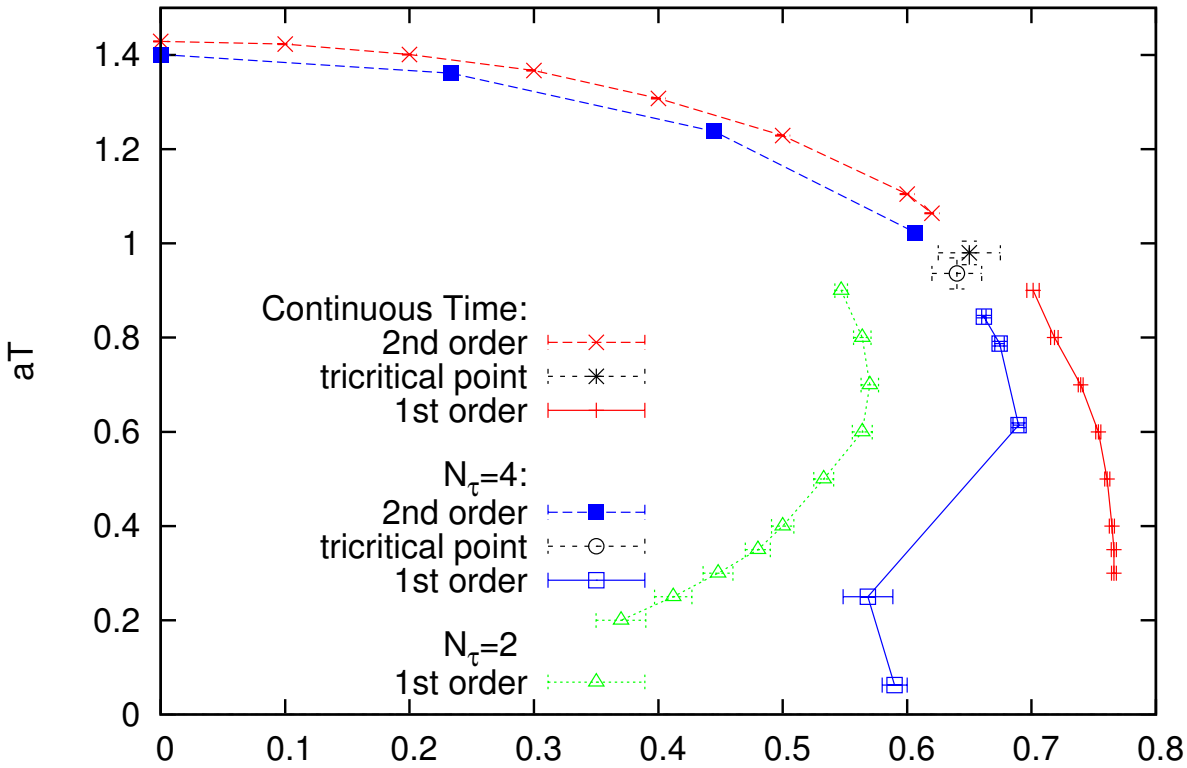
SC-QCD Phase Diagram

Studied via CT-Worm algorithm: arXiv:1111.1434 [hep-lat]
Comparison of phase diagram with $N_\tau = 4$ data (M. Fromm, 2010):

- CT-data compared to $N_\tau = 4$ data for identification

$$a\mu = \gamma^2 a_\tau \mu$$

- behavior at low μ agrees well, location of TCP agrees within errors
- no re-entrance is seen at small temperatures



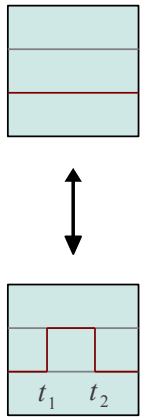
What is Diagrammatic Monte Carlo?

