Comments on lattice chiral symmetry and minimal doubling

Michael Creutz

Brookhaven Lab

Outline

- Chiral symmetry in mass parameter space
- The anomaly and lattice complications
- Topology and doublers
- Minimal doubling
- Point split operators
- Counterterms

Frame the discussion in the path integral formulation

$$Z = \int (dA)(d\psi)(d\overline{\psi}) \exp(-S_g(A) + \overline{\psi} D \psi) = \int (dA)|D(A)| e^{-S_g(A)}$$

- Concentrate on the determinant of the Dirac matrix $|D(A)|$

"Continuum" picture

$$\begin{aligned} D &= K + M \\ K &= \gamma_\mu(\partial_\mu + igA_\mu) = -K^\dagger = -\gamma_5 K \gamma_5 \\ M^\dagger &= M = \gamma_5 M \gamma_5 \end{aligned}$$

- introduce Θ later

Chiral symmetry as a symmetry in parameter space

- $[K, \gamma_5]_+ = 0$ implies

$$K = e^{i\omega_i \tau_i \gamma_5} K e^{i\omega_i \tau_i \gamma_5}$$

- since $(\mathrm{Tr})\tau_\mathrm{i} = 0$

$$|e^{i\omega_i \tau_i \gamma_5}| = \exp(i\omega_i \mathrm{Tr}\,(\tau_i \gamma_5)) = 1$$

- therefore

$$|K + M| = |K + e^{i\omega_i \tau_i \gamma_5} M e^{i\omega_i \tau_i \gamma_5}|$$

Physics unchanged with a modified mass matrix

$$M \longrightarrow e^{i\omega_i \tau_i \gamma_5} M e^{i\omega_i \tau_i \gamma_5}$$

Combined with the vector counterpart

$$M \longrightarrow e^{-i\omega_i \tau_i} M e^{i\omega_i \tau_i},$$

$SU(N_f) \otimes SU(N_f)$ symmetry

- symmetry of the massive theory in parameter space.

Specific example

- for two degerate flavors the mass terms

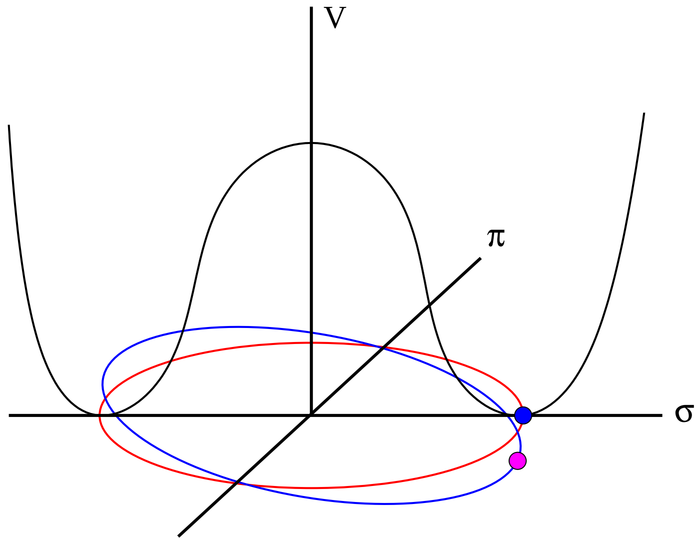
$$m\overline{\psi}\psi$$

$$-m\overline{\psi}\psi$$

$$im\overline{\psi}\tau_3\gamma_5\psi$$

- all give equivalent theories

Wine bottle analogy $V \sim (\sigma^2 + \vec{\pi}^2 - v^2)^2$



- all tilt directions are equivalent
- basis for twisted mass

lattice artifacts improved

The anomaly

$M \longrightarrow e^{i\theta\gamma_5} M$ not a valid symmetry

- mixes σ with η'
- η' gets extra mass from topology

  details depend on specific cutoff

  involve complications in defining γ_5

Fujikawa relates fermion measure and the index theorem

- in gauge field with topology, $K(A)$ has zero modes
- eigenstates of γ_5
- winding number $\nu = n_+ - n_-$

Use K to cutoff high modes of the Dirac operator

- zero modes contribute to $\mathrm{Tr}\ \gamma_5$

$$\mathrm{Tr}\gamma_5 \rightarrow \mathrm{Tr}(\gamma_5 e^{-K^\dagger K/\Lambda^2})$$
$$= \sum_i \langle \psi_i | \gamma_5 e^{-K^\dagger K/\Lambda^2} | \psi_i \rangle$$
$$= \nu \sim \int F\tilde{F}$$

