General Relativity Applied to Non-Gravitational Physics

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General Relativity, its mathematical techniques and conceptual framework are by now part of the tool kit of (almost) all theoretical physicists and at least some pure mathematicians. They have become part of the natural language of physics.

Indeed parts of the subject are passing into Mathematics departments.

It is natural therefore to ask to what extent can they illuminate other (non-relativistic) areas of physics.

It is also the case that the relentless onward progress of technology makes possible analogue experiments illustrating basic ideas in General Relativity.

In this talk I will illustrate this ongoing proces of Unification

As a topical example of the relentless progress of technology last month * saw the demonstration in the laboratory some 40 years after the original prediction ! † of a very basic mechanism in semi-classical General Relativity: amplification of vacuum fluctuations in a time-dependent environment.

This is the basis of all we believe about inflationary perturtbations, Hawking evaporation, Black Hole information "Pardadox?" and much of AdS/CFT etc etc.

*Wilson et al. Observation of the Dynamical Casimir effect, *Nature* 479 (2011) 376-379

†G. T Moore, Quantum Theory of Eletromagnetic Fields in Variable Length One-Dimensional Cavity. *J. Math Phys* **11** (1970) 2379-2691, S.A. Fulling Radiationa from a moving mirror in two dimensional spacetime : Conformal Anomaly, *Proc Roy Soc A* **348** (1976) 393-414

The idea of finding analogue models for General Relativistic effect is not new, but the pace has hotted up recently.

Some important early work was done on cosmic strings modelled by vortices in superfluid Helium 4 and by Volovik *, who noted that the order parameter of some phases of superfluid Helium 3 is a triad $\mathbf{e}_i$ such that $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$.

More recently, the empahisis has shifted to the optics of metamaterials and most recently to graphene .

There are also interesting analogies in liquid crystals.

*G. Volovk, *The Universe in a Helium Droplet* Oxford University Press (2003)

Let's start with a very simple and sadly topical* example: Shallow Water Waves If $\eta = \eta(t, x, y)$ is the height of the water above its level when no waves are present and $h = h(x, y)$ the depth of the water, then shallow water waves satisfy the non-dispersive wave equation: †

$$(gh\eta_x)_x + (gh\eta_y)_y = \eta_{tt} \,,$$

where g is the acceleration due to gravity. From now on we adopt units in which $g = 1$ The wave operator coincides with the covariant D' Alembertian

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\eta) = 0 \,,$$

with respect to the $2+1$ dimensional spacetime metric

$$ds^2 = -h^2 dt^2 + h(dx^2 + dy^2) \,.$$

*Tsunami

†Einstein Equivalence Principle

Applying Ray theory and Geometrical Optics one writes

$$\eta = Ae^{-i\omega(t-W(x,y))},$$

where $A(x,y)$ is slowly varying. To lowest order W satisfies the Hamilton-Jacobi equation

$$(\frac{\partial W}{\partial x})^2 + (\frac{\partial W}{\partial y})^2 = \frac{1}{h},$$

and the rays are solutions of

$$\frac{dx}{dt} = h\frac{\partial W}{\partial x}.$$

Given any static spacetime metric

$$ds^2 = -V^2 dt^2 + g_{ij} dx^i dx^j \, ,$$

the projection $x^i = x^i(t)$ of light rays, that is characteristic curves of the covariant wave equation or the Maxwell or the Dirac equations, onto the spatial sections are geodesics of the Fermat or optical metric given by

$$ds_0^2 = \frac{g_{ij}}{V^2} dx^i dx^j$$

In the special case of shallow water waves, the rays are easily seen to be geodesics of the metric

$$ds_o^2 = \frac{dx^2 + dy^2}{h} \, .$$

For a linearly shelving beach

$$h \propto y \qquad y > 0\,.$$

the rays are cycloids, and all ray's strike the shore, i.e. $y = 0$, orthogonally. For a quadratically shelving beach,

$$h \propto y^2 \qquad y > 0\,,$$

the rays are circles centred on the shore at $y = 0$, and again every ray intersects the shore at right angles.

In fact the optical metric in this case is

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

which is Poincaré 's metric of constant curvature on the upper half plane. If x is periodically identified, one obtains the the metric induced on a tractrix of revolution in $\mathbf{E}^3$,sometimes called the Beltrami Trumpet (i.e. $H^2/\beta Z$. Note that the embedding can never reach the conformal boundary at $y = 0$.

The optical time for rays to reach the shore in the second example above is infinite. This reminds one of the behaviour of event horizons. In fact there is a rather precise correspondence. The Droste-Schwarzchild metric in isotropic coordinates (setting $G = 1 = c$) is

$$ds^2 = -\frac{(1-\frac{m}{2r})^2}{(1+\frac{m}{2r})^2}dt^2 + (1+\frac{m}{2r})^4(dx^2+dy^2+dz^2)\,.$$

with $r = \sqrt{x^2+y^2+z^2}$. The isotropic radial coordinate r is related to the Schwarzschild radial coordinate R by

$$R = r(1+\frac{m}{2r})^2\,.$$

The Event Horizon is at $R = 2m$, $r = \frac{m}{2}$ If we restrict the Schwarzschild metric to the equatorial plane $z = 0$ we obtain

$$ds^2 = -\frac{(1-\frac{m}{2r})^2}{(1+\frac{m}{2r})^2}dt^2 + (1+\frac{m}{2r})^4(dx^2+dy^2)\,.$$

The optical metric is

$$ds_0^2 = \frac{(1+\frac{m}{2r})^6}{(1-\frac{m}{2r})^2}(dx^2+dy^2)\,.$$

and

$$h = \frac{(r-\frac{m}{2})^2 r^4}{(r+\frac{m}{2})^6}\,.$$

we get the analogue of a black hole : a circularly symmetric island whose edge is at $r=\frac{m}{2}$ and away from which the beach shelves initially in a quadratic fashion and ultimately levels out as $r\to\infty$. Since

$$\frac{1}{h}\frac{dh}{dr} = \frac{2}{r-\frac{m}{2}} + \frac{4}{r} - \frac{6}{r+\frac{m}{2}} > 0$$

the beach shelves monotonically.

To obtain a cosmic strings for which the optical metric is a flat cone with deficit angle $\delta = \frac{2\pi p}{p+1}$ one needs a submerged mountain with

$$h \propto (x^2 + y^2)^{\frac{p}{p+1}},$$

As $p = \infty$, we get a parabola of revolution and the optical metric approaches that of an infinitely long cylinder. If $p = 1$ the mountain is conical, like a submerged volcano. In physical coordinates x, y the rays are bent, but one may introduce coordinates in which it is flat:

$$ds^2 = d\tilde{r}^2 + \tilde{r}^2 d\tilde{\phi}^2, \qquad 0 \le \tilde{\phi} \le \frac{2\pi}{p+1}$$

In these coordinates the rays are straight lines.