Importance sampling of diagrams described in terms of a perturbative series:

- each term in the diagrammatic expansion of the partition function is represented by a **world line configuration**
- perturbative series may not converge, but at any finite volume and temperature, only a finite number of orders contribute
- CT-Worm is one of the DMC techniqes, others are the loop cluster algorithm (Evertz et al.), stochastic series expansion (Sandvik)

Here: make use of insertions/shifts of world line decorations:

- Problem: **vanishing acceptance rate:**
  $p = (t\delta\tau)^2 \to 0 \quad \Rightarrow \quad p_{\text{ins}} = \min(1, p) \to 0$
- Solution: **integrate** over all possible insertions within interval Λ:
  $p = \int_0^\Lambda d\tau_1 \int_{\tau_1}^\Lambda d\tau_2 t^2 = \frac{\Lambda^2 t^2}{2} \neq 0 \quad \Rightarrow p_{\text{ins}} = \min(1, p) \neq 0$



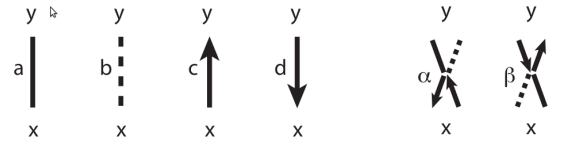
Motivation: Generalization of SC-QCD to 2 chiral flavors!

Aim: obtain phase diagram for 2-flavor SC-QCD, where **pion exchange** may play a crucial role for nuclear transition

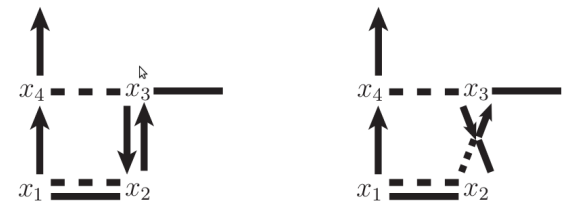
- but: at present, no 2-flavor formulation for staggered SC-QCD suitable for MC present
- already the mesonic sector has a severe (unphysical) **sign problem** in dimer representation
- 2 new types of flavored dimers give negative sign in mesonic loops already for U(2).

Observation in continuous time formulation:

- **flux representation** for 2 different flavors can be composed such that **cancellations appear**
- first step in this direction:
  Monte Carlo for insertion/removal of rectangles rather than worm algorithm



$$a = \bar{u} u_x \bar{u} u_y$$
$$b = \bar{d} d_x \bar{d} d_y$$
$$c = \bar{u} d_x \bar{d} u_y$$
$$d = \bar{d} u_x \bar{u} d_y$$
$$\alpha = \bar{u} d_x \bar{d} u_x \bar{u} u_y \bar{d} d_y$$
$$\beta = \bar{u} u_x \bar{d} d_x \bar{u} d_y \bar{d} u_y$$

