Chiral symmetry breaking in QCD and finite-volume effects

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Renormalization Group Approach from Ultra Cold Atoms to the Hot QGP, Kyoto, August 26 2011
Hadronic matter under extreme conditions

• heavy-ion collisions

[STAR experiment, RHIC, BNL]

• hadronic matter at finite temperature and density
QCD: chiral symmetry and confinement

• SU(3) gauge theory of strongly interacting quarks and gluons

\[
S_{\text{QCD}} = \int d^4 x \left\{ \bar{\psi} \left( i \slashed{D} - m \right) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\}
\]

• chiral flavor symmetry: symmetry for 2 massless quarks (flavors (u, d))

\[
SU(2)_L \times SU(2)_R
\]

left-handed quarks

\[
\psi_L \rightarrow U_L \psi_L
\]

right-handed quarks

\[
\psi_R \rightarrow U_R \psi_R
\]
Spontaneous chiral symmetry breaking

- chiral symmetry is spontaneously broken in the QCD ground state
- order parameter: chiral condensate
  \[ \langle \bar{\psi} \psi \rangle = \langle \psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L \rangle \neq 0 \]
- Goldstone bosons of spontaneous symmetry breaking: pions are the (almost massless) low-energy degrees of freedom
- low-momentum QCD: theory of weakly interacting pions
Deconfinement: $Z(3)$ symmetry breaking

- Polyakov loop acts as order parameter
  \[
  \Phi(\vec{x}) = \frac{1}{N_c} \text{Tr} \left[ \exp i \int_0^{1/T} d\tau A_4(\vec{x}, \tau) \right]
  \]

- $Z(3)$ center symmetry
  \[
  \Phi \to z\Phi, \quad z = \exp \left[ i \frac{2\pi n}{N_c} \right]
  \]

  de-confined phase: $Z(3)$ symmetry spontaneously broken
  \[
  \langle \Phi \rangle \neq 0
  \]

  confined phase: $Z(3)$ symmetric
  \[
  \langle \Phi \rangle = 0
  \]

- Polyakov loop corresponds to free energy of a free static quark:
  \[
  \Phi \sim \exp \left( -F_q/T \right)
  \]
Chiral symmetry restoration and deconfinement

- Chiral quark condensate
  - finite quark mass
  - strictly speaking, not a true phase transition
  - crossover

- Polyakov loop
  - crossover in presence of quarks

[Image of graphs showing the dependence of the Polyakov loop and chiral quark condensate on temperature.]

Phase transitions in QCD

\[ T_{\chi}^{n_f=2} \sim 175 \text{ MeV} \]

\[ T_d \sim 270 \text{ MeV} \]

\[ SU(2)_L \times SU(2)_R \rightarrow O(4) \]

\[ SU(3)_L \times SU(3)_R \]

\[ T_{\chi}^{n_f=3} \sim 155 \text{ MeV} \]

\[ m_{PS}^{\text{crit}} \sim 2.5 \text{ GeV} \]

\[ m_{PS}^{\text{crit}} \sim 200 \text{ MeV} \]

[F. Karsch and E. Laermann, arXiv:hep-lat/0305025.]
Methods for studying QCD

• QCD: perturbative at large momentum scales, non-perturbative at low momentum scales Q

• chiral perturbation theory: Effective Field Theory systematic expansion of low-energy QCD

• lattice QCD: simulation of the gauge theory in finite Euclidean space-time volume
Methods for studying QCD

• Dyson-Schwinger Equations for QCD
  [C. Fischer and R. Alkofer, J. Lücker, ...]

• functional Renormalization Group methods
  – for models, including confinement [B.-J. Schaefer, T. Herbst, A. Tripolt, V. Skokov, B. Friman, K. Redlich ...]
  – for QCD  [J. Braun, H. Gies, L. Haas, L. Fister, Pawlowski, ... ]

• (chiral) model calculations for transition regions
  → this talk
QCD in a finite volume

- finite volume introduces an additional scale $1/L$: probes different momentum regions, like temperature $T$

- no real phase transitions, critical long-range behavior affected by the volume size

- quark boundary conditions: what to choose in a finite spatial volume? periodic or anti-periodic?

finite-volume effects in the analysis?
Topics

• Critical scaling for the chiral phase transition in infinite and finite volume

• Phenomenological implications of finite-volume effects for the QCD phase diagram

long-range fluctuations: Renormalization Group (RG) methods
The quark-meson model