- expand in powers of K

at K^4 ultraviolet divergence cancels $1/\Lambda^4$

In the path integral, configurations weighted by $\left| e^{i\theta \mathrm{Tr}\gamma_5} \right| = e^{i\theta N_f \nu}$

Leaving the usual CP violating parameter Θ of QCD

$m \leftrightarrow -m$ theories not equivalent for N_f odd

- -1 is not in $SU(2N+1)$
- $-m\overline{\psi}\psi$ gives the $\Theta = \pi$ theory
- spontaneous CP violation expected $(N_f > 1)$

  purely non-perturbative

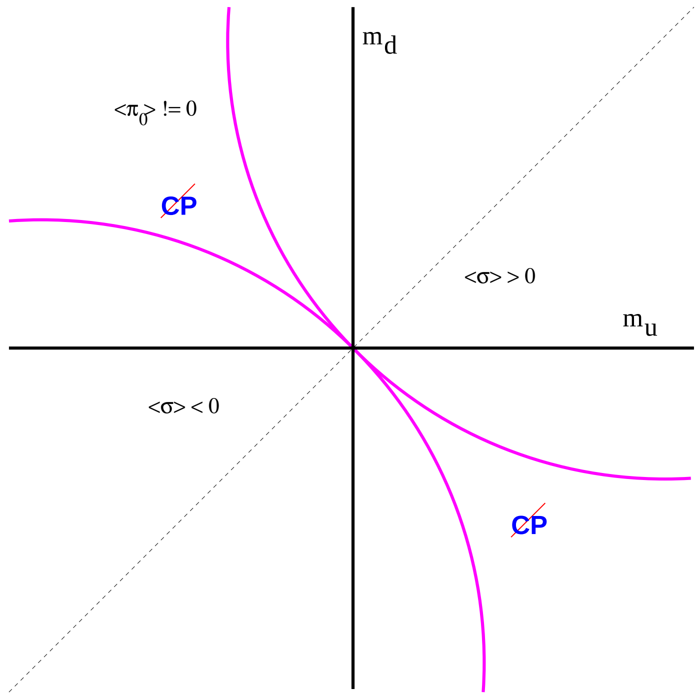
Two light flavors with non-degenerate mass

$$m_{\pi_0}^2 \sim \frac{m_u+m_d}{2} - c\,(m_u - m_d)^2 + O(m_q^3)$$

- c is a "low energy constant"
- associated with isospin breaking and $\pi_0\ \eta\ \eta'$ mixing

  lowers m_{π_0}

Negative $m_{\pi_0}^2$ can give pion condensation



Mass gap does not vanish when only one quark is massless

- no singularity along $m_u = 0$ axis
- scale dependence in defining a massless up quark

Isospin breaking

$$\frac{m^2_{\pi_0}}{m^2_{\pi_+}} = 1 - 2C\frac{(m_d - m_u)^2}{m_d + m_u}$$

- in continuum limit either

  C diverges

  quark mass ratios not constant

Mass independent regularization tricky

- not natural with lattice regulator
- perturbative matching requires caution

Lattice complications

Lattice at finite volume has no infinities

- how can an anomaly appear?

Consider any lattice Dirac operator D

- assume gamma five hermiticity $\gamma_5 D \gamma_5 = D^\dagger$

  all operators in practice satisfy this (except twisted mass)

Divide D into hermitean and antihermitean parts

$$K = (D - D^\dagger)/2$$

$$M = (D + D^\dagger)/2$$

Then

$$[K, \gamma_5]_+ = 0$$

$$[M, \gamma_5]_- = 0$$

$M \to e^{i\theta\gamma_5} M$ an exact symmetry of the determinant

- Where is the anomaly?

Naive fermions solve this with doublers

- half use γ_5 and half $-\gamma_5$
- the naive chiral symmetry is actually flavored

This carries through to staggered and minimally doubled actions

How about Wilson fermions?

