M-theory

and

exact results in SUSY gauge theories

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16 years

after the 2^{nd} superstring revolution

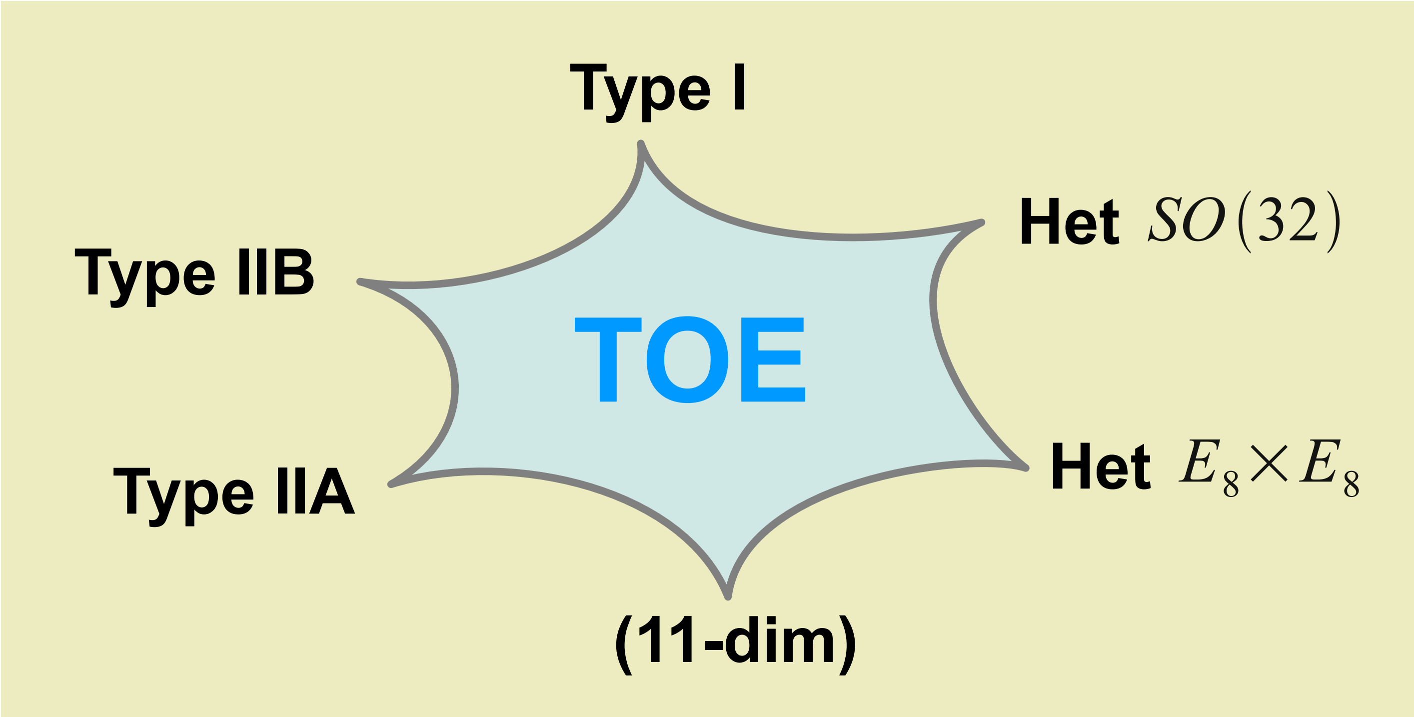
Superstring Revolution II ('95)

Dualities : Quantum equivalence relations among different 10-dim superstring theories

Branes : Spatially extended solitons in superstring theories

Duality Web

connects 5 superstring theories.

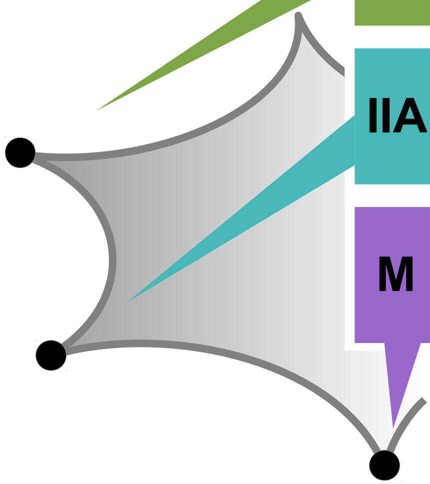


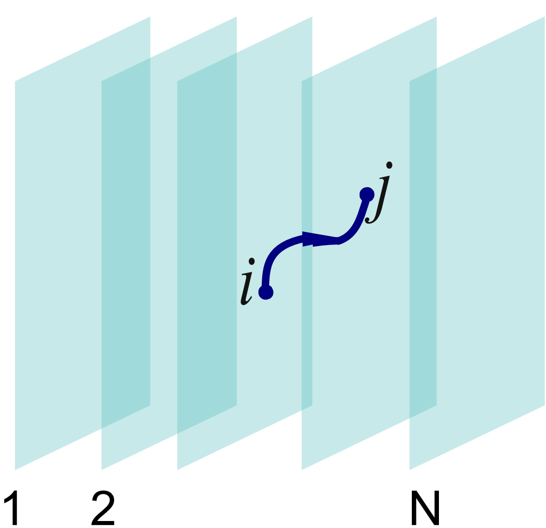
11 dimensions emerge

at a corner of the web = **M-theory**

p-brane = (p+1)-dim solitonic object

p	0	1	2	3	4	5
IIB		D-string string		D3-brane		D5-brane NS5-brane
IIA	D-particle	string	D2-brane		D4-brane	NS5-brane
M			M2-brane			M5-brane





Dynamics of **Dp-branes**

= open string theory

= (p+1)-dim SUSY gauge theory

Other branes are harder to understand.

