Understanding the entanglement between confinement and chiral-symmetry-breaking from QCD

Kei-Ichi Kondo

(Chiba Univ., Japan)

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e-Print: arXiv:1005.0314 [hep-th] $\rightarrow$ Phys.Rev. D82, 065024 (2010) (23pages).
Toward a first-principle derivation of confinement and chiral-symmetry-breaking crossover transitions in QCD.

e-Print: arXiv:1103.3829 [hep-th] $\rightarrow$ Phys.Rev. D (Rapid Communication), in press.
A low-energy effective Yang-Mills theory for quark and gluon confinement.

§ Introduction

The purpose of my talk is to understand the entanglement betweeen confinement/deconfinement and chiral-symmetry-breaking/restoration transitions at finite temperature (and finite density) from the first principle, i.e., directly from QCD.

We discuss how to derive a low-energy effective model which enables us to discuss both transitions directly from QCD on equal footing by performing the functional integration according to the idea of the Wilsonian renormalization group (RG).

First, we know that the Nambu-Jona-Lasinio (NJL) model

$$\mathscr{L}_{NJL} := \bar{\psi}(i\gamma^\mu\partial_\mu - \hat{m}_0)\psi + \frac{G}{2}[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2],$$

is a very good low-energy effective model with an UV cuttoff Λ^{NJL} to study the chiral-symmetry-breaking/restoration transition. See e.g. [Kunihiro-Hatsuda, 1994]

However, it is not obvious to see
(i) how the NJL model is derived from QCD,
(ii) how the NJL model can be improved so as to discuss confinement.

Recently, an improvement of the NJL model was achieved, the so-called the Polyakov-loop-extended NJL (PNJL) model [Fukushima, 2004],...
PNJL=gauged NJL Lagrangian + an effective potential for the Polyakov loop $\Phi[\mathscr{A}]$:

$$\mathscr{L}_{PNJL} := \bar{\psi}[i\gamma^\mu(\partial_\mu - ig\mathscr{A}_\mu) - \hat{m}_0]\psi + \frac{G}{2}[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2] - U(\Phi[\mathscr{A}], \Phi^*[\mathscr{A}]).$$

The PNJL model is successful phenomenology...

However, we still have the questions:
(i) how the NJL part is derived from QCD,
(ii) why the PNJL model can be a good low-energy-effective model also for confinement in the light of QCD.

According to the Wilsonian RG, we must perform the functional integration over high-energy modes in the range $k^2 \leq p^2 \leq \Lambda^2$ to obtain a low-energy-effective model described by the effective action Γ_k, which is valid for $p^2 \leq k^2$.

What d.o.f. should be identifid with the high-enegy mode?

How to treat both transitions on equal footing (to discuss their entanglement) ?

Of course, this task is too difficult to do in a straightforward way. So, we need a guiding principle.

We start from QCD defined by quarks and gluons:

$$\begin{aligned}\mathscr{L}_{QCD}[\psi, \bar{\psi}, \mathscr{A}] =& \mathscr{L}_q + \mathscr{L}_{YM} \\ \mathscr{L}_q =& \bar{\psi}(i\gamma^\mu D_\mu[\mathscr{A}] - \hat{m}_q + \mu_q \gamma^0)\psi = \bar{\psi}(i\gamma^\mu(\partial_\mu - ig\mathscr{A}_\mu) - \hat{m}_q + \mu_q\gamma^0)\psi, \\ \mathscr{L}_{YM} =& -\frac{1}{2}\mathrm{tr}(\mathscr{F}_{\mu\nu}[\mathscr{A}]\mathscr{F}^{\mu\nu}[\mathscr{A}]) = -\frac{1}{2}\mathrm{tr}[(\partial_\mu\mathscr{A}_\nu - \partial_\nu\mathscr{A}_\mu - ig[\mathscr{A}_\mu, \mathscr{A}_\nu])^2].\end{aligned} \tag{1}$$

We wish to obtain the effective action $\Gamma^{\mathrm{eff}}_{\mathrm{QCD}}(\sigma, L)$ or the effective potential $V^{\mathrm{eff}}_{\mathrm{QCD}}(\sigma, L)$ written in terms of two collective variables:

- the chiral condensate

$$\sigma(x) := \left\langle \bar{\psi}(x)\psi(x) \right\rangle \tag{2}$$

- the Polyakov loop average

$$L(\boldsymbol{x}) := \langle \mathrm{tr}\{P(\boldsymbol{x})\}/\mathrm{tr}\{\mathbf{1}\}\rangle\,, \ P(\boldsymbol{x}) := \mathscr{P}\exp\left[ig\int_0^{1/T} d\tau \mathscr{A}_4(\boldsymbol{x}, \tau)\right]. \tag{3}$$

My stragegy for the RG study in QCD

We start from fundamental fields, quark ψ & gluon $\mathscr{A}$, at high energy $p = \Lambda$.

$$S_\Lambda^{\rm QCD}[\psi, \mathscr{A}] \Longrightarrow S_\Lambda^{\rm QCD'}[\psi, \mathscr{V}, \mathscr{X}] \quad \text{(Step 1: Change of variables)}$$

$$\downarrow \text{(Step 2: Integration over } \mathscr{X}(p)\text{: } k \le p \le \Lambda)$$

$$* \, S_k^{\rm eff}[\psi, \mathscr{V}] \quad k \simeq M \simeq \Lambda_{\rm NJL} \to \text{NJL interactions}$$

$$\downarrow \text{(Step 3: Integration over } \psi(p)\text{: } k \le p \le \Lambda)$$

$$\Longrightarrow S_k^{\rm eff}[\sigma, L] \quad (k \le M)$$

$$\downarrow \text{Wetterich eq.(Step 4: Integration over } \sigma, L\text{: } 0 \le p^2 \le k^2)$$

$$S_{k=0}^{\rm eff}[\sigma, L] \to V^{\rm eff}(\sigma, L)$$

We ends with collective fields, meson: σ & Polyakov loop ($\simeq$ glueball) L, at low energy.
The effective potential $V^{\rm eff}(\sigma, L)$ of two order parameters:
chiral condensate $\sigma \sim < \bar\psi\psi >$, Polyakov loop average $L[\mathscr{A}]$,

It is important to discuss
(i) why we need change of variables $\mathscr{A} \to \mathscr{V}, \mathscr{X}$,
(ii) why $\mathscr{X}$ can be identified with a high-energy mode.

Difficulties in performing the RG approach of QCD

1. Integration of the high-energy mode is quite difficult to perform exactly.

2. The low energy physics should be described by the collective field variables σ and L rather than the fundamental field variables ψ and $\mathscr{A}$.

Assumptions

1. $\tilde{\mathscr{X}}(p \in [0, k])$ for $k \leq M$ has no concern with the RG evolution (no direct relevance to confinement):
$$\tilde{\mathscr{X}}(p \in [0, k]) = 0 \quad \text{for} \quad k \leq M$$

2. $\tilde{\mathscr{X}}(p \in [k, \Lambda])$ for $k > M$ can be integrated out by the Gaussian integration; 1-loop approximation is applied.

3. $\tilde{\mathscr{X}}(p \in [k, \Lambda])$ (and $\tilde{\mathscr{X}}(p \in [0, k])$) is not contained in the Polyakov loop operator.

The 1-loop perturbative calculations can be a seed for nonperturbative dynamics

In the high-energy region $g^2 \ll 1$,the perturbative calculation is valid.