- doublers given masses of order the cutoff
- the rotation $M \to e^{i\omega_i\gamma_5} M$ also rotates their phases

Physical Θ is a relative angle

- independently rotate the fermion mass and the Wilson term

  Seiler and Stamatescu

The overlap operator

- eigenvalues on a circle
- each zero eigenmode has counterpart on the opposite side
- rotation of Hermitean part rotates heavy mode as well
- anomaly brings in $\hat\gamma_5$

  $\nu = \mathrm{Tr}(\gamma_5 + \hat\gamma_5)/2$

Message for continuum QCD:

- physical Θ can be moved around
- placed on any one flavor at will

Θ can be entirely moved into the top quark phase

- some aspects of the top quark are relevant to low energy physics!
- decoupling theorems don't apply non-perturbatively

Topology and doublers

Above effectively a restatement of Nielsen-Ninomiya

Consider the quark propagator at large momentum

- $D = i\not{p} + O(m, gA_\mu)$

  a gauge choice needed to keep things smooth

Study the leading part $D = i\not{p}$

- maintain $[i\not{p}, \gamma_5]_+ = 0$

Convenient gamma matrix convention

$$\gamma_5 = \sigma_3 \otimes 1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma_4 = \sigma_2 \otimes 1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\vec{\gamma} = \sigma_1 \otimes \vec{\sigma} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

Then $i\not{p} = \begin{pmatrix} 0 & z(p) \\ -z^*(p) & 0 \end{pmatrix}$

with z a quaternion

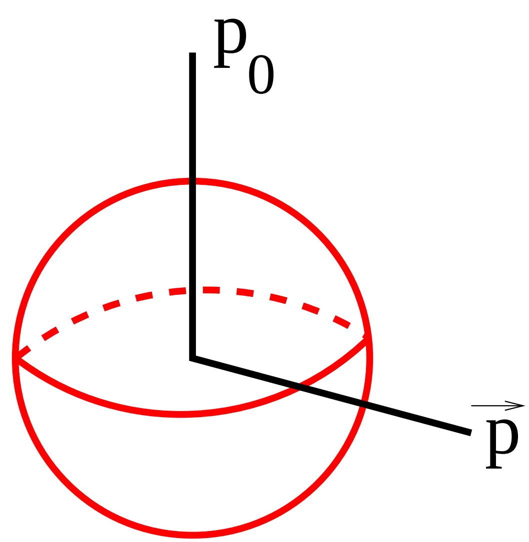
$$z(p) = p_4 + i\vec{p} \cdot \vec{\sigma} = |p| \; g(p) \qquad g(p) \in SU(2) \sim S_3$$

For the path integral we want

$$|D| = z^* z = |z|^2 = p_4^2 + \vec{p}^2$$

Consider a surface of constant $|D|$

- a sphere in four dimensions of constant radius $|p|$



Over this sphere, z wraps non-trivially

- around a similar sphere in quaternionic space

On the lattice

Momentum bounded to the Brilloin zone

$$-\pi/a < p_\mu \leq \pi/a$$

- the propagator must be periodic in momentum space.

Problem

- at the edge of the Brilloin zone

  periodicity forbids wrapping

The propagator must unwrap somewhere

- if z remains finite, there must be another zero.

Not a bad thing

- anomaly says one flavor QCD has no chiral symmetry
- with N_f flavors chiral symmetry is $SU(N_f) \otimes SU(N_f) \otimes U(1)_B$

  not $U(N_f) \otimes U(N_f)$.

Note: zeros at non-zero momentum not a problem

- redefining $\psi(x) \to e^{i\alpha_\mu x_\mu}\psi$
- moves a zero at $p_\mu = 0$ to $p_\mu = \alpha_\mu$

can always translate our zeros anywhere in momentum space.

Nielsen-Ninomiya conclusion:

- local lattice action that anticommutes with γ_5
- must have an even number of fermion species

Note: 3 is also bad

Wilson: unwrapping occurs at the doublers

- doublers given mass by Wilson term

Overlap: unwrapping occurs at the opposite side of the overlap circle

- each zero mode has a corresponding large real eigenvalue

Staggered: four tastes unwrap each other

- rooting: unwrapping forces unphysical singularities

  at $m_q = 0$ even with non-degenerate quarks

SLAC fermions: unphysical singularities in momentum space

Minimal doubling

- 2 species unwrap each other

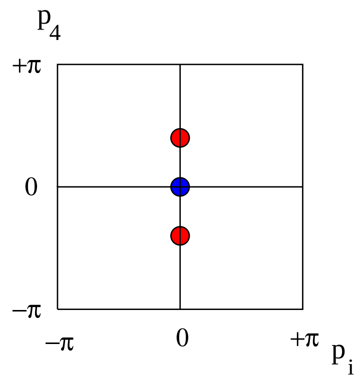
Minimal doubling examples

(units with $a = 1$)