Gauge Theories

can describe D-brane dynamics

D-branes

can engineer SUSY gauge theories

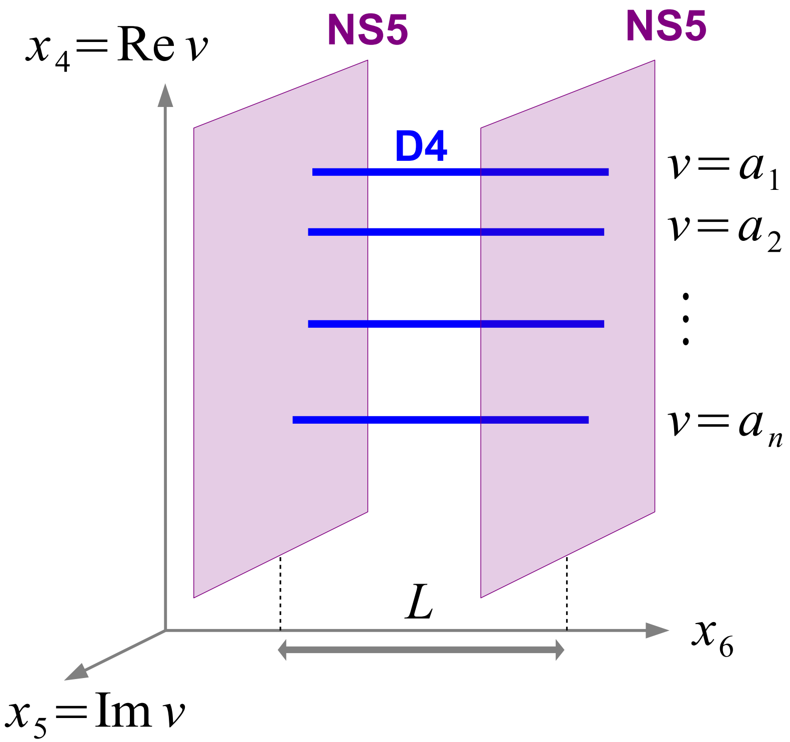
in different dimensions,

and visualize their strong coupling behaviors

The two subjects have been influenced from each other

for the last 15 years.