§ Step 1: Changing variables to reformulate QCD

We decompose $\mathscr{A}$ into two parts, $\mathscr{V}$ and $\mathscr{X}$, a la Cho-Duan-Ge-Faddeev-Niemi: [Cho,1980][Duan-Ge, 1979][Faddeev-Niemi,1999]

$$\mathscr{A}_\mu(x) = \mathscr{V}_\mu(x) + \mathscr{X}_\mu(x), \tag{1}$$

⊙ Remarkable properties of new variables

• The action rewritten in terms of new variables has no $O(\mathscr{X})$ and $O(\mathscr{X}^3)$ terms

$$\mathscr{L}_q = \bar{\psi}[i\gamma^\mu(\partial_\mu - ig\mathscr{V}_\mu) - \hat{m}_q + i\gamma^0\mu_q]\psi + g\bar{\psi}\gamma^\mu T^A \psi \mathscr{X}_\mu^A, \tag{2}$$

$$\mathscr{L}_{YM} = -\frac{1}{4}\mathscr{F}_{\mu\nu}^A[\mathscr{V}]^2 - \frac{1}{2}\mathscr{X}^{\mu A} Q_{\mu\nu}^{AB} \mathscr{X}^{\nu B} - \frac{1}{4}(ig[\mathscr{X}_\mu, \mathscr{X}_\nu])^2, \tag{3}$$

$$Q_{\mu\nu}^{AB}[\mathscr{V}] := -(D_\rho[\mathscr{V}]D^\rho[\mathscr{V}])^{AB} g_{\mu\nu} + 2gf^{ABC}\mathscr{F}_{\mu\nu}^C[\mathscr{V}], \tag{4}$$

The variable $\mathscr{V}$ do not necessarily satisfy the field equation (of motion).

• Only $\mathscr{V}_\mu^A$ is responsible for the Wilson loop operator and the Polyakov loop operator: [Cho,2000][Kondo,2008]

$$W_C[\mathscr{A}] = W_C[\mathscr{V}], \quad L[\mathscr{A}] = L[\mathscr{V}] \tag{5}$$

From these requirements, the decomposition is uniquely determined for $G = SU(2)$:

$$\mathscr{V}_\mu(x) = c_\mu(x)\boldsymbol{n}(x) + ig^{-1}[\boldsymbol{n}(x), \partial_\mu \boldsymbol{n}(x)],$$
$$c_\mu(x) = \mathscr{A}_\mu(x) \cdot \boldsymbol{n}(x), \quad \mathscr{X}_\mu(x) = ig^{-1}[D_\mu[\mathscr{A}]\boldsymbol{n}(x), \boldsymbol{n}(x)], \tag{6}$$

if a unit vector field $\boldsymbol{n}(x)$ called the color field is given by solving defining equations:

(i) covariant constantness of $\boldsymbol{n}(x)$ in $\mathscr{V}_\mu(x)$: $$D_\mu[\mathscr{V}]\boldsymbol{n}(x) = 0 \tag{7}$$

(ii) orthogonality of $\mathscr{X}_\mu(x)$ to $\boldsymbol{n}(x)$: $$\mathscr{X}_\mu(x) \cdot \boldsymbol{n}(x) = 0 \tag{8}$$

This can be used to define new field variables as change of variables,

$$\mathscr{A}_\mu(x) \to \boldsymbol{n}(x), c_\mu(x), \mathscr{X}_\mu(x) \tag{9}$$

if the color field is given as a functional of $\mathscr{A}_\mu(x)$, i.e., $\boldsymbol{n}(x) = \boldsymbol{n}_{\mathscr{A}}(x)$. The relationship between $\mathscr{A}_\mu(x)$ and $\boldsymbol{n}(x)$ is given by (solving) the reduction condition,

$$\boldsymbol{\chi} := [\boldsymbol{n}(x), D_\mu[\mathscr{A}]D_\mu[\mathscr{A}]\boldsymbol{n}(x)] = 0. \tag{10}$$

[Kondo,Murakami& Shinohara, hep-th/0504107,hep-th/0504198][Kondo,hep-th/0609166]

First, we identify $\mathscr{X}_\mu$ with the "high-energy" mode in the range $p^2 \in [M^2, \Lambda^2]$ where M is the infrared (IR) cutoff and Λ is the ultraviolet (UV) cutoff as the initial value for the Wilsonian RG.

The identification of $\mathscr{X}_\mu$ with a high-energy mode is done based on the facts:

(1) [gauge-inv. mass term for $\mathscr{X}$] In sharp contrast to the field $\mathscr{A}_\mu$, we can introduce a gauge-invariant "mass term" for the $\mathscr{X}$ gluon:

$$\frac{1}{2}M^2 \mathscr{X}_\mu^A(x)\mathscr{X}_\mu^A(x). \tag{11}$$

We expect that the $\mathscr{X}_\mu$ gluon can acquire the (gauge-invariant) mass dynamically.

(2) [exponential fall-off of $\mathscr{X}$ correlator] the correlator $\langle \mathscr{X}_\mu^A(x)\mathscr{X}_\mu^A(y)\rangle$ exhibits the exponential fall-off like a massive correlator with mass M [Shibata et al, 2007]: See Fig. 1

$$\langle \mathscr{X}_\mu^A(x)\mathscr{X}_\mu^A(y)\rangle \simeq \frac{\text{const.}}{|x-y|^{3/2}} e^{-M|x-y|}, \quad M = 1.2 \sim 1.3 \text{ GeV}. \tag{12}$$

The variable $\mathscr{X}$ decouples in the long distance (or low energy).
The variable $\mathscr{V}$ has the correlator $\langle \mathscr{V}_\mu^A(x)\mathscr{V}_\mu^A(y)\rangle$ behaving just like $\langle \mathscr{A}_\mu^A(x)\mathscr{A}_\mu^A(y)\rangle$, and dominates in the long distance. This is also the case for the variable c_μ.