One could multiply these examples to cover such things as cosmic strings, moving water and vortices. To take into account the fact that the earth is round we replace $\mathbf{E}^2$ by S^2

$$dx^2 + dy^2 \rightarrow d\theta^2 + \sin^2\theta d\phi^2 \tag{1}$$

which gives Einstein's Static Universe in $2+1$ dimensions.

To take into account that it is rotating, we replace the static, i.e. time-reversal invariant metric by a stationary metric

$$d\theta^2 + \sin^2\theta d\phi^2 \rightarrow d\theta^2 + \sin^2\theta(d\phi - \Omega dt)^2 \tag{2}$$

All of this can be illustrated using a (possibly rotating) ripple tank.

Let's pass from hydrodynamics to to Optics and Maxwell's equations.

Maxwell's source-free equations in a medium are

$$\begin{aligned} \mathrm{curl}\,\mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}\,, & \mathrm{div}\,\mathbf{B} = 0\,, \\ \mathrm{curl}\,\mathbf{H} &= +\frac{\partial \mathbf{D}}{\partial t}\,, & \mathrm{div}\,\mathbf{D} = 0\,, \end{aligned}$$

or if *

$$\begin{aligned} F &= -E_i dt \wedge dx^i + \frac{1}{2}\epsilon_{ijk} B_i dx^j \wedge dx^k \\ G &= H_i dt \wedge dx^i + \frac{1}{2}\epsilon_{ijk} D_i dx^j \wedge dx^k \end{aligned}$$

$$\boxed{dF = 0 = dG}$$

$^*\epsilon_{ijk} = \pm, 0$

In what follows it will be important to realise that these equations hold in any coordinate system and they do not require the introduction of a spacetime metric.

However to “close the system”, one must relate F to G by means of a “constitutive equation”.

If the medium is assumed to be static, and linear then

$$\boxed{D_i = \epsilon_{ij} E_j \qquad B_i = \mu_{ij} H_j}$$

where ϵ is the dielectric permittivity tensor and μ_{ij} the magnetic permeability tensor. If they are assumed symmetric $\epsilon_{ij} = \epsilon_{ji} \quad \mu_{ij} = \mu_{ji}$ then $\mathcal{E} = \frac{1}{2}\Big(E_i D_i + H_i B_i\Big)$ may be regarded as the energy density and $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ the energy current or Poynting Vector since Maxwell's equations imply

$$\boxed{\mathrm{div}\,\mathbf{S} + \frac{\partial \mathcal{E}}{\partial t} = 0\,.}$$

“In olden days a glimpse of stocking was thought of as something shocking” and certainly μ_{ij} and ϵ_{ij} were assume positive definite “but now” , with the advent of Nanotechnology and the construction of metamaterials “ anything goes” . As long ago as 1964, V.G. Vestilago* pointed out that isotopic substances with with $\mu_{ij} = \mu\delta_{ij}\,, \quad \epsilon_{ij} = \epsilon\delta_{ij}$ and or which

$$\boxed{\mu < 0\,, \quad \epsilon < 0}$$

give rise to left-handed light moving in a medium with a negative refractive index

Sov. Phys. Usp. **10** (1968) 509-514

In 2001 R.A. Shelby, D.R. Smith and S. Schutz * produced this effect for microwave frequencies.

In 2002 D.R. Smith, D. Schurig and J.B. Pendry † appeared to have produced this effect in the laboratory.

Science* **292 (2001) 77-79

†*App Phys Lett* **81** (2002) 2713-2715

Assuming a spacetime dependence proportional to an arbitrary function of $\mathbf{k} \cdot \mathbf{x} - \omega t$, with $\omega > 0$ one finds

$$\mathbf{k} \times \mathbf{E} = \omega \mathbf{B}\,, \qquad \mathbf{k} \times \mathbf{H} = -\omega \mathbf{D}\,.$$

$$\mathbf{k} \times \mathbf{E} = \omega \mu \mathbf{H}\,, \qquad \mathbf{k} \times \mathbf{H} = -\omega \epsilon \mathbf{E}$$

It is always the case that $(\mathbf{E}, \mathbf{H}, \mathbf{S})$ form a right handed orthogonal triad but if both μ and ϵ are negative then $(\mathbf{E}, \mathbf{H}, \mathbf{k})$ give form a left-handed orthogonal triad and so $\mathbf{S}$ and $\mathbf{k}$ are anti-parallel rather than parallel as is usually the case. Since the wave vector $\mathbf{k}$ must be continuous across a junction between a conventional medium and and an exotic medium with $\mu < 0, \epsilon < 0$, this give rise to backward bending light.

The speed of propagation $v = \frac{1}{n}$, where n is the refractive index is given by

$$v^2 = \frac{\omega^2}{\mathbf{k}^2} = \frac{1}{\mu\epsilon}$$

with is natural to take the negative square root to get the refractive index

$$\boxed{n = -\frac{1}{\sqrt{\mu\epsilon}}\,.}$$

Given a spacetime metric $g_{\mu\nu}$ one has a natural way of specifying a constitutive relation:

$$\boxed{G = \star_g F}$$

where $\star_g$ denotes the Hodge dual with respect to the spacetime metric g such that $\star_g \star_g = -1$. If

$$ds^2 = -V^2(x^k)dt^2 + g_{ij}(x^k)dx^i dx^j$$

Tamm [*], Skrotskii [†] and Plebanski [‡]showed that

$$\boxed{\mu_{ij} = \epsilon_{ij} = \sqrt{\frac{\det g_{lm}}{V^2}}\, g^{ij}\,,}$$

[*] I. E. Tamm, *Zh. Rus. Fiz.-Khim. Obshchestva, Otd. Fiz.* **56** , 248 (1924)

[†] G.V. Skrotskii, *Dokl. Akad. Nauk SSSR* **114**, 73 (1957) [Soviet Physics Doklady 2, 226 (1957)]

[‡] J. Plebanski, *Phys. Rev.* **118** , 1396 (1960)

A medium with $\mu_{ij} = \epsilon_{ij}$ is said to be impedance matched. A similar result holds for resistivity problems such as that of Calderon * encountered oil prospecting

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{j} = 0\,, \qquad \mathbf{E} = -\boldsymbol{\nabla}\,\phi\,, \qquad j_i = \sigma_{ij} E_j}$$

$$\boxed{\partial_i \sigma_{ij} \partial_j \phi \Rightarrow \nabla_g^2 \phi = 0 = \frac{1}{\sqrt{g}} \phi_i \Big(\sqrt{g} g^{ij} \partial_j \phi \Big)}$$

with

$$\boxed{\sigma_{ij} = \sqrt{g} g^{ij}\,, \qquad g_{ij} = (\det \sigma_{ij}) \rho_{ij}}$$

*A.P. Calderon, On an Inverse boundary value problem, *Seminar in Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro,1980 Soc Mat Rio Janeiro* (1980 65-73

If

$$\sigma_{ij} = \frac{1}{z}\delta_{ij}$$

we get Poincaré metric on upper half space model of hyperbloic or Lobachevsky space H^2 .