- a model for chiral symmetry breaking
- no gauge degrees of freedom

\[
\Gamma_\Lambda[\bar{\psi}, \psi, \sigma, \pi] = \int d^4x \left\{ \bar{\psi}(i \slashed{\partial}) \psi + g \bar{\psi} (\sigma + i \gamma_5 \vec{\tau} \cdot \vec{\pi}) \psi 
+ \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{1}{2} (\partial_\mu \pi)^2 + U_\Lambda (\sigma, \sigma^2 + \pi^2) \right\}
\]

- chiral symmetry breaking: \( SU(2) \times SU(2) \to SU(2) \)
as \( O(4) \to O(3) \) (meson sector) \( \langle \sigma \rangle \neq 0 \)

- specify effective action for the model at initial scale \( \Lambda \)
- use functional Renormalization Group
(Wetterich equation) to obtain effective action

Renormalization Group calculation

- RG flow equation with $3d$ optimized cutoff function: change of effective potential with change of RG scale $k$

\[ k \partial_k U_k(\phi^2) = k^5 \left[ \frac{3}{E_\pi} \left( \frac{1}{2} + n_B(E_\pi) \right) B_p(kL) + \frac{1}{E_\sigma} \left( \frac{1}{2} + n_B(E_\sigma) \right) B_p(kL) 
- \frac{2N_c N_f}{E_q} \left( 1 - n_F(E_q, \mu) - n_F(E_q, -\mu) \right) B_p(kL) \right] \]

- volume dependence encoded in mode-counting functions

- depends on choice of boundary conditions (for the spatial directions)

→ see also talk by M. Scherer
Critical behavior, scaling and universality

- Critical point (e.g. second-order phase transition)
- Diverging correlation length $\xi \sim 1/M_\sigma \to \infty$
- Critical long-range fluctuations dominate the physics of a system close to the critical point
- Universality classes: the same critical behavior is observed in different systems with the same symmetries and dimensionality
- Critical behavior is characterized by a few critical exponents $\beta, \gamma, \delta, \nu$ and scaling functions

Finite volume cuts off long-range fluctuation and impacts critical scaling behavior!
Scaling behavior in infinite volume

- two relevant couplings: temperature $T$ and symmetry breaking (quark mass $m$) $H$

\[ t = \frac{(T - T_c)}{T_0}, \quad h = \frac{H}{H_0} \]

- Scaling function for the order parameter $M(f_\pi, \langle \bar{\psi}\psi \rangle)$:

\[ M(t, h = 0) = (-t)^\beta \quad M(t = 0, h) = h^{1/\delta} \]

\[ M(t, h) = h^{1/\delta} f_M(z) \quad z = \frac{t}{h^{1/(\beta\delta)}} \]

- Scaling function for the susceptibility $\chi_\sigma$:

\[ \chi_\sigma(t, h) = \left. \frac{\partial M}{\partial H} \right|_{t, h} = \frac{1}{H_0} h^{1/\delta - 1} f_\chi(z) \]
Susceptibility $\chi_{\sigma}$ from the model: $O(4)$

- susceptibility $\chi_{\sigma}$ for small values of $m_\pi < 0.9$ MeV

\[ t = (T - T_c)/T_0 \]

\[ z = t/h^{1/\beta\delta} \]

- rescaled susceptibility $\chi_{\sigma} H_0 h^{1-1/\delta}$

Susceptibility $\chi_\sigma$ from the model: $O(4)$

- rescaled susceptibility $\chi_\sigma H_0 h^{1-1/\delta}$ for realistic values of $m_\pi$

- apparent scaling for large quark masses

- differs from actual scaling function!

- sensitivity to quark mass larger than in QCD

Infinite volume scaling in finite volume?

- rescaled susceptibility $\chi_\sigma H_0 h^{1-1/\delta}$ in finite volume
- susceptibility decrease $\chi_\sigma \sim L^2$ susceptibility increase

- $m_\pi = 75$ MeV
- deviations from infinite-volume scaling for $L < 6$ fm
- effects probably weaker in lattice QCD

Scaling behavior in finite volume

- correlation length is cut off by finite volume size $L$
- Volume size $L$ appears as additional relevant coupling
- Finite-size scaling functions with additional scaling variable

- Finite-size scaling function for the order parameter:
  \[ M(t, h, L) = L^{-\beta/\nu} Q_M(z, \bar{h}) \]
  \[ \bar{h} = hL^{\beta\delta/\nu} \]

- Finite-size scaling function for the susceptibility:
  \[ \chi_\sigma(t, h, L) = L^{\gamma/\nu} Q_\chi(z, \bar{h}) \]
Finite size scaling: Susceptibility $\chi_\sigma$