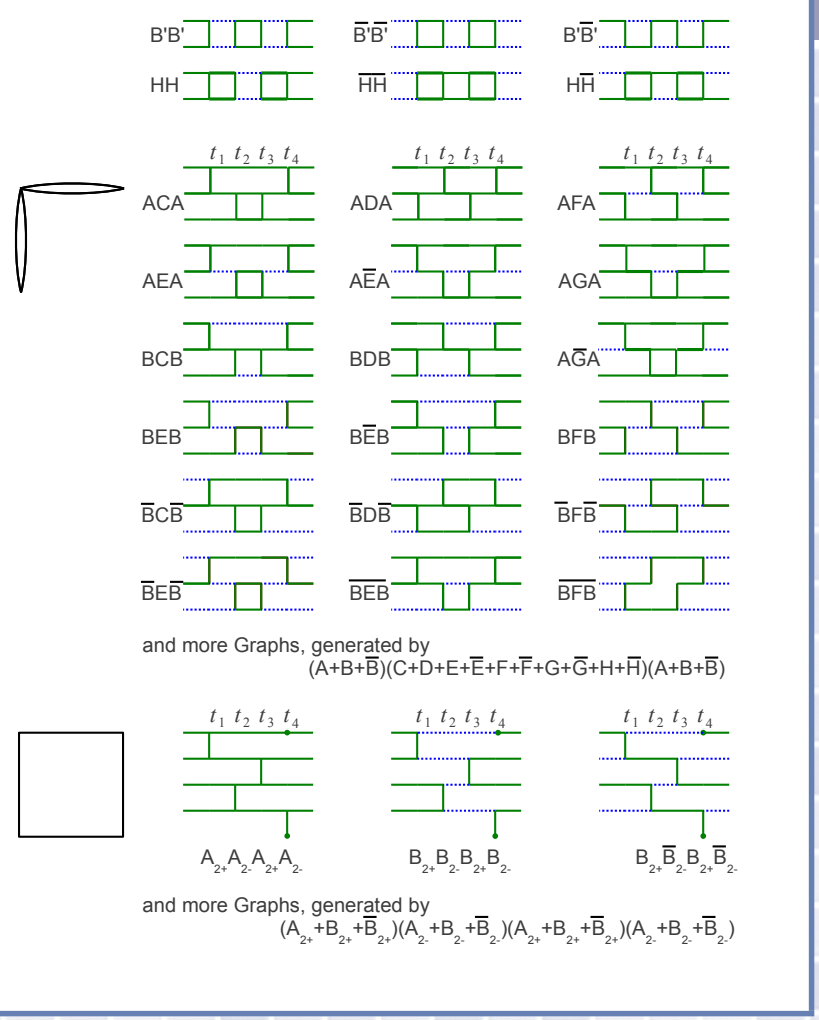


(Michael Fromm, PhD Thesis 2010)

The Hopping parameter Expansio

Diagrammatic expansion for SC-QCD amounts to **hopping parameter expansion** in $\beta/2$ on N_s^3 **with time ordering** of L- and T-vertices:

- label spatial dimers with time index $i = 1, \ldots \kappa$ and attach at its ends vertices with weight v_L, v_T
- enumeration of Γ_κ, i. e. all valid configurations consistent with parity (e. g. even/odd interval lengths, in sketch: even intervals between L- and T-vertices highlighted) not feasible



The Hopping parameter Expansion of CT-SC-QCD

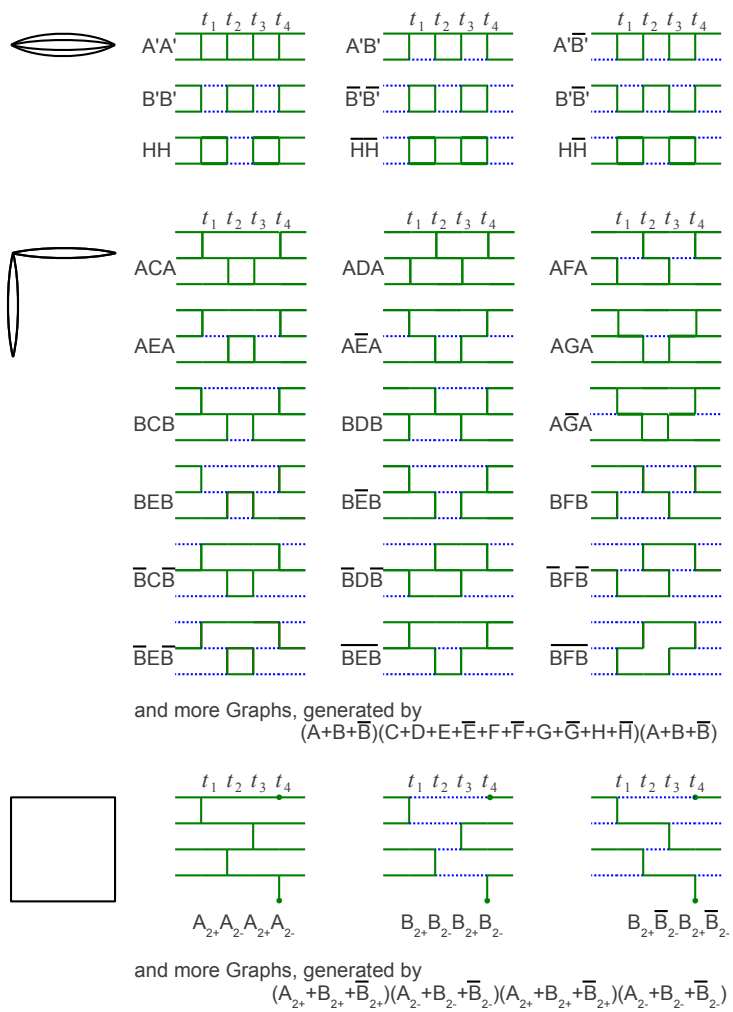
Diagrammatic expansion for SC-QCD amounts to **hopping parameter expansion** in $\beta/2$ on N_s^3 **with time ordering** of L- and T-vertices:

- label spatial dimers with time index $i = 1, \ldots \kappa$ and attach at its ends vertices with weight v_L, v_T
- enumeration of Γ_κ, i. e. all valid configurations consistent with parity (e. g. even/odd interval lengths, in sketch: even intervals between L- and T-vertices highlighted) not feasible

Data structure used in DMC for sampling Γ_κ:

- state vector $\chi(x, t = 0) \in \{0, \ldots N_c\}$
- set of time ordered spatial dimers $\{\langle x_1^{V_1}, y_1^{W_1}\rangle, \langle x_2^{V_2}, y_2^{W_2}\rangle, \ldots \langle x_\kappa^{V_\kappa}, y_\kappa^{W_\kappa}\rangle\}$ with $V_i, W_i \in \{L, T\}$

Some time-ordered diagrams of order $\kappa = 4$:



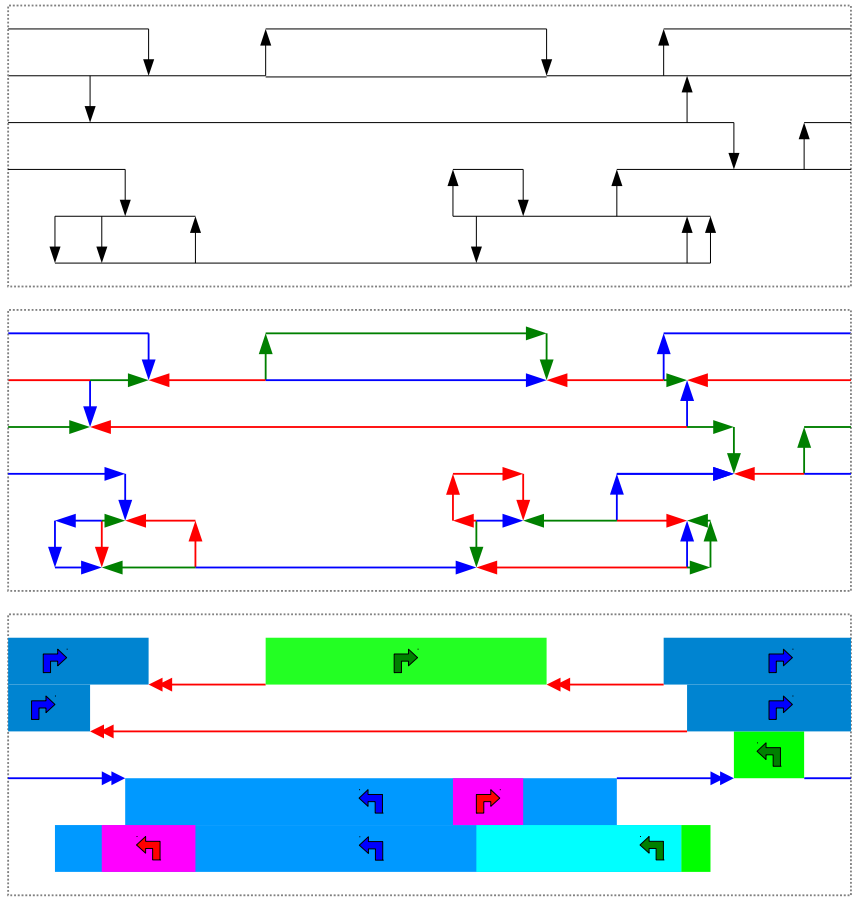
Flux Representation

Combinatorics of diagrams in Γ_κ goverened by assignment of

- emission/absorption events to vertices, or
- equivalently: even/odd lengths of time intervals