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{z^2}$$

The conformal boundary is a perfect conductor.

In physics we may choose either the West Coast signature convention $-+++$, so that $g_{tt} < 0$ and g_{ij} is positive definite or the East Coast convention $+---$ for which $g_{tt} > 0$ and g_{ij} is negative definite. By Sylvester's Law of Inertia the signature is locally constant, however Running between the East Coast and the West coast there must be a curve on which the spactime signature flips (as originally suggested in a different context by Arthur Eddington in 1922). Clearly light passing from Coast to Coast will get bent back.

By Fermat's Principle electromagnetic waves move along geodesics of the optical metric

$$\boxed{ds_0^2 = V^{-2} g_{ij} dx^i dx^j}$$

but this is invariant under signature change.

If time reversal symmetry is broken a Stationary metric may be cast in three different forms *

$$
\begin{aligned}
ds^2 &= -U(dt+\omega_i dx^i)^2 + \gamma_{ij}dx^i dx^j \\
&= U\Big\{ -(dt - b_i dx^i)^2 + a_{ij}dx^i dx^j \Big\} \\
&= \frac{U}{1-h_{ij}W^iW^j}\Big\{ -dt^2 + h_{ij}(dx - W^i dt)(dx^j - W^j dt) \Big\}
\end{aligned}
$$

Fermat's Principle for light rays now generalises to Zermelo's Problem : minimize the travel time of a boat moving with fixed speed wrt a Riemannian metric h_{ij} in the presence of a "wind" W^i.

*G.W. Gibbons, C. A. R. Herdeiro and M. C . Stationary Metrics and Optical Zermelo-Randers-Finsler Geometry. *Phys.Rev.* **D 79** :044022,2009. e-Print: arXiv:0811.2877 [gr-qc]

One may also think of the problem as one of a particular type of Finsler Geometry considered first by Randers with a Finsler function of homogeneous degree one in velocity $v^i = \frac{dx^i}{d\lambda}$ defining a line element $ds = F d\lambda$, given by

$$\boxed{F = \sqrt{a_{ij} v^i v^j} + b_i v^i \, .}$$

Alternatively one may think of a charged particle of unit mass and unit charge , moving on a Riemannian manifold with metric a_{ij} and magnetic field $B_{ij} = \partial_i b_j - \partial_i b_j$. In General Relativity, this is Gravito-Magnetism verified recently by the GPB satellite experiment.

In the absence of time reversal symmetry there is a magneto-electric effect first predicted by L. Landau and E. M. Lifshitz in 1956 and exhibited for instance by Cr_2O_3 .

$$\boxed{B_i = \mu_{ij} H_j + \alpha_{ji} E_j \,, \qquad D_i = \epsilon_{ij} E_j + \alpha_{ij} H_j}$$

$$\boxed{\mathcal{E} = \frac{1}{2}\mu_{ij} H_i H_j + \alpha E_i H_j + \frac{1}{2}\epsilon_{ij} E_i E_j}$$

If we take as constitutive relation $G = \star_g F$, then μ_{ij}, ϵ_{ij} and α_{ij} may be read off form the spacetime metric.

In a moving medium, a typical sound or light wave satisfies

$$\boxed{\Big[(\partial_t - W^i\partial_i)^2 - h^{ij}\partial_i\partial_j\Big]u = 0\,.}$$

The *rays* solve the Zermelo problem with wind W^i. For sound waves this is known to explain the curious (and irritating) propagation of traffic noise. The rays behave like charged magnetic particles , the magnetic field being given by the vertical gradient of the horizontal wind. Of course a vertical gradient in temperature and hence refractive index will also provide an anti-mirage effect. This produces a curve metric h_{ij}. Claude Warnick and I have recently modelled this by a charged particle moving in a magnetic field on the upper half plane. *

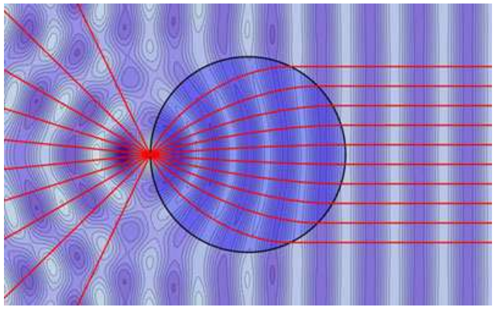
*The Geometry of sound rays in a wind. *Contemp.Phys.* **52** :197-209,2011.: arXiv:1102.2409 [gr-qc], Traffic Noise and the Hyperbolic Plane. *Annals Phys.* **325** :909-923,2010. arXiv:0911.1926 [gr-qc]

Designing Invisibility Cloaks, analogue black holes etc using Metamaterials and Transformation Optics. The basic idea is to start with a metric (it could be flat) and read off ϵ_{ij} and μ_{ij} . The metric could even be flat and obtained by a local diffeomorphism from the flat metric by which a beam or pencil of parallel straight lines in Cartesian coordinates are taken to the desired set of light rays in an *impedance matched* metamaterial medium. This technique has been much exploited by Pendry, Leonhardt and their collaborators and followers recently.

As pointed ut by Uhlmann and others, similar problems arise in Calderon's inverse problem: given a measuremant of $\mathbf{E}$ and ϕ on the boundary of some domain, can you determine uniquely the conductivity in the interior or can a reservoir of oil be invisible to the prospector?

In general one needs anisotropic materials.

To obtain an isotropic metamaterial medium the local diffeos should be conformal. The oldest and best known example of this is Maxwell's Fish Eye Lens which makes use use of Hipparchus's stereographic projection. This is the basis of the Luneburg Lens *



*R. K. Luneburg, Mathematical Theory of Optics. Providence, Rhode Island: Brown University. (1944) pp. 189 - 213.