Karsten (1981); Wilczek (1987)

$$K = i \sum_{i=1}^{3} \gamma_i \sin(p_i) + \frac{i\gamma_4}{\sin(\alpha)} \left(\sum_{\mu=1}^{4} \cos(p_\mu) - \cos(\alpha) - 3 \right)$$

- propagator poles along p_4 axis at $\vec{p} = 0$, $p_4 = \pm\alpha$



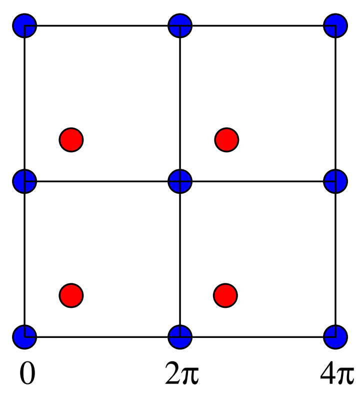
- strictly antihermitean kinetic term, exactly anticommutes with γ_5.
- similar to a Wilson term for space momenta with an $i\gamma_4$ factor

Tatsu Misumi

- "twisted ordering"

$$\begin{aligned} K = & i\gamma_1(\sin(p_1) + \cos(p_2) - 1) \\ & i\gamma_2(\sin(p_2) + \cos(p_3) - 1) \\ & i\gamma_3(\sin(p_3) + \cos(p_4) - 1) \\ & i\gamma_4(\sin(p_4) + \cos(p_1) - 1) \end{aligned}$$

- propagator poles at $p = (0,0,0,0)$ and $p = (\pi/2, \pi/2, \pi/2, \pi/2)$



- $1 \to C$ shifts distance between poles

Graphene motivated: generalize $z = e^{ip_1} + e^{ip_2} + 1$ to

$$\begin{aligned} z = {} & e^{+i\sigma_1 p_1} + e^{+i\sigma_1 p_2} + e^{-i\sigma_1 p_3} + e^{-i\sigma_1 p_4} \\ & + e^{+i\sigma_2 p_1} + e^{-i\sigma_2 p_2} + e^{-i\sigma_2 p_3} + e^{+i\sigma_2 p_4} \\ & + e^{+i\sigma_3 p_1} + e^{-i\sigma_3 p_2} + e^{+i\sigma_3 p_3} + e^{-i\sigma_3 p_4} \\ & - 12C \end{aligned}$$

- minimal doubling for $\frac{1}{2} < C < 1$.
- poles along major diagonal $p_\mu = p_\nu = \arccos(C)$

$C = 1/\sqrt{2}$ gives orthogonal lattice

$C = \cos(\pi/5)$ gives 4d analogue of graphene/diamond

Point splitting

One field gives two particles

- separate them by combining fields at nearby points

For the Karsten Wilczek form above

- start with free fields momentum space

$$u(q) = \frac{1}{2}\left(1 + \frac{\sin(q_4+\alpha)}{\sin(\alpha)}\right)\psi(q+\alpha e_4)$$

$$d(q) = \frac{1}{2}\,\Gamma\left(1 - \frac{\sin(q_4-\alpha)}{\sin(\alpha)}\right)\psi(q-\alpha e_4)$$

- factor of $\Gamma = i\gamma_5\gamma_4$ corrects for opposite chirality of one mode

Go to position space and insert gauge fields

$$u_x = \frac{e^{i\alpha x_4}}{2}\left(\psi_x + i\frac{e^{-i\alpha}U_{x,x-e_4}\psi_{x-e_4} - e^{i\alpha}U_{x,x+e_4}\psi_{x+e_4}}{2\sin(\alpha)}\right)$$

$$d_x = \frac{\Gamma e^{-i\alpha x_4}}{2}\left(\psi_x - i\frac{e^{-i\alpha}U_{x,x-e_4}\psi_{x-e_4} - e^{i\alpha}U_{x,x+e_4}\psi_{x+e_4}}{2\sin(\alpha)}\right)$$