Example: Brane construction of 4D N=2 SUSY YM



Open strings on n D4-branes

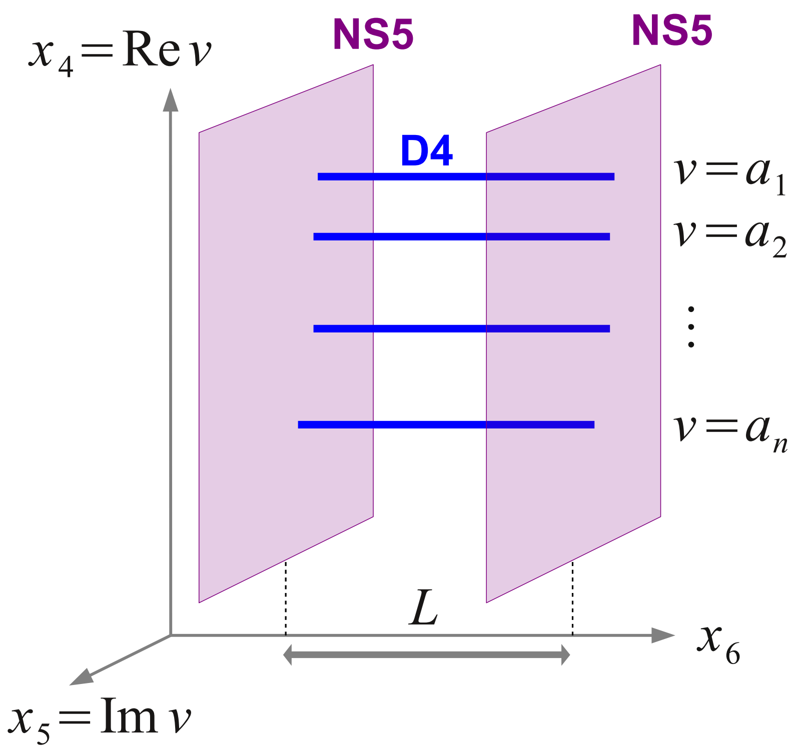
= SU(n) gauge theory

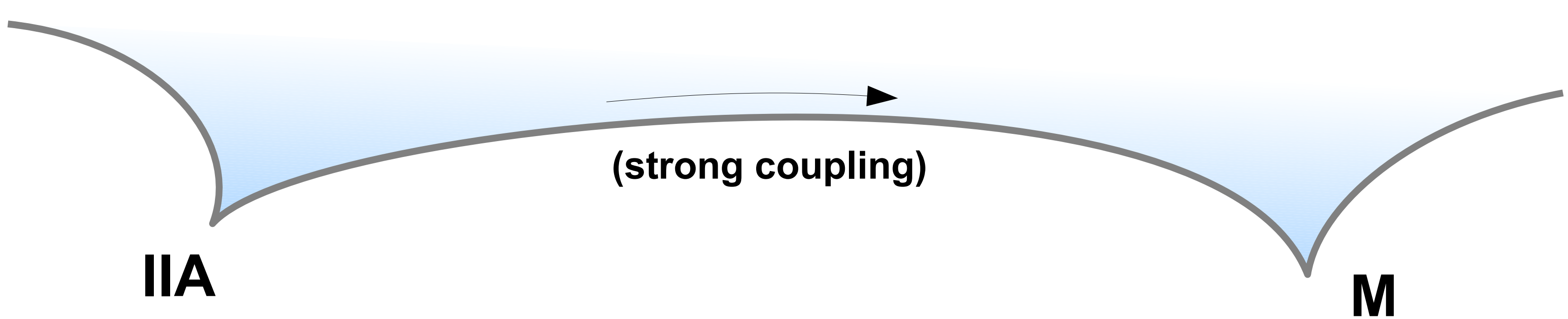
$$\left(a_1, \cdots, a_n\right)$$

= position of D4-branes

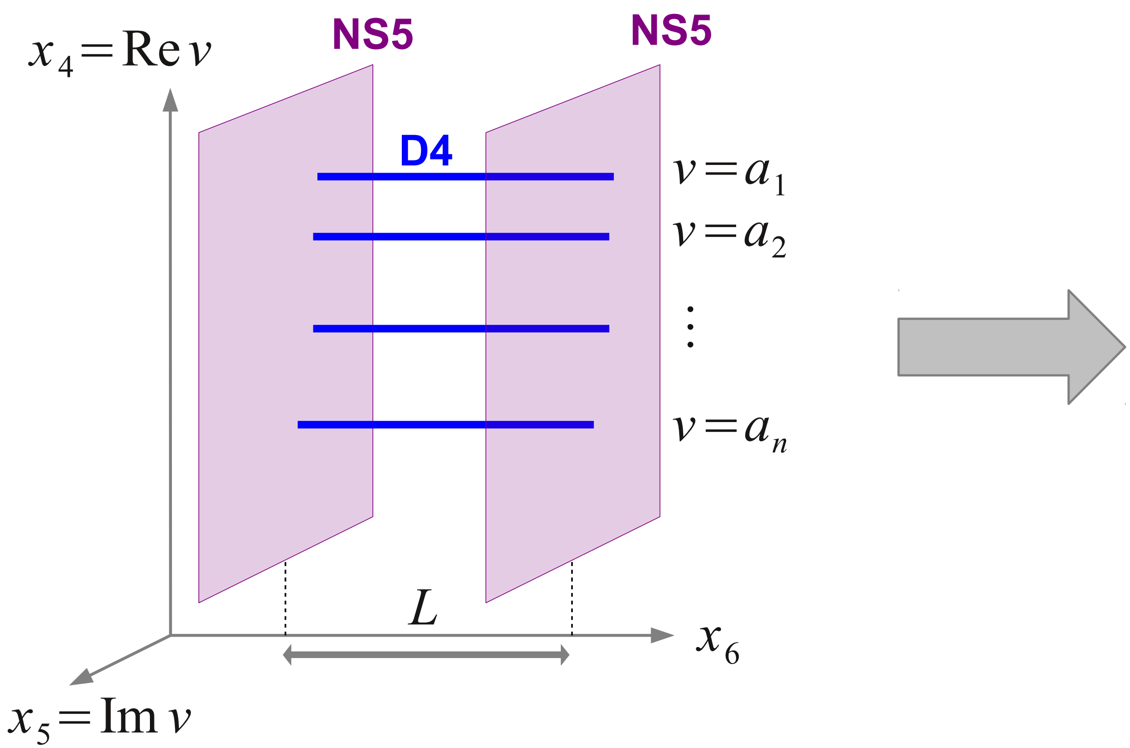
= label of vacua of gauge theory

Example: Brane construction of 4D N=2 SUSY YM