A variant due to Minano * pulls back the round metric on $S^2, (\theta, \phi)$ to $R^2(x, y)$ using

$$x = \left(\frac{1 - \sin\theta}{\cos\theta}\right)^{\frac{1}{p}} \cos(\frac{\phi}{p}) \,, \qquad \left(\frac{1 - \sin\theta}{\cos\theta}\right)^{\frac{1}{p}} \sin(\frac{\phi}{p})$$

to get

$$ds_0^2 = d\theta^2 + \cos^2 d\phi^2 = n^2(dx^2 + dy^2) \,, \qquad n = 2p^2 \frac{r^{p-1}}{r^{2p} + 1}$$

**Optical Express* **14** (2006) 9627-9635

To get a black hole start with Droste-Schwarzschild in isotropic co-ordinates

$$
\begin{aligned}
ds^2 &= -\frac{(1-\frac{M}{2|\mathbf{x}|})^2}{(1+\frac{M}{2|\mathbf{x}|})^2}dt^2 + (1+\frac{M}{2|\mathbf{x}|})^4 d\mathbf{x}^2\,, \\
n = \mu = \epsilon &= (1+\frac{M}{2|\mathbf{x}|})^3(1-\frac{M}{2|\mathbf{x}|})^{-1}\,.
\end{aligned}
$$

The original cloak construction by Uhlmann works like this. We consider a spherical shell or solid annulus $a < r < 2a$ in r, θ, ϕ space and map it onto the punctured disc $0 < \tilde{r} < 2a$ by

$$\tilde{r} = 2(r - a) \qquad \tilde{\theta} = \theta \qquad \tilde{\phi} = \phi$$

The map is the identity: $r = \tilde{r}$ for $r > 2a, \tilde{r} > 2a$. Now pull back the flat metric $d\tilde{r}^2 + \tilde{r}^2(d\tilde{\theta}^2 + \sin^2\tilde{\theta} d\tilde{\phi}^2)$ and straightlines in $\tilde{r}, \tilde{\theta}, \tilde{\phi}$ space

$$ds^2 = 4dr^2 + 4(r-a)^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$\epsilon = \mu = \mathrm{diag}(2(r-a)^2 \sin\theta, 2\sin\theta, \frac{2}{\sin\theta})$$

No light ray (or electric current) enters the solid ball $r < a$.

The metric

$$ds^2 = -(\frac{r}{R})^{2p}dt^2 + B^2dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

(with R and B constants) arises in Generla Relativity in a number of contexts

- $p = 0$ and $B = \sqrt{1 - 8\pi G\eta^2}$, gives the Barriola-Vilenkin Global Monopole.

- $p = \frac{2\gamma}{1+\gamma}$, $B = \frac{\sqrt{1+6\gamma+\gamma^2}}{1+\gamma}$, gives Bisnovatyi-Kogan Zeldovich's gas sphere Here, γ is the constant ratio of pressure to density of. the gas for which

$$P = \frac{\gamma^2}{1 + 6\gamma + \gamma^2}\frac{1}{2\pi r^2}$$

Tippett has considered the case $p = 1 - s$ $B = s$, is, for $r < R$ to get Tippett's interior cloaking metric. For $r > R$, the exterior cloaking metric has $p = 0$, and $B = 1$ and hence is flat. Note that Tippett assumes that $s > 1$.

If $p > 0$, the origin $r = 0$ is an infinite redshift surface, while if $p < 0$ it is an infinite blueshift surface. The former is the case for the Bisnovatyi-Kogan Zeldovich gas sphere, while, since $s > 1$, the latter is the case for the cloaking metric. The optical metric is

$$\boxed{ds_o^2 = B^2\Big(\frac{R}{r}\Big)^{2p} dr^2 + R^2\Big(\frac{R}{r}\Big)^{2p-2}\Big(d\theta^2 + \sin^2\theta d\phi^2\Big)}$$

If $\rho = \left(\frac{r}{R}\right)^{1-p}$

$$ds_o^2 = R^2\Big\{(B')^2 d\rho^2 + \rho^2\Big(d\theta^2 + \sin^2\theta d\phi^2\Big)\Big\}$$

with $B' = \frac{B}{|1-p|}$. If $p < 1$, ρ increases as r increases, if $p > 1$, then ρ decreases as r-increases. This is a a cone over a 2-sphere.

The equatorial section $\theta = \frac{\pi}{2}$ has metric

$$R^2 \Big\{ (B')^2 d\rho^2 + \rho^2 d\phi^2 \Big\} = (RB')^2 \Big\{ d\rho^2 + \rho^2 (d\frac{\phi}{B'})^2 \Big\}$$

with $\phi \in (0, 2\pi]$, $\frac{\phi}{B'} \in (0, \frac{2\pi}{B'}]$ and is is a flat cone with deficit angle $\delta = (\frac{1}{B'} - 1)2\pi = (\frac{|1-p|}{B} - 1)2\pi$ Remarkably case of the interior cloaking metrics has

$$B = s\,, \quad 1 - p = s\,, \qquad \Longrightarrow \qquad B' = 1\,,$$

Thus the equatorial optical metric is globally flat, both inside and outside. The geodesics are therefore straight lines as are all meridional sections $\phi =$ constant and therefore in each meridional plane we have

$$\boxed{\rho \cos\theta = c\,.}$$

For $r > R$ we have therefore

$$r \cos\theta = Rc = b\,,$$

where we identify the constant Rc with the impact parameter. For $r < R$ we have

$$\boxed{\Big(\frac{r}{R}\Big)^s \cos\theta = \frac{b}{R}\,, \qquad \Longrightarrow \qquad r = R\Big(\frac{b}{R\cos\theta}\Big)^{\frac{1}{s}}\,.}$$

The geodesics passing through the interior which would, as described in (ρ, θ, ϕ) coordinates, be straight lines parallel to the axis of symmetry are, as described in (r, θ, ϕ) coordinates, are radially outwards compared with straight lines, thus giving the impression of cloaking. This accords with Figure 2 of *.

*B. Tippett, arXiv:1108.3793

If $p < 1$, then ρ increases as r increases, while if $p > 1$, then ρ decreases and r increases. Thus in general the interior metric will be conical and as long as $p < 1$ and $B\prime > 1$ One may then envisage meridional or equatorial cross-sections of the the optical manifold as an extended . flat plane $\rho > 1$ with a central conical central mountain $0 < \rho < 1$. In (ρ, θ, ϕ) coordinates the geodesics are straight lines, but, in contrast with the case considered by Tippett , they become deflected as they pass over the mountain, since like travellers in a mountainous landscape they avoid the summit. If one then maps back to the "physical coordinates" (t, r, θ, ϕ) one obtains a cloaking effect. All of this is very similar to the theory of lensing by cosmic strings or the motion of electrons in graphene with pentagonal or heptagonal defects.