Observation: emission-absorption ordering induces **orientation on rectangles**:

- spatial dimers have orientation from emission to absorption site
- solid lines can be consistently oriented (colors for illustration)
- 2-dim. SC-QCD partition function can be conceived as composed of **static lines** and **oriented rectangles**



re-expressing a 2-dim. configuration in terms of oriented rectangles and oriented static lines

Algorithmic Details (1)

Apply Metropolis-Hastings Alogrithm to SC-QED:

- configuration with κ spatial dimers has weight $w(C^{(\kappa)}) = \frac{1}{\kappa!}\left(\frac{\beta}{2}\right)^{\kappa}$, κ always even
- insertion: $p_{\text{ins}}\left(C_1^{(\kappa)} \to C_2^{(\kappa+2)}\right) = \min\left[1, \frac{\mathrm{w}(\mathrm{C}_2^{(\kappa+2)})}{\mathrm{w}(\mathrm{C}_1^{(\kappa)})} \frac{\mathrm{A}_{\text{rem}}(\mathrm{C}_2^{(\kappa+2)} \to \mathrm{C}_1^{(\kappa)})}{\mathrm{A}_{\text{ins}}(\mathrm{C}_1^{(\kappa)} \to \mathrm{C}_2^{(\kappa+2)})}\right]$
- removal: $p_{\text{rem}}\left(C_1^{(\kappa)} \to C_2^{(\kappa-2)}\right) = \min\left[1, \frac{\mathrm{w}(\mathrm{C}_2^{(\kappa-2)})}{\mathrm{w}(\mathrm{C}_1^{(\kappa)})} \frac{\mathrm{A}_{\text{ins}}(\mathrm{C}_2^{(\kappa-2)} \to \mathrm{C}_1^{(\kappa)})}{\mathrm{A}_{\text{rem}}(\mathrm{C}_1^{(\kappa)} \to \mathrm{C}_2^{(\kappa-2)})}\right]$
- **proposal probabilities $A_{\text{ins}}(C_1 \to C_2)$, $A_{\text{rem}}(C_2 \to C_1)$ not indepenent of C_1, C_2**, need to be calculated for each update in an efficient way
- proposal probabilities depend on the way how insertion, removal is proposed
- several options how to implement rectangle updates, here: split into two parts, **insertion/removal of pair** (easy part, local), moving them apart (**shift update** difficult part, non-local)

Algorithmic Details (2)

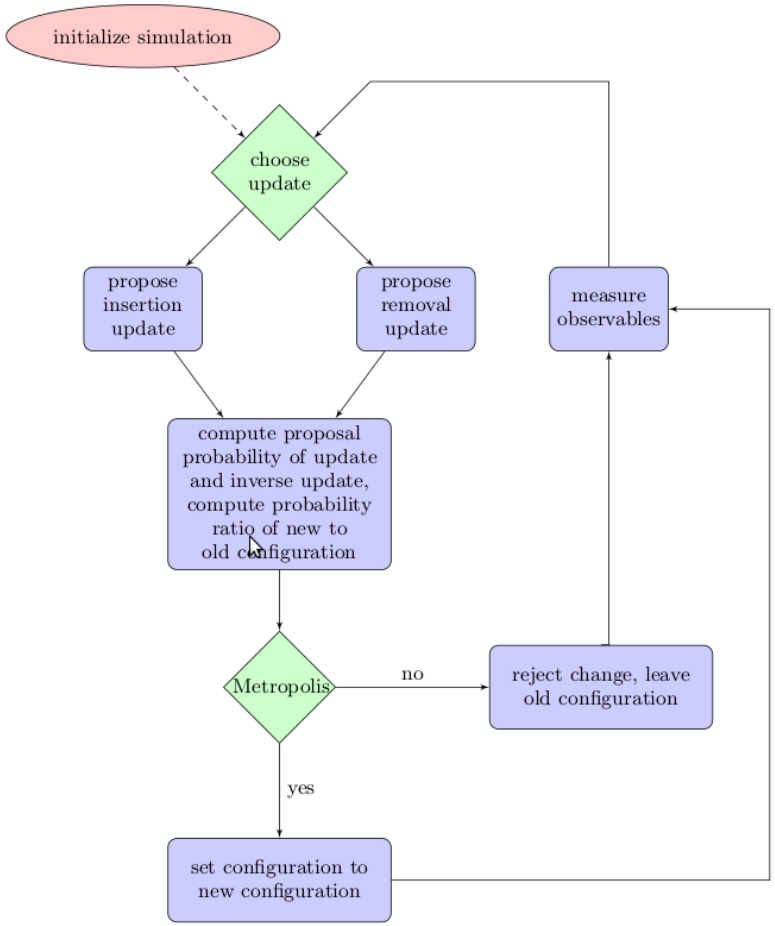
Apply Metropolis-Hastings Alogrithm to SC-QED:

- insertion update: $\kappa \mapsto \kappa + 2$:
  - choose a specific insertion for C_1
  - determine number n_p^{ins} of possible insertions (cheap: at each bond and t_i insertion is possible if parity agrees)
  - determine number of distinct insertions n_c^{ins} which would gives the same new config. C_2

  $\Rightarrow \quad A_{\mathrm{ins}}(C_1 \to C_2) = n_c^{\mathrm{ins}} / n_p^{\mathrm{ins}}$
  - determine number n_p^{rem} of possible removals in C_2 (cheap: computed along with possible insertions)
  - determine number of distinct removals n_c^{rem} which would gives the same old config. C_1

  $\Rightarrow \quad A_{\mathrm{rem}}(C_2 \to C_1) = n_c^{\mathrm{rem}} / n_p^{\mathrm{rem}}$
- removal update: $\kappa \mapsto \kappa - 2$: similar to the above

DMC flow chart:
insertion and **removal** updates for **pairs** of oppositely oriented spatial dimers

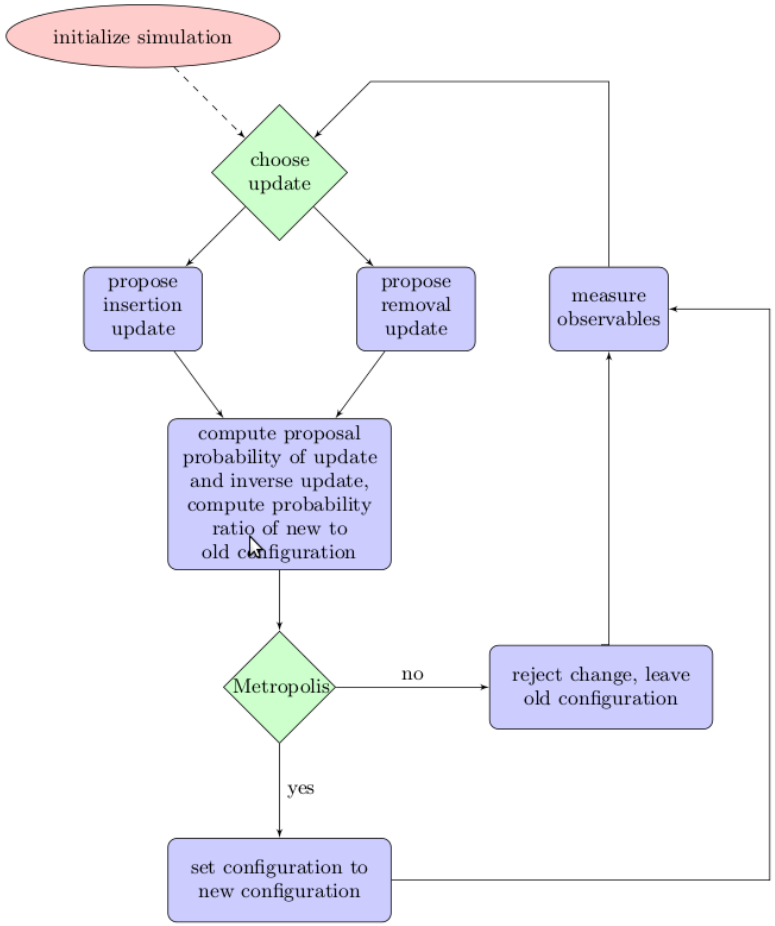


Algorithmic Details (2)