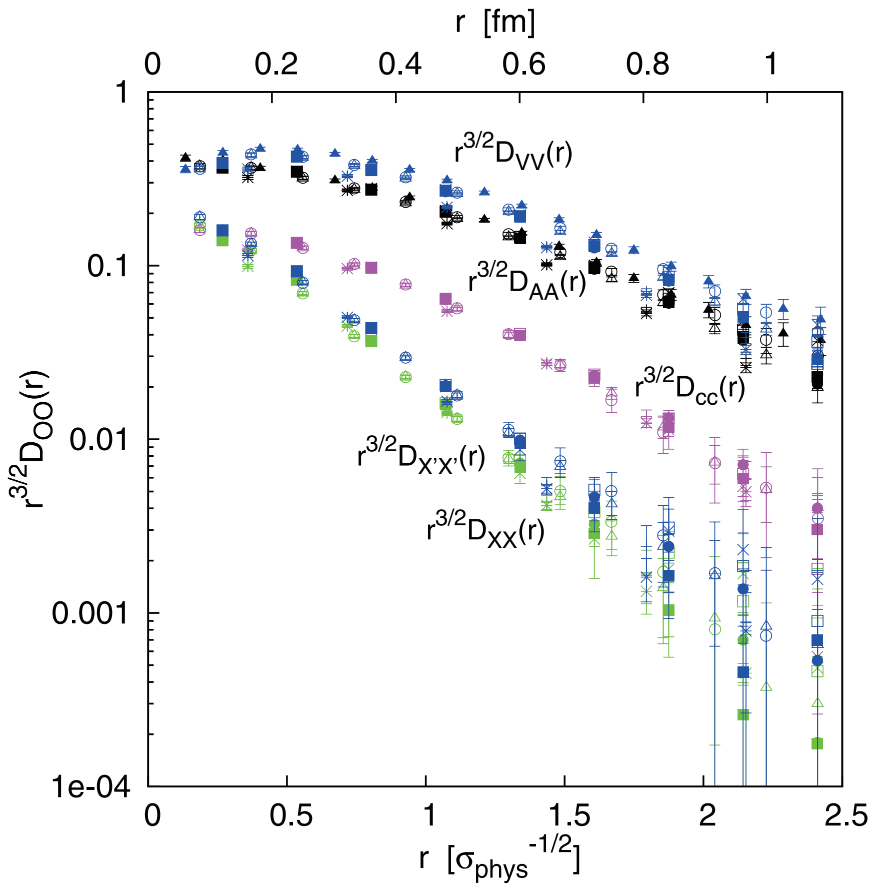


Figure 1: Logarithmic plots of the rescaled correlation function $r^{3/2}D_{OO}(r)$ as a function of r for $O = \mathbb{V}^A_\mu, \mathbb{A}^A_\mu, c_\mu, \mathbb{X}^A_\mu$ (and $\mathbb{X}'^A_\mu$) from above to below, using the same colors and symbols as those in Fig. **??**. Here two sets of data for the correlation function $D_{XX}(x-y)$ are plotted according to the two definitions of the $\mathbb{X}^A_\mu$ field on a lattice.

(3) [gauge-inv, condensate of mass dimension-2] The mass term or the mass scale M can originate from a gauge-invariant version of vacuum condensation of "mass dimension-2" [Gubarev, Stodolsky & Zakharov, 2001][Gubarev & Zakharov, 2001] [Kondo, 2001, 2003],

$$\langle \mathscr{X}_\mu^A(x)\mathscr{X}_\mu^A(x)\rangle \neq 0 \tag{13}$$

In fact, it is examined [Kondo,2006] that this condensation can be generated through self-interactions $O(\mathscr{X}^4)$ among $\mathscr{X}_\mu$ gluons,

$$\frac{1}{4}(ig[\mathscr{X}_\mu(x),\mathscr{X}_\nu(x)])^2 \to \frac{1}{2}M_X^2\mathscr{X}_\mu(x)\mathscr{X}_\mu(x), \quad M_X^2 \simeq g^2\left\langle \mathscr{X}_\nu^B(x)\mathscr{X}_\nu^B(x)\right\rangle, \tag{14}$$

Remark: The value (12) is nearly equal to the earlier result of the off-diagonal "gluon mass" M_A in the Maximally Abelian (MA) gauge for $SU(2)$ case, $M_A \simeq 1.2$ GeV. [Amemiya and Suganuma, 1999][Bornyakov et al, 2003]

Reformulation of quantum Yang-Mills theory in terms of new variables:

⊙ Path-integral continuum formulation:
SU(2) [K.-I. Kondo, T. Murakami and T. Shinohara, [hep-th/0504107], Prog. Theor. Phys. 115, 201–216 (2006).]
[K.-I. Kondo, T. Murakami and T. Shinohara, [hep-th/0504198], Eur. Phys. J. C 42, 475–481 (2005).]
SU(N) [K.-I. Kondo, T. Shinohara and T. Murakami, e-Print: arXiv:0803.0176 [hep-th], Prog. Theor. Phys. **120**, No.1, 1–50 (2008).]

⊙ Lattice gauge formulation:
SU(2) [S. Kato, K.-I. Kondo, T. Murakami, A. Shibata, T. Shinohara and S. Ito, [hep-lat/0509069], Phys. Lett. B **632**, 326–332 (2006).
SU(3) [K.-I. Kondo, A. Shibata, T. Shinohara, T. Murakami, S. Kato and S. Ito, arXiv:0803.2451[hep-lat], PLB**669**, 107-118 (2008).]
SU(N) [A. Shibata, K.-I. Kondo, and T. Shinohara, arXiv:0911.5294 [hep-lat].]

§ Wetterich RG flow equation

The flow equation for the k(RG scale)-dependent effective action Γ_k in the Wilsonian renormalizaton group [C. Wetterich, Phys. Lett. B**301**, 90 (1993)]

$$\partial_k \Gamma_k[\phi] = \frac{1}{2}\mathrm{STr}\left\{ \left[\frac{\overrightarrow{\delta}}{\delta\phi^\dagger}\Gamma_k[\phi]\frac{\overleftarrow{\delta}}{\delta\phi} + R_k \right]^{-1} \cdot \partial_k R_k \right\}. \tag{1}$$

The effective action Γ_k can be obtained by the Legendre transform from

$$Z_k[J] := \int \mathcal{D}\phi e^{-S[\phi]-\Delta S_k[\phi]+\int J\phi}, \quad \Delta S_k[\phi] := \int \frac{d^D q}{(2\pi)^D}\frac{1}{2}\tilde{\phi}(-q)R_k(q)\tilde{\phi}(q). \tag{2}$$

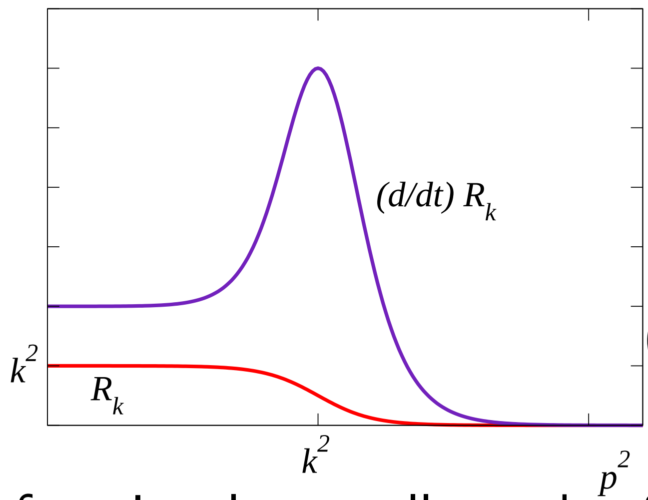
where $\Delta S_k[\phi]$ is the regulator term which is quadratic in ϕ.

The regulator function $R_k(q)$ should satisfy the three requirements:

1. The role of an IR regularization:

$$\lim_{q^2/k^2 \to 0} R_k(q) > 0, \tag{3}$$

The regulator term can be viewed as a momentum-dependent mass term. For instance, if $R_k \sim k^2$ for $q^2 \ll k^2$, it screens the l ,
$m^2 \sim k^2$.



2. Recovery of the standard form for $k \to 0$:

$$\lim_{k^2/q^2 \to 0} R_k(q) = 0. \tag{4}$$

We automatically recover the standard generating functional as well as the full effective action in this limit: $\lim_{k\to 0} Z_k[J] = Z[J]$ and $\lim_{k\to 0} \Gamma_k = \Gamma$.

3. The functional integral is dominated by the stationary point of the action in the limit $k \to \infty$:

$$\lim_{k^2 \to \Lambda^2 \to \infty} R_k(q) \to \infty, \tag{5}$$

This justifies the use of the saddle-point approximation which filters out the classical field configuration and the bare action, $\Gamma_{k\to\Lambda} \to S + \mathrm{const.}.$

See sketch of a regulator function $R_k(p^2)$ (lower curve) and its derivative $\partial R_k(p^2)$ (upper curve).

Whereas the regulator provides for an IR regularization for all modes with $p^2 \lesssim k^2$, its derivative implements the Wilsonian idea of integrating out fluctuations within a momentum shell near $p^2 \simeq k^2$.