Another possibly are Hyperbolic Metamaterials for which ϵ_{ij} is an indefinite matrix. The dispersion relation for a bi-refringent medium with $\mu_{ij} = \delta_{ij}$ is a quartic cone of two sheets:

$$\Big(\frac{k_x^2}{n_o^2} + \frac{k_y^2}{n_o^2} + \frac{k_z^2}{n_o^2} - \frac{\omega^2}{c^2}\Big)\Big(\frac{k_x^2}{n_e^2} + \frac{k_y^2}{n_e^2} + \frac{k_z^2}{n_0^2} - \frac{\omega^2}{c^2}\Big) = 0 \tag{3}$$

with $n_o^2 = \epsilon_z \, , n_e^2 = \epsilon_x = \epsilon_y$. Exceptional electromagnetic waves in a *uniaxial* thus obey obey

$$\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{1}{\epsilon_1}\frac{\partial^2 E}{\partial z^2} + \frac{1}{\epsilon_2}\Big(\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2}\Big) \tag{4}$$

The idea is * that dipole-moments in some crystals such as α *quartz* interact with lattice vibrations to form *phonon-polariton modes* called *restrahlen bands* in the mid infra red region for which both ϵ_1 and ϵ_2 can both become negative. Moreover because of crystal anisotropy ϵ_1 and ϵ_2 change sign at slightly different temperatures.

This would allow effective two-time physics .

*I. I. Smolyaninov,Virtual Black Holes in Hyperbolic Metamaterials *J Optics* **13** (2011) 125101 [arXiv.org:1101.5625[physics.optics]] I. I. Smolyaninov, Optical models of the big bang and non-trivial spacetime metrics based on metamaterials *Phys Rev Lett* **105** (2010) 067402 [arXiv:0908:2407[physics.optics]] I. I. Smolyaninov, Metamaterial "Multiverse", *J.Optics* **13** (2011) :024004 [arXiv:1005.1002[physics.optics]] I. I. Smolyaninov and E. E. Narimanov Metric Signature Transitions in Optical Metamaterials *Phys Rev Letts* **105** (2010) 067402[arXiv:1007.1130[physics.optics]]

In a model in a layered composite dielectric material

$$\epsilon_2 = n_m + (1 - n_m)\epsilon_d\,, \qquad e_1 = \frac{\epsilon_m \epsilon_d}{(1 - n_m)\epsilon_m + n_m \epsilon_d} \tag{5}$$

where the subscripts d and m stands for dielectric and metal respectively and ϵ_m is frequency dependent and can be negative. n_m is the volume fraction of metal. In a simple Drude model

$$\epsilon_m = 1 - \frac{\omega_p^2}{\omega^2 + i\omega\gamma} \tag{6}$$

with $\frac{\gamma}{\omega_p}$ is small. If $n_m << 1$ we have

$$\epsilon_2 \approx \epsilon_d - \frac{n_m \omega_p^2}{\omega^2 + i\omega\gamma}\,, \qquad \epsilon_1 \approx \epsilon_d\,. \tag{7}$$

Rather than consider artificial impedance matched or hyperbolic metamaterials, we may consider realistic substances such as chiral nematics in their helical phase *. Up to a divergence the Frank-Oseen Free energy is

$$\boxed{F = \frac{1}{2} \int (|\nabla^q \mathbf{n}|^2 - \lambda(\mathbf{n} \cdot \mathbf{n} - 1)\, d^3\, x\,,.}$$

$$\boxed{\nabla^q_i n_j = \partial_j n_j + q\epsilon_{ijk} n_k}$$

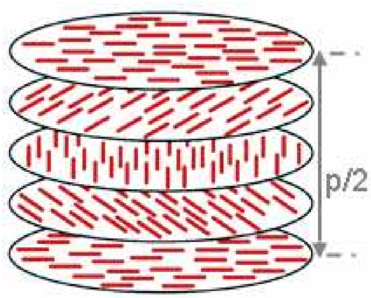
is an Euclidean metric preserving connection with torsion . The free energy density would vanish if $\mathbf{n}$ were covariantly constant with respect to ∇^q, $\nabla^q_i n_j = 0$. But rather like an anti-ferromagnet it is frustrated

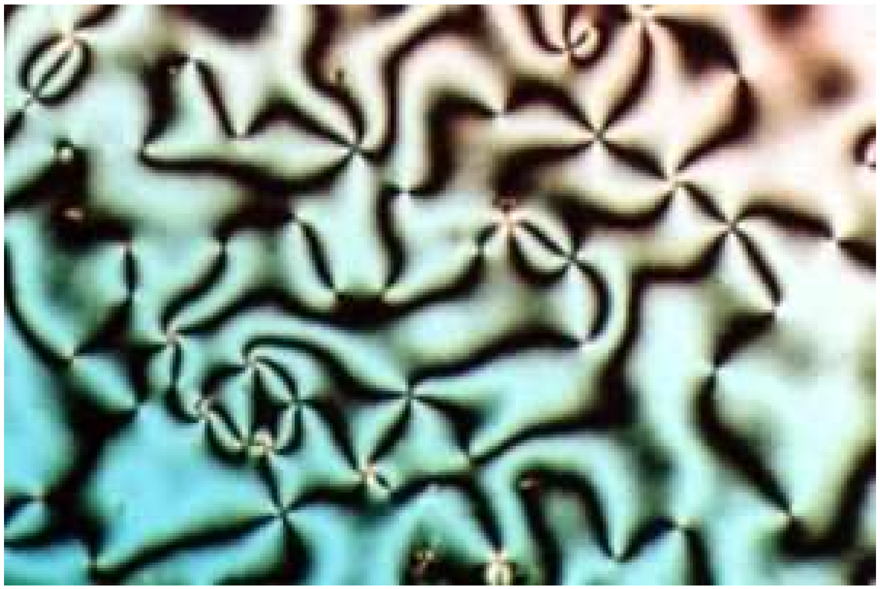
*GWG and C. Warnick, arXiv:1106.2423, The helical phase of chiral nematic liquid crystals as the Bianchi VII(0) group manifold, *Phys Rev* **E** in press since .

$$\boxed{(\nabla_i^q \nabla_j^q - \nabla_j^q \nabla_i^q) n_k \neq 0 \,.}$$

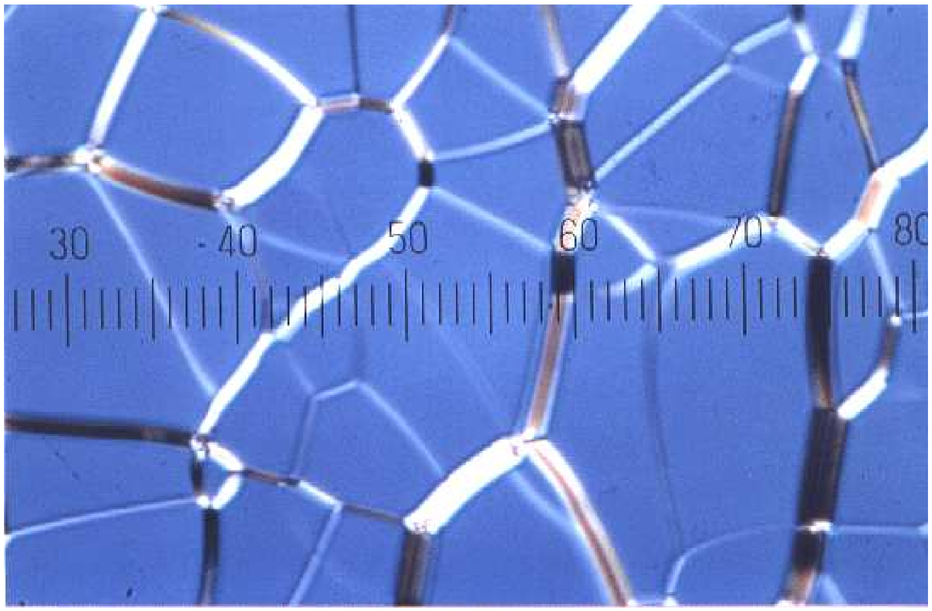
The substance may adopt a compromise configuration called the Helical Phase which satisfies the second order equations but not the first order Bogomolnyi type equation