Meson operators ($\alpha = \pi/2$):

$$\pi_0(x) = \frac{i}{16}(4\overline{\psi}_x\gamma_5\psi_x + \overline{\psi}_{x-e_4}\gamma_5\psi_{x-e_4} + \overline{\psi}_{x+e_4}\gamma_5\psi_{x+e_4}$$
$$- \overline{\psi}_{x+e_4}UU\gamma_5\psi_{x-e_4} - \overline{\psi}_{x-e_4}UU\gamma_5\psi_{x+e_4})$$

$$\eta'(x) = \frac{1}{8}(\overline{\psi}_{x-e_4}U\gamma_5\psi_x - \overline{\psi}_x U\gamma_5\psi_{x-e_4}$$
$$+ \overline{\psi}_{x+e_4}U\gamma_5\psi_x - \overline{\psi}_x U\gamma_5\psi_{x+e_4})$$

- π_0 connects sites of the same parity
- η' connects even and odd sites; no on-site contribution

Tatsu and Taro: use $\overline{u}u$ and $\overline{d}d$ fields to split masses

- variation on the Wilson term.
- use this form to study eigenvalue flow and the index theorem

Counterterms

All actions pick a special direction

- s_μ points between the two zeros
- hypercubic symmetry broken

Capitani, Weber, Wittig, MC

- introduces new perturbative counterterms
- the lattice gets distorted

  the distance between the zeros

  fermion speed of light

  gluon speed of light

Three relevant operators

$$\psi\gamma_\mu s_\mu \psi$$
$$\psi\gamma_\mu s_\mu s_\nu \partial_\nu \psi$$
$$F_{\mu\nu}F_{\mu\rho}s_\nu s_\rho$$

One operator of naive dimension 3, similar to Wilson

- two of dimension 4

Are the counterterms really necessary?

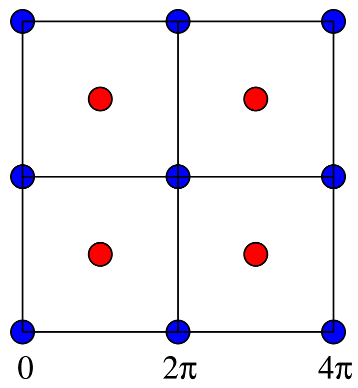
- Need to remove the special direction

Not yet realized

Two possible approaches:

Zeros half way around Brilloin zone

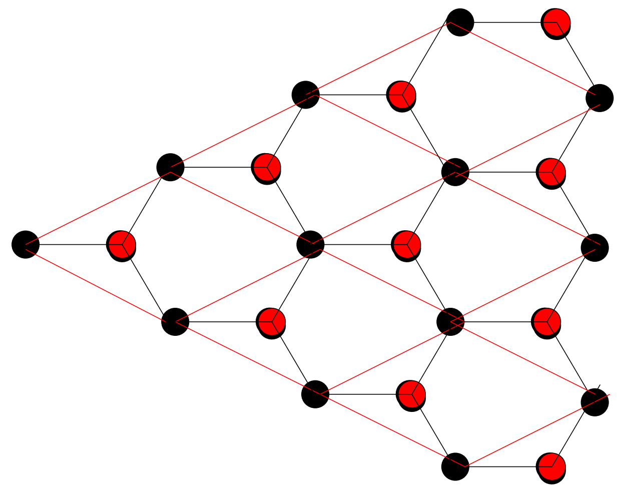
- $p_\mu = 0$ and $p_\mu = p_\nu = +\pi$
- zeros in a body centered hyper-cubic arrangement



- adjusting C above gives extra zeros before reaching this

Distort lattice to form a hyperdiamond

- make $0, 0, 0, 0$ to p, p, p, p distance equal p, p, p, p to $2\pi, 0, 0, 0$
- each site has five equidistant neighbors
- analog of graphene in 2-d



- $C = \cos(\pi/5)$ reaches these points
- but the action as we travel along different bonds not equivalent
- brings in higher harmonics

Can non-nearest hoppings fix these examples?

- can we remain local over hypercubes?

Summary

Chiral symmetry is a symmetry in mass parameter space

Breaking of continuum chiral symmetry gives rich physics

- spontaneous
- anomaly
- masses

Understanding this on the lattice is challenging but instructive

Minimal doubling maintains some chiral symmetry

- fast for simulations
- avoids rooting
- counterterm questions remain