The choice of the regulator function: An optimal choice is [Litim,2000]

$$R_k(p) = (k^2 - p^2)\theta(k^2 - p^2) = \begin{cases} k^2 - p^2 & (p < k) \\ 0 & (p > k) \end{cases}, \tag{6}$$

while a smooth regulator function is given by

$$R_k(p) = p^2 \frac{\exp(-p^2/k^2)}{1 - \exp(-p^2/k^2)} = \begin{cases} k^2 & (p \ll k) \\ 0 & (p \gg k) \end{cases}. \tag{7}$$

An approximate solution is given by

$$\Gamma_k[\phi] \simeq \frac{1}{2} \mathrm{Tr} \ln \left[D_{\phi\phi} + R_k \right], \tag{8}$$

if $\frac{\overrightarrow{\delta}}{\delta\phi^\dagger}\Gamma_k[\phi]\frac{\overleftarrow{\delta}}{\delta\phi} = D_{\phi\phi}$ is k-independent. This is the one-loop expression modified by the regulator function R_k.

§ Step 2: Integrating out $\mathscr{X}$ as a high-energy mode

First, neglecting $O(\mathscr{X}^4)$ terms, the integration over $\mathscr{X}_\mu^A$ can be done by the Gaussian integration as the first approximation of solving the Wetterich eq.

$$
\begin{aligned}
S_{\rm eff}^{\rm QCD} =& S_{\rm eff}^{\rm glue} + S_{\rm eff}^{\rm gNJL}, \\
S_{\rm eff}^{\rm glue} :=& \int d^D x \frac{-1}{4} \mathscr{F}_{\mu\nu}^2[\mathscr{V}] + \frac{1}{2} \mathscr{V}_\mu^A \delta^{AB} g_{\mu\nu} R_k^{\mathscr{V}} \mathscr{V}_\nu^B + \frac{i}{2} \ln\det \tilde{Q}[\mathscr{V}]_{\mu\nu}^{AB} - i \ln\det \tilde{G}[\mathscr{V}]^{AB}, \\
S_{\rm eff}^{\rm gNJL} :=& \int d^D x\ \bar{\psi}(i\gamma^\mu \mathcal{D}_\mu[\mathscr{V}] - \hat{m}_q + i\gamma^0 \mu + R_k^\psi)\psi \\
& + \int d^D x \int d^D y \frac{g^2}{2} \mathscr{J}_A^\mu(x) \tilde{Q}^{-1}[\mathscr{V}]_{\mu\nu}^{AB}(x,y) \mathscr{J}_B^\nu(y), \quad \mathscr{J}^{\mu A} := g\bar{\psi}\gamma^\mu T^A \psi \qquad (1) \\
\tilde{Q}_{\mu\nu}^{AB}[\mathscr{V}] :=& -(D_\rho[\mathscr{V}] D^\rho[\mathscr{V}])^{AB} g_{\mu\nu} + 2g f^{ABC} \mathscr{F}_{\mu\nu}^C[\mathscr{V}] + \delta^{AB} g_{\mu\nu} R_k^{\mathscr{X}}, \qquad (2) \\
\tilde{G}^{AB}[\mathscr{V}] :=& -(D_\rho[\mathscr{V}] D^\rho[\mathscr{V}])^{AB} + \delta^{AB} R_k^{\mathscr{C}}, \qquad (3)
\end{aligned}
$$

The nonlocal 4 fermion-interaction is generated. Here $(Q^{-1})_{\mu\nu}^{AB}(x,y)$ is the $\mathscr{X}$ field correlator. The range of the nonlocality is determined by the correlation length ξ, which is characteristic of the color exchange through gluon fields $\mathscr{X}$. This correlation length ξ is the inverse of the effective mass M_X. [$R_k^\psi, R_k^{\mathscr{V}}, R_k^{\mathscr{X}}, R_k^{\mathscr{C}}$: regulator functions]

Second, by including $O(\mathscr{X}^4)$ terms, the effective mass M_X is estimated.

• $\mathscr{X}_\mu^A$ decouple in the low-energy regime by acquiring the (gauge-invariant) mass dynamically. In fact, the correlator $\left\langle \mathscr{X}_\mu^A(x)\mathscr{X}_\mu^A(y)\right\rangle$ behaves like a massive propagator with mass M_X. The scale confinement sets in is

$$M_X = 1.2 \sim 1.3 \text{ GeV} \tag{4}$$

[A. Shibata et al, [0706.2529[hep-lat]], Phys. Lett. B **653**, 101–108 (2007).]
(cf. [K. Amemiya and H. Suganuma, hep-lat/9811035, Phys.Rev.D60, 114509 (1999).]

• The NJL model is valid below the UV cutoff $\Lambda_4^{\rm NJL} = 1.4$GeV ($\Lambda_3^{\rm NJL} = 0.6$GeV). [Kunihiro-Hatsuda, 1994][Klevansky,1992]

$$\sqrt{p^2} \lesssim \Lambda_4^{\rm NJL} = 1.4 \text{GeV}, \quad |\boldsymbol{p}| \lesssim \Lambda_3^{\rm NJL} = 0.6 \text{GeV}, \tag{5}$$

$\Longrightarrow$ Integration over the high-energy mode $\mathscr{X}$ in QCD results in the NJL model is based on the observation:

$$M_X \simeq \Lambda_4^{\rm NJL}, \tag{6}$$

M_X is identified with the ultraviolet cutoff $\Lambda^{\rm NJL}$ below which the effective NJL model works well.

(1) We adopt the Polyakov gauge for the background field $\mathscr{V}_0^A(x)$,

$$\mathscr{V}_0^A(x) = \delta_{A3} c_0(\boldsymbol{x}) \Longrightarrow \partial_0 \mathscr{V}_0^3(x) = 0, \tag{7}$$

(2) We expand the theory around the non-trivial background for $\mathscr{V}_0$:

$$\mathscr{V}_0^A(x) = c_0(\boldsymbol{x})\delta^{A3} = g^{-1}T\varphi\delta^{A3} + v_0(\boldsymbol{x})\delta^{A3}, \quad \mathscr{V}_j^A(x) = 0 + v_j^A(x), \tag{8}$$

and take into account the expansion up to quadratic in the fluctuation fields v_0, v_j.

For the Polyakov loop in the Polyakov gauge

$$P(\boldsymbol{x}) = \exp\left[ig\beta\mathscr{V}_4^3(\boldsymbol{x})\frac{\sigma_3}{2}\right] = \exp\left[i\varphi\frac{\sigma_3}{2}\right] \Longrightarrow L = \frac{1}{2}\mathrm{tr}(P) = \cos\frac{\varphi}{2}, \tag{9}$$

We choose a gauge to simplify the calculations. In what follows, we take the unitary-like gauge

$$n^A(x) \to \delta_{A3}, \tag{10}$$

which reproduces the same action as that of the original theory in the MA gauge.