$$\boxed{\mathbf{n} = (\cos(pz), \sin(pz), 0)}$$





A nematic liquid crystal seen through cross polarisers. It appears dark in places where the director is oriented along one of the polarizer axes. The points where the dark ares converge are disclinations.



Chiral nematic in its helical phase or Grandjean texture seen through cross polarisers. The director is parallel to the substrate plane and the axis perpendicular to it. The white lines are disclinations.

Optics in a nematic liquid crystal is governed by Fermat's principle using the Joets-Ribotta metric

$$ds_o^2 = n_e^2 d\mathbf{x}^2 + (n_o^2 - n_e^2)(\mathbf{n} \cdot d\mathbf{x})^2$$

where n_o is the refractive index of the ordinary ray and n_e that of the extra-ordinary ray.

Introduce 3 one-forms with Maurer-Cartan relations

$$\begin{aligned} \lambda^1 &= \cos(pz)dx + \sin(pz)dy\,, & d\lambda^1 = \lambda^3 \wedge \lambda^2 \\ \lambda^2 &= \cos(pz)dx - \sin(pz)dy\,, & d\lambda^2 = \lambda^3 \wedge \lambda \\ \lambda^3 &= pdz\,, & d\lambda^3 = 0\,. \end{aligned}$$

we find the Joets-Ribotta metric is

$$\boxed{ds_0^2 = n_o^2(\lambda^1)^2 + n_e^2(\lambda^2)^2 + \frac{n_e^2}{p^2}(\lambda^3)^2\,.}$$

This is a left-invariant metric on $\tilde{E}(2)$, the universal cover of the two-dimensional Euclidean group $E(2)$ whose Lie algebra $e(2)$ is of Type VII_0 in Bianchi's classification.

Thus the helical phase of chiral nematic crystals gives rise to a static Bianchi VII_0 cosmology :

$$ds^2 = -dt^2 + n_o^2(\lambda^1)^2 + n_e^2(\lambda^2)^2 + \frac{n_e^2}{p^2}(\lambda^3)^2 \,.$$

and one may, and we did, use all the standard tools of General Relativistic cosmology to describe its optical and electromagnetic properties, including solving Maxwell's equations, applying the Floquet Bloch theorem and the associated Mathieu Hill equation .

Gravitational Kinks

The Topology of a Lorenztian metric may be (partially) captured by a direction field n^i. Given a Riemannian metric g^R_{ij}, and a unit direction field n^i such that $g^R_{ij}n^i n^j = 1$ we may construct a Lorentzian metric g^L_{ij} via

$$\boxed{g^L_{ij} = g^R_{ij} - \frac{1}{\sin^2\alpha} n_i n_j\,, \qquad g^{ij}_L = g^{ij}_R - \frac{1}{\cos^2\alpha} n^i n^i\,, \qquad n_i = g^R_{ij} n^j}$$

Conversely given g^L_{ij} and g^R_{ij} we may reconstruct n_i up to a sign. Fixing the sign amounts to fixing a time orientation In what follows we wil choose g^R_{ij} to be the usual flat Euclidean metric.

$$\boxed{ds^2_L = g^{ij}_L dx^i dx^j = d\mathbf{x}^2 - \frac{1}{\cos^2\alpha}(\mathbf{n}\cdot d\mathbf{x})^2}$$

Given a closed surface enclosing a domain D,Finkelstein and Misner quantified the notion of tumbling light cones the light cone tips over on $\Sigma = \partial D$ by introducing a kink number which counts times how many times the light cone tips over on $\Sigma = \partial D$. The outward unit normal $\boldsymbol{\nu}$ and gives a 2-dimensional cross section of the four-dimensional bundle $S(\partial D = \Sigma)$ of unit 3-vectors over $\partial D = Sigma$. In the orientable case, the director field gives another 2-dimensional cross section of $S(\Sigma)$. The kink number kink(Σ, g^L) is number of intersections of these two sections with attention paid to signs. In the non-orientable case, one considers the bundle of directions. If the Lorentzian metric is non-singular we have

$$\boxed{\chi(D) = \text{kink}(\partial D, g^L)\,.}$$

For planar domains kink$(\partial D, g^L)$ is the obvious winding number.

$$\boxed{disclination \quad line \qquad \mathbf{n} = (\cos(s\phi), \sin(s\phi), 0)\,, \qquad \phi = \arctan(\frac{y}{x})}$$

$s \in \mathbf{Z} \cup \in \mathbf{Z} + \frac{1}{2}$. If s is half integral, then then we just have a direction field, not a vector field.

$$\boxed{\mathbf{n} \cdot d\mathbf{x} = \cos((s-1)\phi)dr + \sin((s-1)\phi)rd\phi\,,}$$

$$\alpha = \frac{\pi}{2}\,, \Rightarrow ds_L^2 = g_{ij}^L dx^i dx^j = -\cos(2(s-1)\phi)\Big(dr^2 - r^2 d\phi^2)\Big) - 2\sin(2(s-1)\phi)$$

Moving around a circle $r = constant$, the radial coordinate is timelike and the angular coordinate spacelike or *vice versa* depending upon the sign of $\cos(2(s-1)\phi)$ (tumbling light cones). $\det g_{ij}^L = -r^2$ and the components g_{ij}^L finite $\Rightarrow$ metric non-singular if $r > 0$

Bloch Walls If parity symmetry holds then a typical free energy functional takes the form

$$\boxed{F[\mathbf{M}] = \frac{1}{2}\int d^x \Big(\alpha_{ij}\partial_i \cdot \mathbf{M}\partial_j \mathbf{M} + \beta_{ij} M_i M_j}$$