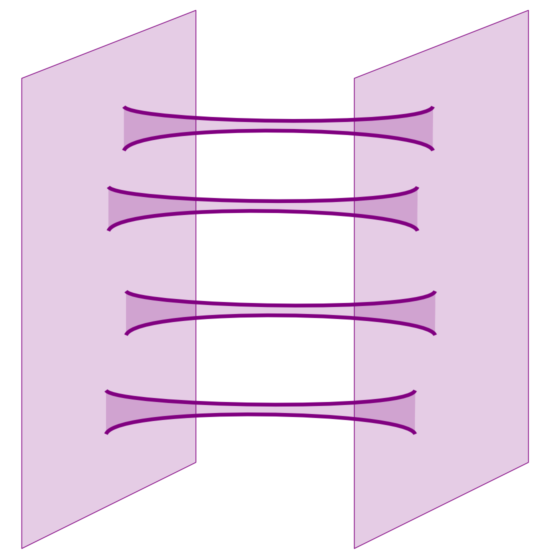


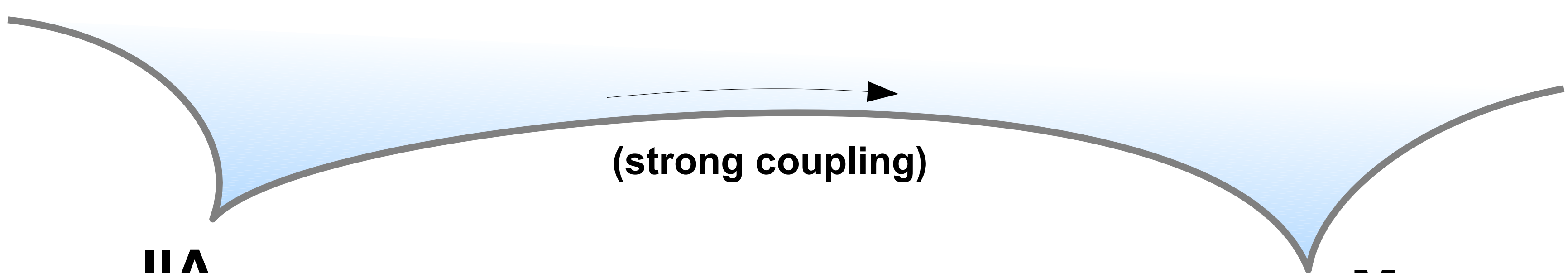


Example: Brane construction of 4D N=2 SUSY YM



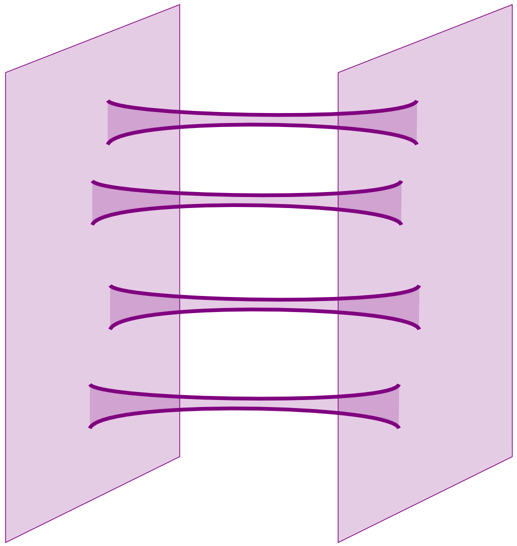
Single smooth M5 wrapping a 2D surface





The shape of the smooth M5-brane

(complex curve) :



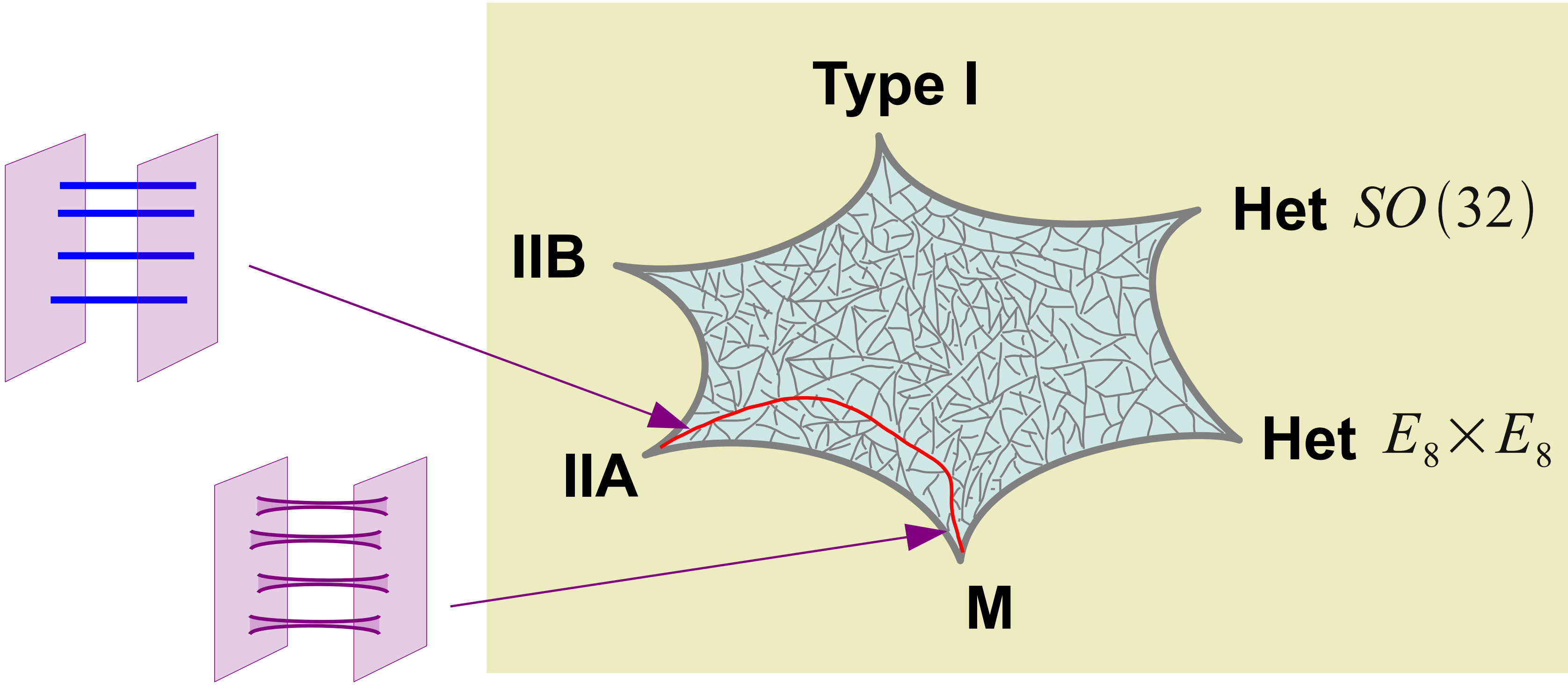
$$t^2 - t\prod_{i=1}^{n}(v - a_i) + \Lambda_{\text{SYM}}^{2n} = 0$$