§ Emergence of (non-local) NJL interactions

The nonlocal current-current 4 quark self-interaction is [nonlocal gauged Thirring model]

$$S_{\text{int}} = \int d^D x d^D y \frac{1}{2} (\bar{\psi}(x)\gamma^\mu T^A \psi(x)) e_a^A(x) g^2 (\tilde{Q}^{-1})^{ab}_{\mu\nu}(x,y)(\bar{\psi}(y)\gamma^\nu T^B \psi(y)) e_b^B(y), \tag{1}$$

In the approximation, $\mathscr{V}_0^A(x) \cong g^{-1} T\varphi \delta_{A3}, \mathscr{V}_j^A(x) \cong 0$ for SU(2). Then $\mathscr{F}^C_{\mu\nu}[\mathscr{V}] = 0$ and

$$\tilde{Q}^{ab}_{\mu\nu} \cong G^{ab} g_{\mu\nu}, \ G^{ab} = -\partial^2 \delta^{ab} + (T\varphi)^2 \delta^{ab} + 2\epsilon^{ab3} T\varphi \partial_0 + \delta^{ab} R_k. \tag{2}$$

The inverse is obtained as

$$\begin{aligned}
(\tilde{Q}^{-1})^{ab}_{\mu\nu}[\mathscr{V}] =& (G^{-1})_{ab} g^{\mu\nu} = \begin{pmatrix} \frac{1}{2}[F_\varphi + F_{-\varphi}] & -\frac{1}{2i}[F_\varphi - F_{-\varphi}] \\ \frac{1}{2i}[F_\varphi - F_{-\varphi}] & \frac{1}{2}[F_\varphi + F_{-\varphi}] \end{pmatrix} g^{\mu\nu}, \\
F_\varphi :=& \frac{1}{(i\partial_\ell)^2 + (i\partial_0 + T\varphi)^2 + R_k} = \frac{1}{(i\partial_\mu)^2 + (T\varphi)^2 + 2T\varphi i\partial_0 + R_k},
\end{aligned} \tag{3}$$

To simplify the calculation, we consider the diagonal parts or take the procedure

$$\frac{g^2}{2}(\tilde{Q}^{-1})^{ab}_{\mu\nu}(x,y) \cong g^{\mu\nu}\delta^{ab}\mathcal{G}(x-y), \tag{4}$$

Then

$$\mathcal{G}(x-y) \cong \frac{g^2}{2}(\tilde{Q}^{-1})^{ab}_{\mu\nu}(x,y)\frac{g_{\mu\nu}}{D}\frac{\delta^{ab}}{2} = \frac{g^2}{2}\frac{\mathrm{tr}(G^{-1})}{2} = \frac{g^2}{4}[F_\varphi + F_{-\varphi}], \tag{5}$$

For $D = 4$, we use the Fierz identity with $\Gamma_\alpha := (\mathbf{1}, i\gamma_5\vec{\tau})$

$$\begin{aligned} S_{\rm int} &= \int d^4x d^4y \sum_\alpha c_\alpha(\bar{\psi}(x)\Gamma_\alpha\psi(y))\mathcal{G}(x-y)(\bar{\psi}(y)\Gamma_\alpha\psi(x)) \\ &= \int d^4x d^4z \sum_\alpha c_\alpha\{\bar{\psi}(x+z/2)\Gamma_\alpha\psi(x-z/2)\}\mathcal{G}(z)\{\bar{\psi}(x-z/2)\Gamma_\alpha\psi(x+z/2)\} \end{aligned} \tag{6}$$

The nonlocal function $\mathcal{G}(z)$ is separated into a coupling constant G times a normalized distribution $\mathcal{C}(z)$:

$$\mathcal{G}(z) := \frac{G}{2}\mathcal{C}(z), \quad \int d^4z\mathcal{C}(z) = 1 \Longrightarrow \frac{G}{2} = \int d^4z\mathcal{G}(z) = \tilde{\mathcal{G}}(p=0), \tag{7}$$

The Fourier transform is

$$\tilde{\mathcal{G}}(p) = \frac{g^2}{2}\frac{\tilde{F}_\varphi(p) + \tilde{F}_{-\varphi}(p)}{2}, \quad \tilde{F}_\varphi(p) = \frac{1}{p^2 + (T\varphi)^2 + 2T\varphi p_0 + R_k(p)}. \tag{8}$$

The IR divergence of $\tilde{F}_{\pm\varphi}(p=0)$ and $G = 2\tilde{\mathcal{G}}(p=0)$ at $T=0$ is avoided by the presense of R_k, $R_k(p) \sim k^2$. To avoid the IR divergence at $k=0$, we add a contribution $M_0^2 \simeq M_X^2$ and replace $F_\varphi(i\partial)$ by

$$F_\varphi \to \frac{1}{(i\partial_\ell)^2 + (i\partial_0 + T\varphi)^2 + M_0^2 + R_k}. \tag{9}$$

The nonlocal gauged NJL model is derived as $\Gamma_\alpha := (\mathbf{1}, i\gamma_5\vec{\tau})$; $G = 2\tilde{\mathcal{G}}(p=0)$

$$\begin{aligned} S^E_{\mathrm{gNJL}} &= \int d^4x \bar{\psi}(x)(i\gamma^\mu \mathcal{D}_\mu[\mathscr{V}] + \hat{m}_q + i\gamma^0\mu + R^\psi_k)\psi(x) + S_{\mathrm{int}}, \\ S_{\mathrm{int}} &= \int d^4x d^4z \frac{G}{2}\mathcal{C}(z)[\bar{\psi}(x+z/2)\Gamma_\alpha\psi(x-z/2)\cdot\bar{\psi}(x-z/2)\Gamma_\alpha\psi(x+z/2)]. \end{aligned} \tag{10}$$

This is similar to the nonlocal NJL model proposed in [T. Hell, S. Rössner, M. Cristoforetti and W. Weise, 0810.1099, Phys. Rev.D**79**, 014022 (2009).]
The standard (local) NJL model follows for the limiting case $\mathcal{C}(z) = \delta^4(z)$.

§ Step 3: Integrating over ψ: collective field

We introduce collective fields σ and π by the auxiliary field method: we insert the unity,

$$1 = \int \mathcal{D}\sigma \mathcal{D}\vec{\pi} \exp\Big\{ - \int d^4 z \mathcal{C}(z) \int d^4 x \frac{1}{2G} |\Phi_\alpha(x) + G\bar{\psi}(x+z/2)\Gamma_\alpha \psi(x-z/2)|^2 \Big\}, \tag{1}$$

where $\Phi_\alpha(x) := (\sigma(x), \vec{\pi}(x))$ and $\Gamma_\alpha := (\mathbf{1}, i\gamma_5 \vec{\tau})$; [we have used $\int d^4 z \mathcal{C}(z) = 1$.]
Then the NJL four-quark interactions are eliminated to yield the Yukawa interaction:

$$Z_{\rm gNJL} := \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-S_{\rm gNJL}} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \int \mathcal{D}\sigma \mathcal{D}\vec{\pi} \exp\{-\tilde{S}_{\rm gNJL}\}, \tag{2}$$

$$\begin{aligned} \tilde{S}_{\rm gNJL} =& \int d^4 x \frac{1}{2G} \Phi_\alpha(x) \Phi^*_\alpha(x) \\ &+ \int d^4 x' \int d^4 y' \bar{\psi}(x') \Big\{ \delta^4(x'-y')[-i\gamma_\mu(\partial_\mu - ig\mathscr{V}_\mu) + \hat{m}_q + i\gamma^4 \mu_q + R_k^\psi] \\ &+ \frac{1}{2}\mathcal{C}(x'-y')\Gamma_\alpha[\Phi_\alpha(\frac{x'+y'}{2}) + \Phi^*_\alpha(\frac{x'+y'}{2})] \Big\} \psi(y'), \end{aligned} \tag{3}$$

with $x' := x + z/2$, $y' := x - z/2$,

We can integrate out quark fields to switch to the collective meson field:

$$Z_{\mathrm{gNJL}} = \int \mathcal{D}\sigma \mathcal{D}\vec{\pi} \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\{-\tilde{S}_{\mathrm{gNJL}}\} = \int \mathcal{D}\sigma \mathcal{D}\vec{\pi} \exp\{-S^{\mathrm{eff}}_{\mathrm{gNJL}}\}, \tag{4}$$

where the bosonised action $S^{\mathrm{eff}}_{\mathrm{gNJL}}$ is

$$\begin{aligned} S^{\mathrm{eff}}_{\mathrm{gNJL}} = & \int d^4x \frac{1}{2G} \Phi_\alpha(x)\Phi^*_\alpha(x) \\ & - \mathrm{Tr}\ln\Big\{\delta^4(x'-y')[-i\gamma_\mu(\partial_\mu - ig\mathscr{V}_\mu) + \hat{m}_q + i\gamma^4\mu_q + R^\psi_k] \\ & \qquad + \frac{1}{2}\mathcal{C}(x'-y')\Gamma_\alpha[\Phi_\alpha(\frac{x'+y'}{2}) + \Phi^*_\alpha(\frac{x'+y'}{2})]\Big\}, \quad G = \frac{g_F^2}{\Lambda^2_{\mathrm{NJL}}} \end{aligned} \tag{5}$$