- susceptibility in finite volume

\begin{figure}
\centering
\includegraphics[width=\textwidth]{susceptibility_plot.png}
\caption{\textbf{Finite size scaling:} Susceptibility $\chi_\sigma$ in dependence on $h$ for $z=0$ and different volumes. (b) Susceptibility $\chi_\sigma$ in dependence on $h$ for $z=\bar{z}$ and different volumes.}
\end{figure}
Finite size scaling: Susceptibility $\chi^\sigma$

- finite-size scaled susceptibility

![Graph showing finite size scaling of susceptibility $\chi^\sigma$. The graph plots $\chi^\sigma$ against $hL^{\beta\delta/\nu}$ for different box sizes $L$. The data points for various $L$ (2 fm, 4 fm, 6 fm, 8 fm, 10 fm, 20 fm, 30 fm) are marked with different symbols and colors. The table shows the values of $M_\pi$ and $M_\pi L$ for different $L$ values: 4 (139, 2.82), 6 (85, 2.59), 8 (60, 2.43), 10 (45, 2.30), 20 (19, 1.94).]
QCD phase diagram

- second-order phase transition for two flavors in the chiral limit
- crossover at finite quark masses for finite temperature at $\mu = 0$
- conventional expectation: first-order phase transition with critical end point
Curvature of the transition line

- at small baryon chemical potential $\mu$, the phase transition line is characterized by the curvature $\kappa$

$$\frac{T_\chi(L, m_\pi, \mu)}{T_\chi(L, m_\pi, \mu = 0)} = 1 - \kappa \left( \frac{\mu}{(\pi T_\chi(L, m_\pi, 0))} \right)^2 + \ldots$$

- “sign problem” in lattice QCD: simulations are difficult at finite $\mu$


differences partially due to finite-volume effects?
Why Finite-volume effects?

• curvature depends on the sensitivity of the system on the chemical potential

\[ \mu = \frac{\partial F}{\partial N_q} \bigg|_{T,V} \]

• sensitivity in turn depends on the "constituent quark mass"

• constituent quark mass affected by volume!
Change of curvature in finite volume

- **periodic** boundary conditions for quarks
- decreasing curvature in intermediate volume
- corresponds to decreasing pion mass/increasing constituent quark mass
- decreased sensitivity to chemical potential

\[
\Delta \kappa = \frac{\kappa(L, m_\pi) - \kappa(\infty, m_\pi)}{\kappa(\infty, m_\pi)}
\]

\[m_\pi = 138 \text{ MeV}\]

[B.-J. Schaefer, J.Braun, B. Klein]
Phase diagram for QCD models in finite volume - qualitative results

- qualitatively clear effects of finite volume on curvature
- phase transition line tends to *flatten* in an intermediate volume range
- curvature increases dramatically for very small volumes

\[ L \to \infty \]
\[ L_c \lesssim L < \infty \]
\[ L < L_c \]
Phase diagram for QCD models in finite volume - first results

- potential discretized on a mesh grid
- first-order phase transition can be determined
- effects on critical point can be determined

[A. Tripolt, B.-J. Schaefer, J. Braun, B. Klein]
Effects of the quark boundary conditions

- Pion mass shift in $V = L^3 \times 1/T$ in quark-meson model
- periodic vs. anti-periodic quark boundary conditions (b.c.)

\[ f_\pi \sim \langle \sigma \rangle \]
\[ \langle \bar{\psi} \psi \rangle \sim \langle \sigma \rangle \]
\[ m_{\pi}^2 = m \frac{\langle \bar{\psi} \psi \rangle}{f_\pi^2} \sim \frac{m}{\langle \sigma \rangle} \]


Quark contributions for a finite volume

- quark momentum modes contributing to the condensate (and the constituent quark mass) in a large finite volume
- zero-mode for **periodic** b.c.
- no zero mode for **anti-periodic** b.c.
Quark contributions for a finite volume

- quark momentum modes contributing to the condensate (and the constituent quark mass) in a small finite volume
- enhancement of the zero-mode contribution $\sim 1/V$ for periodic b. c.

\[ \frac{2\pi}{L} \]
Comparison to Chiral Perturbation Theory

- comparison of model results with **anti-periodic** boundary conditions from RG to ChPT in NNLO

ChPT data thanks to G. Colangelo


![Graph](image)

agreement only for **this** choice of boundary conditions!

keep boundary conditions in mind for the **finite-volume** analysis of lattice QCD results
Conclusions

• Scaling functions from the functional renormalization group for the analysis of the QCD chiral phase transition: Results from a model for the chiral phase transition

• Finite-size effects in lattice simulations can lead to significant deviations from expected scaling behavior

• Additional finite-volume effects for curvature of transition line is expected

• choice of spatial quark boundary condition is important!
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