$$\Lambda_{\text{SYM}}^{2n} = \exp(-L/g_s)$$

= **Seiberg-Witten curve** for N=2 SUSY YM with G=SU(n), which encodes everything about low-energy dynamics

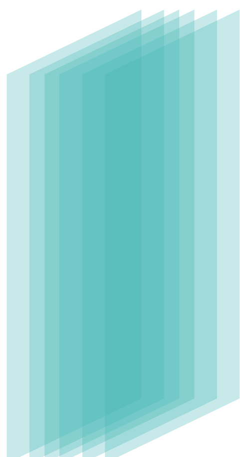
The mesh of the web

became very, very fine after 16 years.



Another big discovery:

AdS/CFT correspondence (1997)



But still,

the (5+1)-dim theory on N flat M5-branes in $\mathbb{R}^{10,1}$

remains mysterious.

“6dim (2,0) theories”

Recent progress
in SUSY gauge theories

Localization principle

= simplification of (path-)integrals due to (super)symmetry.

Non-zero contributions arises only from **fixed points**.

Application to **SW theories** (=4D N=2 SUSY gauge theories):

-- Partition function on "Omega-background" $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ (Nekrasov '02)

-- Partition function & Wilson loop on S^4 (Pestun '07)

. . . proved a long-standing conjecture

Circular Wilson loop in N=4 SYM

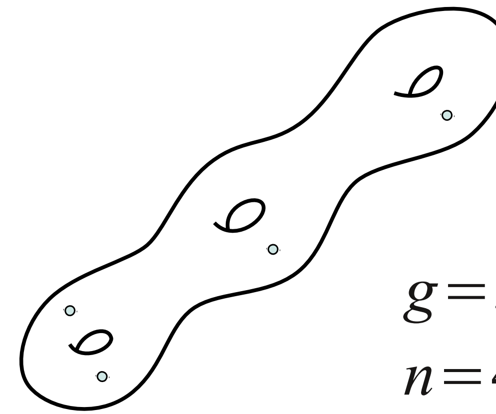
= Gaussian matrix integral

(Erikson-Semenoff-Zarembo, Drukker-Gross, . . .)

Application to M5-brane physics

Gaiotto's theory $T(N, \Sigma_{g,n})$

= SW theory describing N M5-branes wrapped on a Riemann surface $\Sigma_{g,n}$

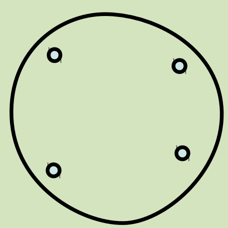


$g=3$

$n=4$

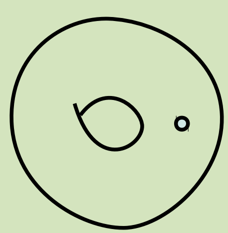
[examples]

2 M5-branes on $\Sigma_{0,4}$



$\Leftrightarrow$ SU(2) SQCD with 4 doublet quarks

2 M5-branes on $\Sigma_{1,1}$



$\Leftrightarrow$ SU(2) SQCD with 1 triplet quark

AGT relation (Alday-Gaiotto-Tachikawa '09)

S^4-partition function of $T\left(N,\Sigma_{g,n}\right)$	=	n-point correlation function of **SU(N) Toda CFT** on Σ_g
(4dim field theory)		**(2dim field theory)**

Comments:

* The relation was first found experimentally.
* Several ideas for proof have been proposed.
* The relation implies

$$\boxed{\text{N M5-branes on } S^4 \;=\; \text{SU(N) Toda CFT}}$$

(not proved yet.)