For simplicity, we restrict to the local NJL interaction $\mathcal{C}(x'-y') = \delta^4(x'-y')$,

$$\begin{aligned} S^{\mathrm{eff}}_{\mathrm{gNJL}} = & \int_0^{1/T} d\tau \int d^3x \frac{\Lambda^2_{\mathrm{NJL}}}{2}(\sigma^2 + \boldsymbol{\pi}\cdot\boldsymbol{\pi}) \\ & - \mathrm{Tr}\ln[-i\gamma_\mu(\partial_\mu - ig\mathscr{V}_\mu) + m_q + \gamma^0\mu + g_F(\sigma + i\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\pi}) + R^\psi_k], \end{aligned} \tag{6}$$

We calculate the $\mathrm{Tr}\ln$ at finite temperature and density, The effective potential at $k=0$ is obtained in the approximation, $\mathscr{V}_\mu^A(x) \cong g^{-1}T\varphi\delta_{A3}\delta_{0\mu}$ at $\pi(x)=0$

$$S^{\mathrm{eff}}_{\mathrm{gNJL}} = \int_0^{1/T} d\tau \int d^3x V^{\mathrm{eff}}_{\mathrm{gNJL}}(\sigma, L), \tag{7}$$

$$\begin{aligned}
V^{\mathrm{eff}}_{\mathrm{gNJL}}(\sigma, L) &= \frac{\Lambda^2_{\mathrm{NJL}}}{2}\sigma^2 - 2N_cN_f\int \frac{d^3p}{(2\pi)^3}E_p \\
&- 2N_fT\int \frac{d^3p}{(2\pi)^3}\{\mathrm{tr}_c \ln[1+Pe^{-(E_p-\mu)/T}] + \mathrm{tr}_c\ln[1+P^\dagger e^{-(E_p+\mu)/T}]\}, \\
(N_c=2) =&\frac{\Lambda^2_{\mathrm{NJL}}}{2}\sigma^2 - 2N_cN_f\int \frac{d^3p}{(2\pi)^3}E_p \\
&- 2N_fT\int \frac{d^3p}{(2\pi)^3}\{\ln[1+2Le^{-(E_p-\mu)/T} + e^{-2(E_p-\mu)/T}] \\
&+ \ln[1+2Le^{-(E_p+\mu)/T} + e^{-2(E_p+\mu)/T}]\}, \quad (N_c=2) \\
E_p &:= \sqrt{\boldsymbol{p}^2 + (m_q+g_F\sigma)^2}, \quad L = \mathrm{tr}(P)/\mathrm{tr}(1) = \cos\frac{\varphi}{2},
\end{aligned} \tag{8}$$

Putting $L\equiv 1$ (deconfinement) recovers the NJL case, $V^{\mathrm{eff}}_{\mathrm{gNJL}}(\sigma, L\equiv 1) = V^{\mathrm{eff}}_{\mathrm{NJL}}(\sigma)$.

We observe the Polyakov-loop dependence of the chiral transition.
In the neighborhood of the chiral transition at T_χ, $\sigma \ll 1$ for $N_c = 2$, $m_q = 0$, $\mu = 0$

$$
\begin{aligned}
V^{\rm eff}_{\rm gNJL}(\sigma, L) =& \frac{1}{2}\Lambda^2_{\rm NJL}\sigma^2 - 2N_c N_f \int \frac{d^3p}{(2\pi)^3} E_p \leftarrow \ T = 0 \ (L\text{-indep.}) \\
& - 2N_f T \int \frac{d^3p}{(2\pi)^3} \{2\ln[1 + 2Le^{-E_p/T} + e^{-2E_p/T}]\} \leftarrow \ T \neq 0 \text{ part } (L\text{-dep.}), \\
=& - \frac{N_f}{2\pi^2}\Lambda^4_{\rm NJL} - \frac{1}{2}\Lambda^2_{\rm NJL}\left(\frac{1}{\pi^2/N_f} - \frac{1}{g_F^2}\right)(g_F\sigma)^2 + \frac{N_f}{8\pi^2}\sigma^4 \ln\frac{4\Lambda^2_{\rm NJL}e^{-1/2}}{\sigma^2} + \dots \\
& + \begin{cases} -N_f\frac{7\pi^2}{90}T^4 + \frac{N_f}{6}T^2\sigma^2 + \dots & (\text{for} L \equiv 1 : \mathrm{NJL}) \\ -N_f\frac{7\pi^2}{1440}T^4 + \frac{N_f}{24}T^2\sigma^2 + \dots & (\text{for} L \equiv 0) \end{cases}
\end{aligned}
\tag{9}
$$

First, we can understand that the chiral symmetry is spontaneously broken at $T = 0$:

$$
g_F \leq g_c \Longrightarrow \sigma = 0, \quad g_F > g_c \Longrightarrow \sigma \neq 0, \ \sigma \simeq \sqrt{\frac{g_F^2 - g_c^2}{-\ln(g_F^2 - g_c^2)}} \quad (g_F \downarrow g_c), \tag{10}
$$

where $g_c^2 = \frac{8\pi^2}{d_q} = \frac{\pi^2}{N_f}$, with quark d.o.f. $d_q = 4N_cN_f$.

Second, we can understand that the chiral symmetry is restored at high temperature:

$$V^{\text{eff}}_{\text{gNJL}}(\sigma, L) = -\frac{N_f}{2\pi^2}\Lambda^4_{\text{NJL}} + \frac{N_f}{8\pi^2}\sigma^4 \ln \frac{4\Lambda^2_{\text{NJL}} e^{-1/2}}{\sigma^2} + ... + \begin{cases} -N_f \frac{7\pi^2}{90} T^4 + \frac{N_f}{6}[T^2 - T_\chi(L\equiv 1)^2]\sigma^2 + O(\sigma^4) & (L \equiv 1) \\ -N_f \frac{7\pi^2}{1440} T^4 + \frac{N_f}{24}[T^2 - T_\chi(L\equiv 0)^2]\sigma^2 + O(\sigma^4) & (L \equiv 0) \end{cases} \tag{11}$$

Chiral symmetry is restored, $\sigma(T) = 0$ for $T \geq T_\chi$, even if $\sigma(T = 0) \neq 0$ with $g > g_c$:

$$\sigma(T) = \begin{cases} 0 & (T \geq T_\chi) \\ \text{const.}[T_\chi(L)^2 - T^2]^{1/2} & (0 \leq T < T_\chi) \end{cases} \tag{12}$$

Even in the first approximation, the chiral-symmetry breaking/restoration transition is well understood. However, T_χ depends on the value of the Polyakov loop $L = L(T)$.

$$T_\chi(L \equiv 1) = \sqrt{\frac{3}{N_f}\left(\frac{1}{g_c^2} - \frac{1}{g_F^2}\right)}, \quad T_\chi(L \equiv 0) = N_c \cdot T_\chi(L \equiv 1), \tag{13}$$

Thus, $N_c \cdot T_\chi^{NJL}$ is an upper bound on the chiral transition temperature T_χ in QCD in the presence of confinement.