In the unixial case with the easy direction along the third direction: $\alpha_{ij} = \mathrm{diag}(\alpha_1, \alpha_1, \alpha_2), \beta_{ij} = \mathrm{diag}(\beta, \beta, 0)$. For a domain wall separating a region $x << -1$ and with $\mathbf{M}$ pointing along the positive 3rd direction, from the rigion $x >> +1$ where it points along the negative 3rd direction

$$\boxed{\mathbf{M} = M(0, \sin\theta(x), \cos\cos\theta(x)) \,, \qquad M = \text{constant}}$$

and finds that θ must satisfy the quadrantal pendulum equation, $l = \sqrt{\frac{\alpha_1}{\beta}}$

$$\boxed{\theta^2 - \frac{1}{l^2}\sin^2\theta = \text{constant}' \,,}$$

If we impose the boundary condition that $\theta \to 0$ as $x \to -\infty$ and $\theta \to \pi$ as $x \to +\infty$, then constant$' = 0$ and

$$\boxed{\cos\theta = -\tanh(\frac{x}{l})}$$

The Lorenzian metric (if $\alpha = \frac{\pi}{2}$) is

$$\boxed{ds^2 = g^L_{ij} dx^i dx^j dx^2 + \cos(2\theta)\Big(dy^2 - dz^2\Big) - 2\sin 2\theta dz dy\,,.}$$

This closely resembles our previous examples and clearly exhibits the phenonemen of tumbling light cones. We note,*en passant* that in principle the tensor α_{ij} could itself vary with position. If so, we might interpret it in terms of an effective metric g_j with inverse g^{ij} and $g = \det g_{ij}$ obeying

$$\alpha_{ij} = \sqrt{g} g^{ij}\,. \tag{8}$$

Example: Liquid Crystal Droplets

The normal $\nu_i = \partial_i S$ to the surface $S =$ constant of a droplet of anisotropic nematic phase inside a domain D with unit outward normal $\boldsymbol{\nu}$ surounded by an isotropic phase satisfies the constant angle condition

$$\boxed{\mathbf{n} \cdot \boldsymbol{\nu} = \cos\alpha = \text{constant}\,.}$$

That is $$\boxed{\boldsymbol{\nu} \cdot \boldsymbol{\nu} - \frac{1}{\cos^2\alpha}(\boldsymbol{\nu} \cdot \mathbf{n})(\boldsymbol{\nu} \cdot \mathbf{n}) = 0 = g_L^{ij}\nu_i\nu_j = g_L^{ij}\partial_i S \partial_j S}$$

The surface ∂D of the droplet ∂D is a *null-hypersurface* or *wave surface* (a solution of the zero rest mass *Hamilton-Jacobi* equation)

Taking the z-coordinate as time so time runs vertically upwards and making the ansatz

$$\boxed{S = \frac{z}{\sin\alpha} + W(x,y)\,, \qquad \nabla W \cdot \nabla W = 1\,.}$$

Simple solutions of this Eikonal equation are given by Sandpiles with

$\frac{\pi}{2}-\alpha$ the angle of repose

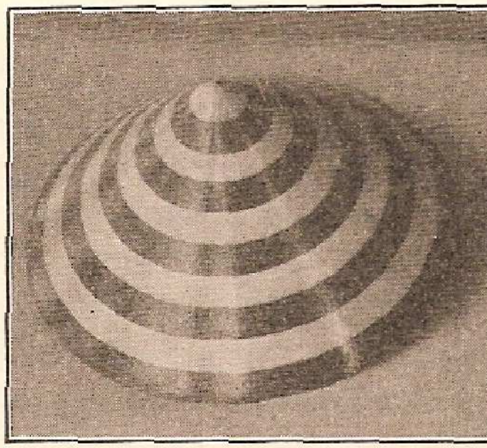


Fig. 149.

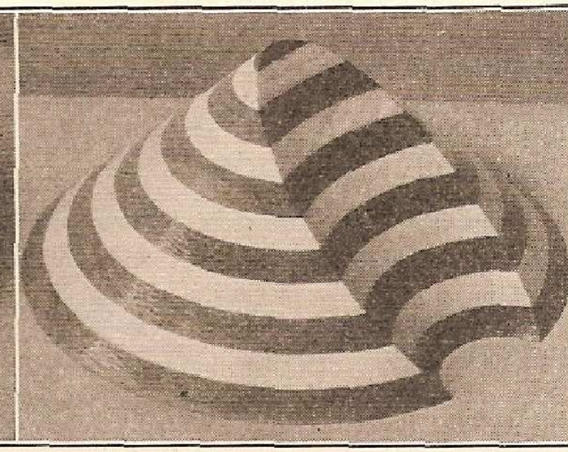


Fig. 150.

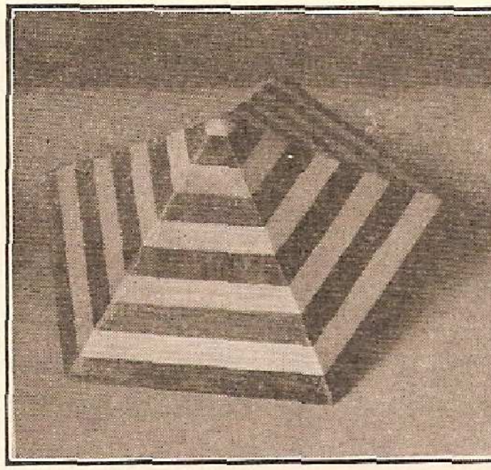


Fig. 151.

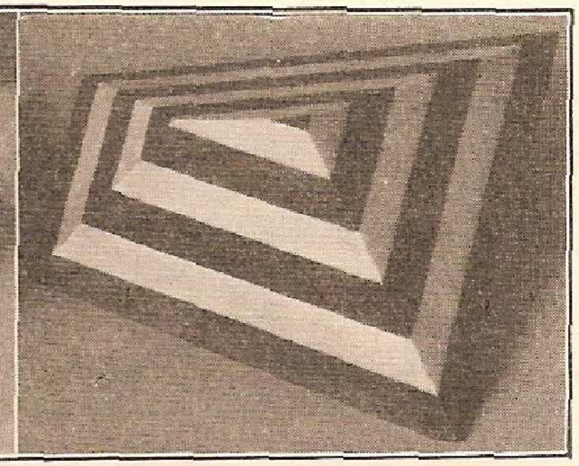


Fig. 152.

Figs. 149 to 152.—Examples of plastic stress function for torsion represented by wooden models for various cross-sections.

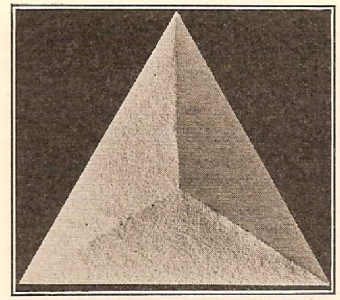


Fig. 162.—Sand heap over equilateral triangle.

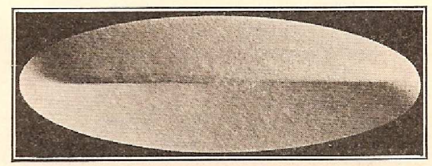


Fig. 163.—Sand heap over an ellipse.