More recent progress:

Localization in 3-dim SUSY theories

SUSY theories on 3-sphere

-- Initiated by Kapustin-Willett-Yaakov ('09)

on **round sphere**

* cf) Sen('87,'90), Romelsberger('05) : 4D theories on $\mathbb{R}\times S^3$

-- Partition function for general N=2 SUSY theories

Jafferis('10), Hama-KH-Lee('10)

-- Generalization to **Non-round spheres**

Hama-KH-Lee('11), Imamura-Yokoyama('11)

SUSY on 3-sphere (or any curved manifold)

. . . in correspondence with **Killing Spinor (KS)**

$$D_\mu \varepsilon \equiv \left(\partial_\mu + \frac{1}{4}\Gamma_{ab}\omega_\mu^{ab}\right)\varepsilon = \Gamma_\mu \tilde{\varepsilon} \quad \text{for some} \quad \tilde{\varepsilon}$$

There are <u>4 KSs</u> on **round 3-sphere.**

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 \quad \text{in} \quad \mathbb{R}^4$$

$$ds^2 = \mu^1\mu^1 + \mu^2\mu^2 + \mu^3\mu^3 \qquad (\ \mu^a : \text{SU(2) LI 1-forms})$$

Generalization 1 (Hama-KH-Lee '11)

Ellipsoid S_b^3 : $b^2\left(x_0^2+x_1^2\right) + b^{-2}\left(x_2^2+x_3^2\right) = 1$

admits no KSs.

But with a suitable background field V_μ turned on,

there are charged KSs.

$$D_\mu \varepsilon^\pm \equiv \left(\partial_\mu + \frac{1}{4}\Gamma_{ab}\omega_\mu^{ab} \underline{\mp iV_\mu}\right)\varepsilon^\pm = \Gamma_\mu \tilde{\varepsilon}^\pm$$

Generalization 2 (Imamura-Yokoyama '11)

<u>Squashed 3-sphere:</u>

$$ds^2 \;=\; \mu^1\mu^1 + \mu^2\mu^2 + s^2\,\mu^3\mu^3 \qquad (s<1)$$

admits no KSs.

But with a suitable background field V_μ turned on,

there are <u>modified KSs</u> satisfying

$$\left(\partial_\mu \;+\; \frac{1}{4}\Gamma_{ab}\,\omega_\mu^{ab}\right)\varepsilon^\pm \;=\; -\frac{is}{2}\Gamma_\mu\varepsilon^\pm \underline{\pm\, tV^\nu\,\Gamma_{\mu\nu}\,\varepsilon^\pm}$$

$$\left(t \equiv \sqrt{1-s^2}\right)$$

$$b \equiv s+it \qquad \left(\,|b|=1\,\right)$$

Interesting questions :

* Any other 3-manifolds admitting SUSY?

* Any other variations of KS equation?

Note the relation (Festuccia-Seiberg '11)

KS equation $\Longleftrightarrow$ Gravitino's transformation law in supergravity

3D N=2 SUSY theories

multiplets:	"gauge"		"matter"	
labelled by	G	Lie algebra	R	Rep of G
			q	U(1) R-charge
fields	σ	real scalar	ϕ	complex scalar
	λ	spinor	ψ	spinor
	A_μ	vector		
	D	aux. scalar	F	aux. scalar

Couplings: YM coupling, Chern-Simons coupling, masses . . .

Partition function depends on <u>some</u> of them.

Computation of partition function using localization principle

SUSY localization principle

Nonzero contribution to SUSY (path-)integrals

localizes to “**fixed points**” satisfying

$$\delta_{\mathrm{SUSY}}\Psi = 0 \quad \text{for all fields} \quad \Psi .$$

For 3D N=2 SUSY theories,

$$\partial_\mu \sigma = A_\mu = \lambda = \phi = \psi = F = 0 \quad \text{at fixed points.}$$

(up to gauge choice)

$$\int D(\text{fields})\, e^{-S} \Longrightarrow \int d\sigma_0 (\cdots)$$

Path integral **Matrix integral**

For integral over everything except for σ_0 ,

Gaussian approximation is exact,

since partition function does not change under the shift

$$S \;\to\; S + t\;\delta_{\text{SUSY}} V \qquad \left(\text{s.t.}\;\; \delta_{\text{SUSY}}^2 V = 0\right)$$

* YM action for gauge multiplet,

* Kinetic action for matter multiplets

are of this class

Result: (Hama-KH-Lee '11)

$$Z_{S^3_b} = \frac{1}{|\mathrm{Weyl(G)}|}\int \prod_{i=1}^{\mathrm{rank}G} \mathrm{d}\sigma_i \cdot \Delta_{\mathrm{gauge}}(\sigma)\Delta_{\mathrm{mat}}(\sigma)\cdot e^{-S_{\mathrm{cl}}}$$

$$\Delta_{\mathrm{gauge}}(\sigma) = \prod_{\alpha\in\Delta_+} \sinh\frac{b\alpha\cdot\sigma}{2}\sinh\frac{\alpha\cdot\sigma}{2b}$$

$$\Delta_{\mathrm{mat}}(\sigma) = \prod_{\rho:\text{weights of }R} s_b\left(\frac{iQ}{2}(1-q)-\rho\cdot\sigma\right)$$