§ Step 4(a): Integrating out L; confinement transition

First, we calculate a part of the effective potential generated by integrating out the $\mathscr{X}$ field. In particular, we calculate it at $k=0$ case, i.e., without the regulator function:

$$\begin{aligned} V_{T,0} =& \lim_{k\downarrow 0} V_{T,k} = \frac{1}{2}\ln\det[Q^{AB}_{\mu\nu}] - \ln\det[G^{AB}] = \ln\det[G^{AB}] \\ =& \mathrm{Tr}\ln[G^{AB}] = T\sum_{n\in\mathbb{Z}}\int\frac{d^3p}{(2\pi)^3}\mathrm{tr}\ln[\boldsymbol{p}^2 - D_0^2]. \end{aligned} \tag{1}$$

If $\mathscr{V}^A_\mu \cong g^{-1}T\varphi\delta_{A3}\delta_{0\mu}$, then $V_{T,0}$ reads [N. Weiss, Phys.Rev. D**24**, 475(1981).]

$$V_W(\varphi) = T^4\left[-\frac{1}{6}(\varphi-\pi)^2 + \frac{1}{12\pi^2}(\varphi-\pi)^4 + \frac{\pi^2}{12}\right] \pmod{2\pi}. \tag{2}$$

The effective potential V_W is g^2 independent and the overall curve scales as T^4.
V_W has symmetries: $V_W(-\varphi) = V_W(\varphi)$ and $V_W(\varphi + 2\pi n) = V_W(\varphi)$.
For the Polyakov loop in the Polyakov gauge

$$P(\boldsymbol{x}) = \exp\left[ig\beta\mathscr{V}^3_4(\boldsymbol{x})\frac{\sigma_3}{2}\right] = \exp\left[i\varphi\frac{\sigma_3}{2}\right] \Longrightarrow L = \frac{1}{2}\mathrm{tr}(P) = \cos\frac{\varphi}{2}, \tag{3}$$

$V_W(\varphi)$ has minima at $\varphi = 2\pi n$. $\Longrightarrow L = \cos\frac{\varphi}{2} = \pm 1$: deconfinement.
$\Longrightarrow V_W(\varphi)$ is valid at very high temperature.

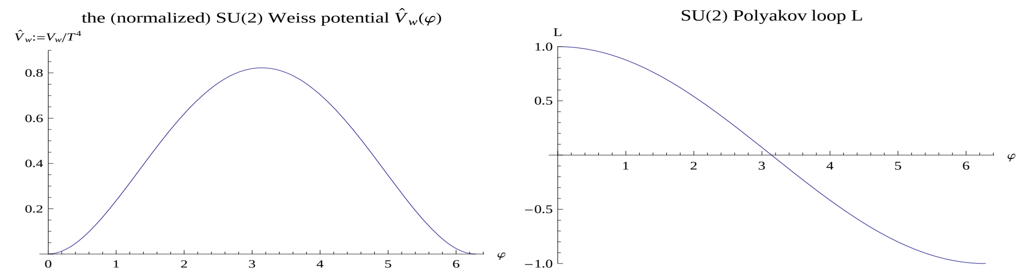
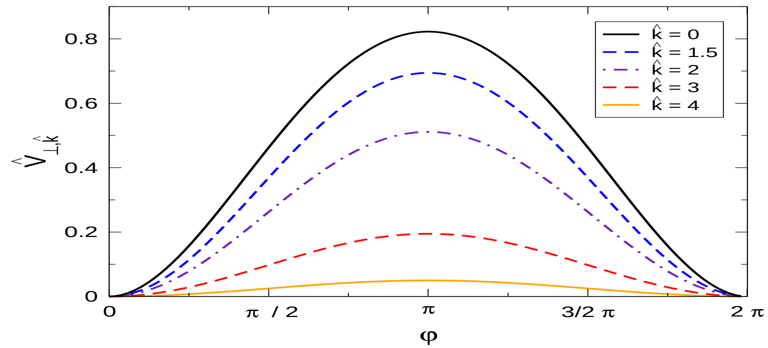


Figure 2: (Left panel) (normalized) SU(2) Weiss potential $\hat{V}_W$. (Right panel) SU(2) Polyakov loop L.

For any k,

$$\begin{aligned} V_{T,k}(L) =& \frac{1}{2}\ln\det[Q^{AB}_{\mu\nu} + \delta^{AB}\delta_{\mu\nu}R_k] - \ln\det[G^{AB} + \delta^{AB}R_k] \\ =& \mathrm{Tr}\ln[G^{AB} + \delta^{AB}R_k] \quad \text{minus of the ghost contribution} \\ =& T\sum_{n\in\mathbb{Z}}\int\frac{d^3p}{(2\pi)^3}\mathrm{tr}\ln[\boldsymbol{p}^2 - D_0^2 + (k^2 - \boldsymbol{p}^2)\theta(k^2 - \boldsymbol{p}^2)] \\ . \end{aligned} \tag{4}$$

$$\lim_{k \downarrow 0} V_{T,k} = V_W, \quad \lim_{k \uparrow \infty} V_{T,k} = 0, \tag{5}$$



$\uparrow$ $\uparrow$ $\uparrow$

$L = 1$ $(\varphi = 0)$ $L = 0$ $(\varphi = \pi)$ $L = -1$ $(\varphi = 2\pi)$

Figure 3: $\hat{V}_{T,k}$ for different values of $\hat{k}$

$T < T_d$

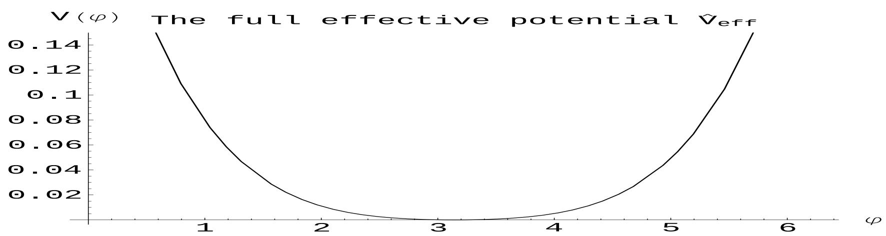


Figure 4: Expected full effective potential at low temperature

The flow equation

$$\partial_k \Gamma_k = \frac{1}{2}\beta \int \frac{d^3p}{(2\pi)^3} \left\{ \left[\frac{\overrightarrow{\delta}}{\delta \mathscr{V}^\dagger} \Gamma_k[\phi] \frac{\overleftarrow{\delta}}{\delta \mathscr{V}} + R_{0,k} \right]^{-1} \partial_k R_{0,k} \right\} + \partial_k V_{T,k}, \quad \beta := \frac{1}{T}, \tag{6}$$

After integrating over the fields other than $\mathscr{V}_0$, we are lead to the effective action of $\mathscr{V}_0$,

$$\begin{aligned} \Gamma_k[\mathscr{V}_0] =& \beta \int d^3x \left\{ -\frac{1}{2} Z_0 \mathscr{V}_0(\boldsymbol{x}) \partial_j \partial_j \mathscr{V}_0(\boldsymbol{x}) + V^{\rm glue}_{{\rm eff},k}[\mathscr{V}_0] \right\}, \\ V^{\rm glue}_{{\rm eff},k}[\mathscr{V}_0] =& V_{T,k}[\mathscr{V}_0] + \Delta V_k[\mathscr{V}_0]. \end{aligned} \tag{7}$$