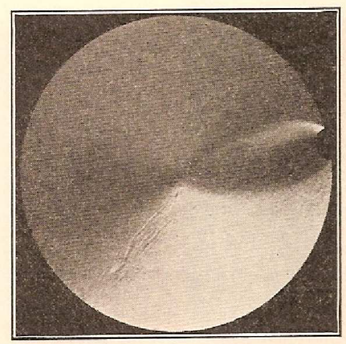


Fig. 164*a*.

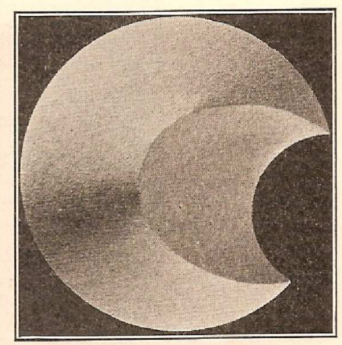
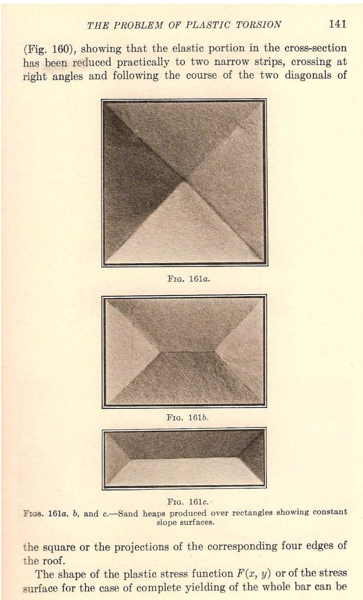


Fig. 164*b*.

Figs. 164*a* and *b*.—Sand heaps over areas bounded by two circular arcs.


THE PROBLEM OF PLASTIC TORSION 141

(Fig. 160), showing that the elastic portion in the cross-section has been reduced practically to two narrow strips, crossing at right angles and following the course of the two diagonals of

Fig. 161*a*.

Fig. 161*b*.

Fig. 161*c*.

Figs. 161*a*, *b*, and *c*.—Sand heaps produced over rectangles showing constant slope surfaces.

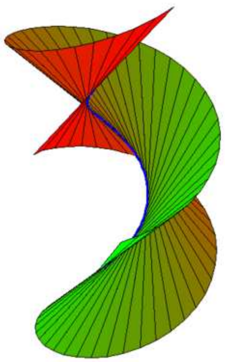
the square or the projections of the corresponding four edges of the roof.

The shape of the plastic stress function $F(x, y)$ or of the stress surface for the case of complete yielding of the whole bar can be

These describe Bitter Domains in a ferromagnetic film with $\mathbf{n} = \frac{\mathbf{M}}{|\mathbf{M}|}$ with normal $\boldsymbol{\nu}$ and boundary condition $\mathbf{M} \cdot \boldsymbol{\nu} = 0$.

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{M} = 0\,, \qquad |\mathbf{M}| = \text{constant}}$$

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{n} \Rightarrow n_x = \partial_y \psi\,, \qquad n_y = -\partial_x \psi \qquad |\nabla \psi| = 1\,.}$$



The axisymmetric solution is the spiral wave surface swept out by the involute of a circle, a helical developable.

$$\boxed{S = \pm\frac{z}{\sin\alpha} + \pm a\left(\sqrt{\frac{r^2}{a^2}-1} - \arctan\left(\sqrt{\frac{r^2}{a^2}-1}\right)\right) \pm a\phi}$$

For the helical phase we make the ansatz

$$\boxed{S = F(z) + x\cos\theta + y\sin\theta}$$

$F(z)$ solves the quadrantal pendulum equation

$$\boxed{\cos^2(\theta - pz) - \cos^2\alpha = (\cos\alpha\frac{dF}{dz})^2 \Rightarrow F = \frac{1}{\cos\alpha}\int dz\sqrt{\cos^2(\theta - pz) - \cos^2\alpha}}$$

The surface is ruled by horizontal straight lines making a constant angle θ with the x-axis and is bounded by $|pz - (\theta + n\pi)| < \alpha$, $n \in Z$ In other words it is horizontal cylinder or tube. The angle of the director $\mathbf{n}$ makes with the fixed direction $(\cos\theta, \sin\theta, 0)$ cannot be less than α.

The hexagonal Graphene “lattice” in $\mathbf{x}$ has a hexagonal Brillouin zone in the dual $\mathbf{p}$-space and is is the sum of two trangular (true) lattices, A and B in $\mathbf{x}$ space. Each lattice has a Fermi surface in $\mathbf{p}$ space and these two Fermi surfaces, governing the conduction and valence bands, touch in two conical Dirac points inside a Brillouin zone. Thus the dispersion relation for small $\mathbf{p}$ is

$$E = \pm|\mathbf{p}|$$

Low energy excitations are governed by

$$E\Psi = \boldsymbol{\sigma} \cdot \mathbf{p}\Psi$$

where the two-component Ψ has two pseudo-spin states.

But this is the massless Dirac equation! * On a curved graphene sheet it bcomes the Dirac equation on a curved surface $\Sigma \subset \mathbf{E}^3$ in Euclidean 3-space with metric

$$ds^2 = -dt^2 + h_{ij} dx^i dx^j \,, \qquad i,j = 1,2$$

where h_{ij} is the induced metric.

Since the massless Dirac equation is conformally invariant we may think of this metric on $R \times \Sigma$ as the optical metric of a static metric with $g_{tt} \neq$ constant.

*cf Semenov *Phys Rev Lett*,(1984)

Various examples have arisen in the literature If $\Sigma = S^2$ we have an approximation for Fullerenes.

If Σ is a Beltrami trumpet with metric of constant negative curvature, we have the near horizon geometry of a 2-dimensional black hole. Unfortunately we cannot find an isometric embeding of $H^2/\mathbf{Z}$ into $\mathbf{E}^3$ all the way down to $y = 0$, the horizon * We may also obtain the optical geometry of the BTZ black hole away from the horizon †

If we dope the graphene in an analogue of a p-n junction we can also obtain negative refractive indices.

*arXiv A. Iorio and G. Lambiase The Hawking-Unruh phenomenon on graphene, 1108.2340cond.mat matrl-sci]]

†GWG and Mirjam Cvetic: to appear

Conclusion and Propects

- In this talk I have described on some areas of non-gravitational physics where analogues of basic ideas in general relativity come into play. They include

- Dynamic Casimir Effect

- Water and sound waves

- Cloaking and other devices using metamaterials

- Nematic liquid crystals

- Graphene

Other areas not covered include

- Bose-Einstein Condensate
- Dirac Metals
- Smectcs and blue phases in liquid crystals