[double-sine function]

$$s_b(x) = \prod_{m,n\geq 0}\frac{mb+nb^{-1}+\frac{Q}{2}-ix}{mb+nb^{-1}+\frac{Q}{2}+ix}, \quad Q\equiv b+b^{-1}$$

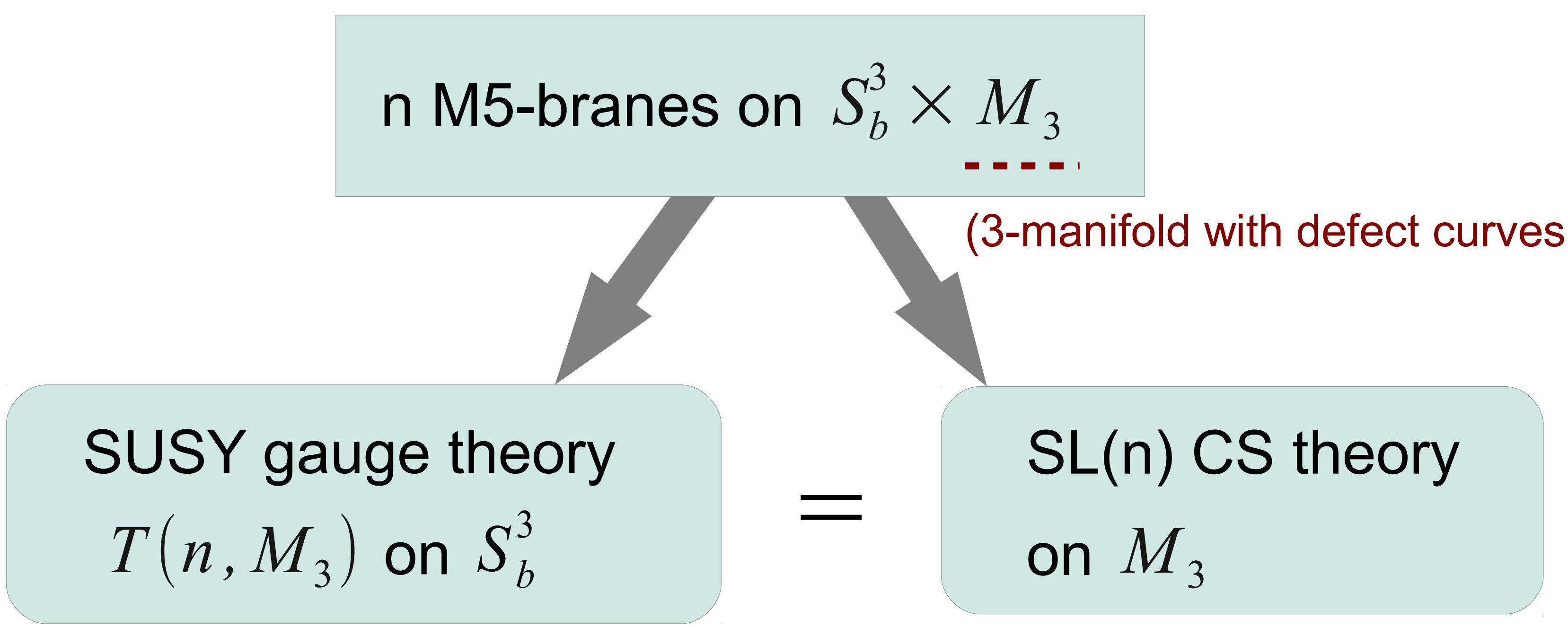
3-dim AGT (Dimofte-Gaiotto-Gukov, Cecotti-Cordova-Vafa)