Apply Metropolis-Hastings Alogrithm to SC-QED:

- static line update: change $k_0(x) \mapsto k_0(x) \pm 1$, if consitent with vertices on site x
- shift update: needed for ergodicity, and in simplified insertion/removal (only pairs): essential!
  - A_{shift} trivial, but shifts may involve large number of flow inversions
  - need to construct clusters which spatial dimers are affected
  - demanding in eihter memory or CPU time

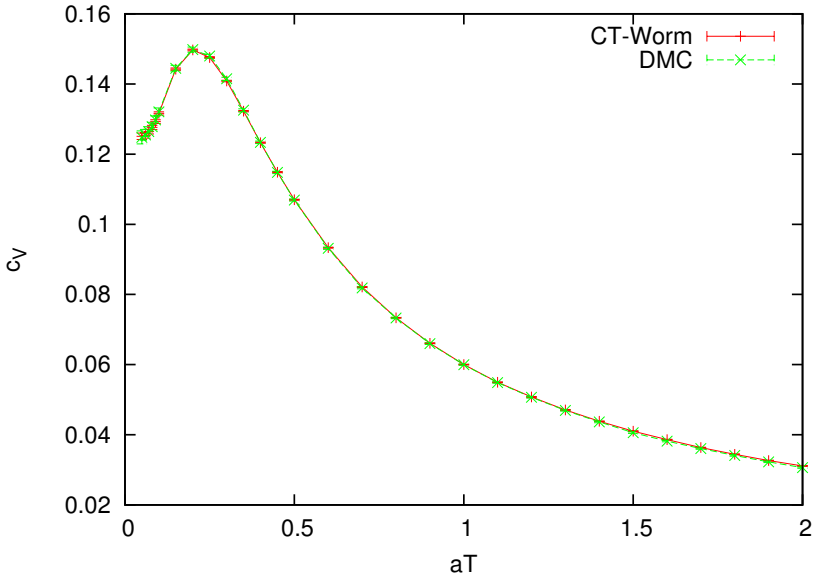
DMC flow chart:
insertion and **removal** updates for **pairs** of oppositely oriented spatial dimers

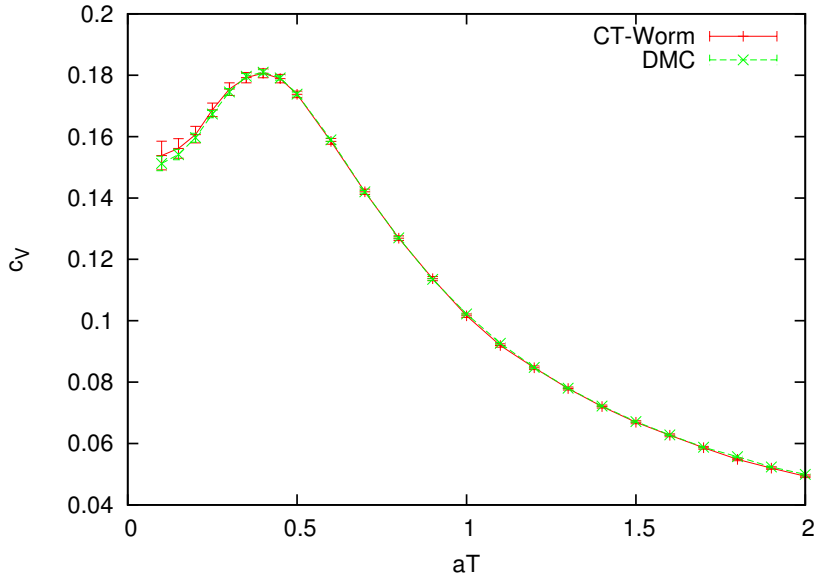


Preliminary Results

So far: only spatial dimer measurements in 1 spatial dimension possible

- First measurement: comparing CT-Worm with DMC
- specific heat (can be obtained from spatial dimers) for U(1) (left) and U(3) (right) on 1+1-dimensional lattice:





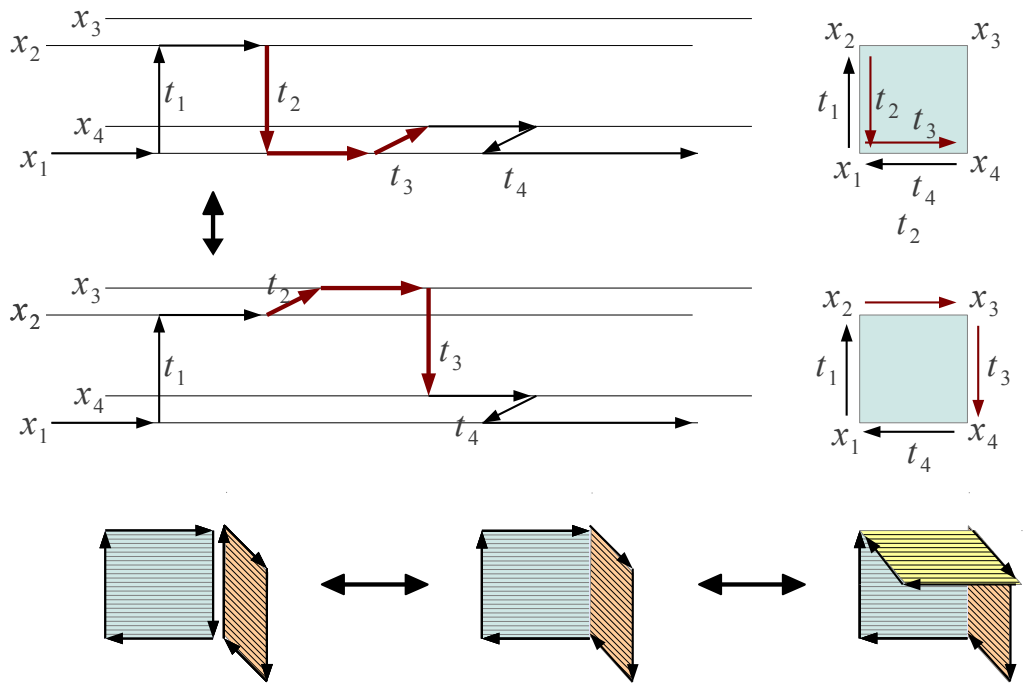
- observation: DMC performs better at small volumes, but not so well on large volumes (improvements on shift update?)
- also DMC becomes demanding in memory for large volumes (for shift updates)

Generalization to higher dimensions:

- so far: every graph composed of only planar rectangles
- in spatial dimensions $d > 1$, algorithm is not ergodic

Flip update:

- does not change order in β
- but might add/remove T-vertices $\rightarrow$ another Metropolis update ...
- sufficient to produce any higher-dimensional closed loops occuring in hopping parameter expansion



Conclusions

Prospects:

- CT partition function: new formulation for analytic treatment
- hope: extend formulation for two flavors (incorporates pion exchange)
- extension to SU(3) with finite baryon chem. potential straight forward

Drawbacks

- generalization to higher dimensions turns out to be very difficult
- not yet possible to study periodic boundary conditions
- in contrast to worm: no 2-point function for free
  (but: cluster-size as a measure for susceptibility?)
- might turn out that worm algorithm is more efficient at large volumes

Why Study Strong Coupling QCD on the Lattice?

Two possible scenarios for the relation between
SC-LQCD (back) and the (L)QCD phase diagram for four flavors (front):