another mysterious relation began to be uncovered,

between

-- 3D N=2 SUSY gauge theories on S_b^3

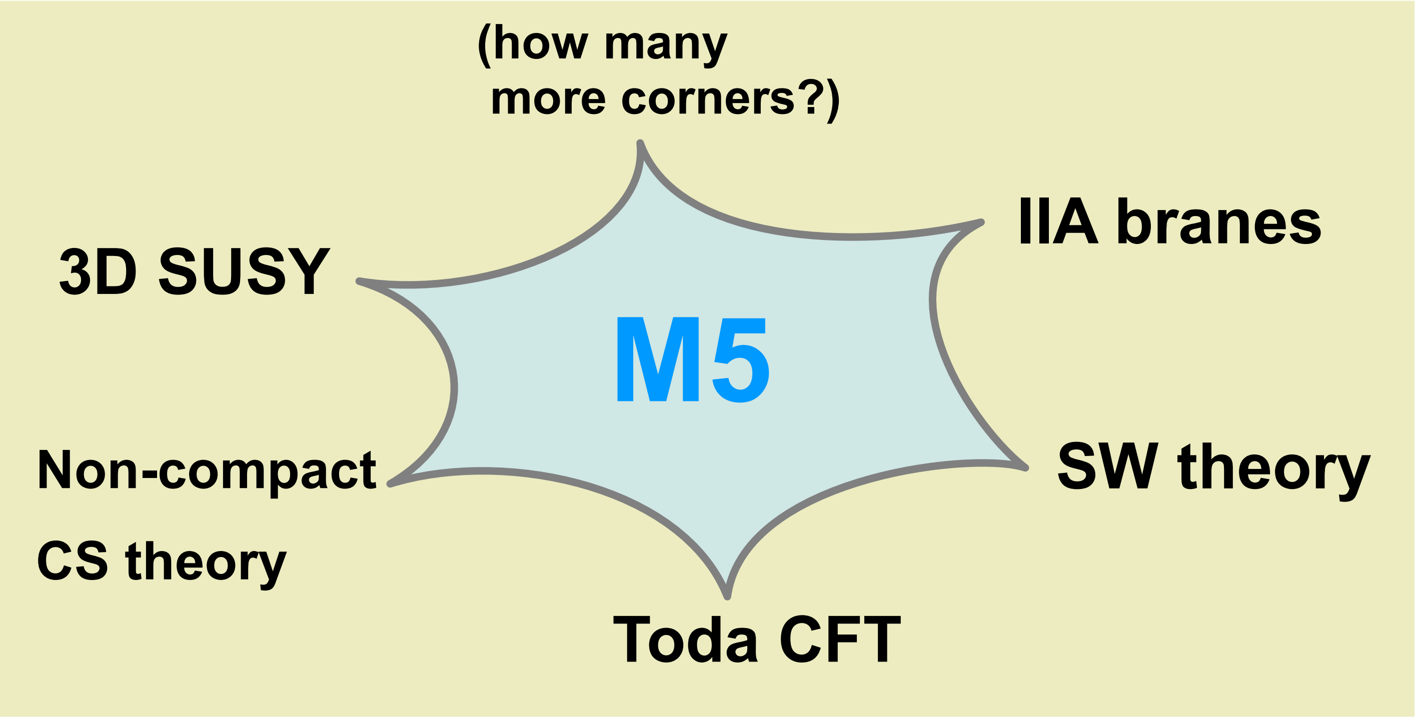
-- SL(n) Chern-Simons path integrals



Summary

M5-branes may provide a new web of duality

among different non-gravitating theories



Studying this web will lead to

a better understanding of M5-branes themselves

Application of the exact 3D result to M2-brane dynamics

“M2-brane mini-revolution” ('08)

New 3D Chern-Simons-matter theories

with extended SUSY were discovered,

and identified with the theory of multiple M2-branes.

ABJM model (Aharony-Bergman-Jafferis-Maldacena '08)

$U(N)_k \times U(N)_{-k}$ Chern-Simons-matter theory

describing N M2-branes in the "**orbifold**" $\mathbb{C}^4/\mathbb{Z}_k$

$$S = \frac{k}{4\pi}\int \mathrm{Tr}\Big(AdA + \tfrac{2}{3}A^3\Big) - \frac{k}{4\pi}\int \mathrm{Tr}\Big(\tilde{A}d\tilde{A} + \tfrac{2}{3}\tilde{A}^3\Big)$$

$+$ (bi-fundamental matters)

(AdS/CFT)

11d SUGRA on $AdS_4 \times S^7/\mathbb{Z}_k \quad (k \ll N^{1/5})$

IIA SUGRA on $AdS_4 \times \mathbb{CP}^3 \quad (k \gg N^{1/5})$

A puzzle: free energy at large N

$\lambda \equiv N/k$: 't Hooft coupling of ABJM model

At weak coupling $(k \gg N)$: $F \sim N^2$

(since ABJM model is a field theory of NxN matrices)

At strong coupling $(k \ll N)$: $F \sim N^{3/2}$

(from SUGRA analysis)

How are these two connected?

Solution

Drukker-Marino-Putrov ('10) studied the formula

for partition function on 3-sphere,

$$Z_{\mathrm{ABJM}}(N,k)$$

$$= \frac{1}{N!^2}\int \prod_{i=1}^{N}\frac{d\mu_i}{2\pi}\prod_{j=1}^{N}\frac{d\nu_j}{2\pi}\prod_{i<j}^{N}\left(2\sinh\frac{\mu_i-\mu_j}{2}\right)^2\prod_{i<j}^{N}\left(2\sinh\frac{\nu_i-\nu_j}{2}\right)^2$$

$$\prod_{i=1}^{N}\prod_{j=1}^{N}\left(2\cosh\frac{\mu_i-\nu_j}{2}\right)^{-2}\exp\left[\frac{ik}{4\pi}\left(\sum_{i=1}^{N}\mu_i^2-\sum_{i=1}^{N}\nu_i^2\right)\right]$$

using techniques of large-N matrix models.

(eigenvalue distribution analysis)

➡ reproduced the SUGRA result $F \simeq \frac{\sqrt{2}\pi}{3}k^{\frac{1}{2}}N^{\frac{3}{2}}.$

Conclusion

* Exact analysis of SUSY gauge theories (**localization**)

* Rigid SUSY theories on compact curved spaces (**spheres**)

    -- Application to multiple **M2-brane** theories

* New relations among QFTs in different dimensions (**AGT, DGG**)

    -- Better understanding of **M5-branes**