Then the flow equation reformulated for ΔV_k with the external input $V_{T,k}$ reads

$$\beta \partial_k (\Delta V_k[\mathscr{V}_0]) = \frac{1}{2}\beta \int \frac{d^3p}{(2\pi)^3} \left\{ \left(\frac{1}{\Gamma_k^{(2)} + R_{0,k}} \right)_{00} \partial_k R_{0,k} \right\}, \tag{8}$$

$$\Gamma_k^{(2)}[\mathscr{V}_0] = \beta \left\{ Z_0 \boldsymbol{p}^2 + \partial^2_{\mathscr{V}_0} (V_{T,k}[\mathscr{V}_0] + \Delta V_k[\mathscr{V}_0]) \right\} \tag{9}$$

By introducing the dimensionless RG scale $\hat{k}$ and the dimensionless potential $\hat{V}$ by

$$\hat{k} := k/T, \quad \hat{V} := V/T^4 \Longrightarrow \partial_{\hat{k}} \Delta \hat{V}_{\hat{k}}[\mathscr{V}_0] = \frac{1}{6\pi^2} \frac{(1+\eta_k/5)\hat{k}^2}{1 + \frac{4\pi\alpha_k}{\hat{k}^2}\partial_\varphi^2(\hat{V}_{T,\hat{k}}[\mathscr{V}_0] + \Delta\hat{V}_{\hat{k}}[\mathscr{V}_0])}, \tag{10}$$

All scales are measured in units of temperature. Here we have introduced the anomalous dimension η_k and the running gauge coupling α_k

$$\eta_k := \partial_t \ln Z_k = -\partial_t \ln \alpha_k, \quad \alpha_k := \frac{g_k^2}{4\pi} = Z_k^{-1}\frac{g^2}{4\pi}, g_k^2 := Z_k^{-1} g^2. \tag{11}$$

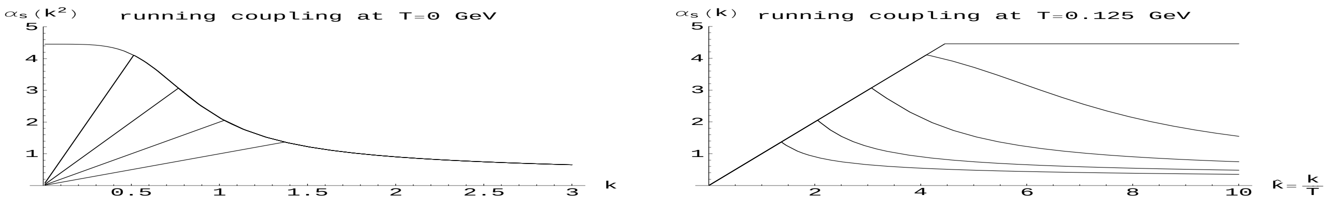


Figure 5: Running gauge coupling at $T = 0.001, 0.125, 0.25, 0.50, 1.0$ GeV. (Left panel) α_k vs. k. (Right panel) α_k vs. $\hat{k}$.

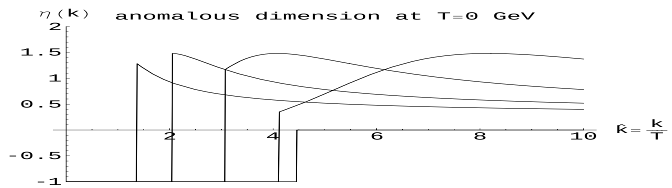


Figure 6: η_k vs. $\hat{k}$ at $T = 0.001, 0.125, 0.25, 0.50, 1.0$ GeV.

The existence of the confinement phase at low temperature has been shown
(1) by solving the Wetterich equation numerically for the first time
[F. Marhauser and J.M. Pawlowski, 0812.1144[hep-ph], unpublished!]

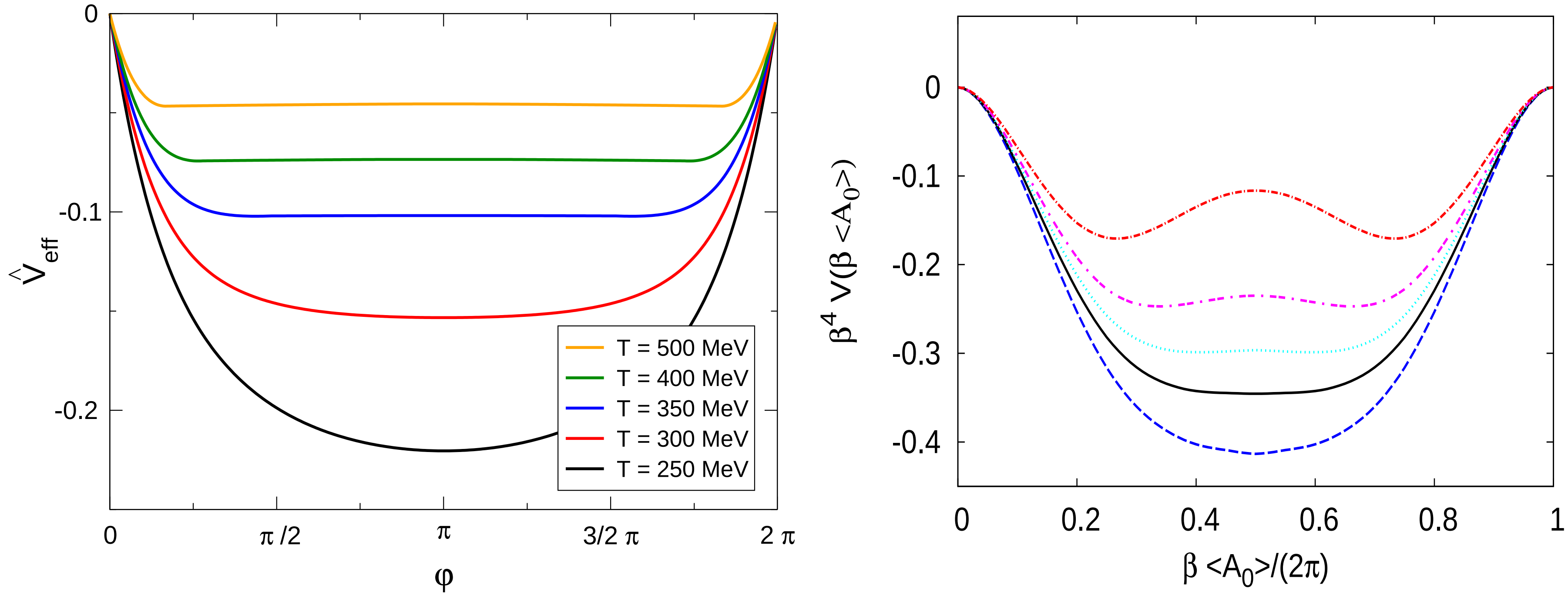


Figure 7: (Left panel) Full effective potential $\hat{V}_{\mathrm{eff}}(\varphi)$, normalised to 0 at $\varphi = 0$ [reprinted from [F. Marhauser and J.M. Pawlowski, 0812.1144[hep-ph].]
(Right panel) Order-parameter potential for SU(2) for various temperatures. For SU(2) we show the potential for $T = 260, 266, 270, 275, 285\,$MeV (from bottom to top). We find $T_c \approx 266\,$MeV for SU(2). [J. Braun, H. Gies and J.M. Pawlowski, 0708.2413[hep-th], Phys.Lett. B684, 262–267 (2010)]

The input for solving the flow equation was just a running gauge coupling constant α_k.

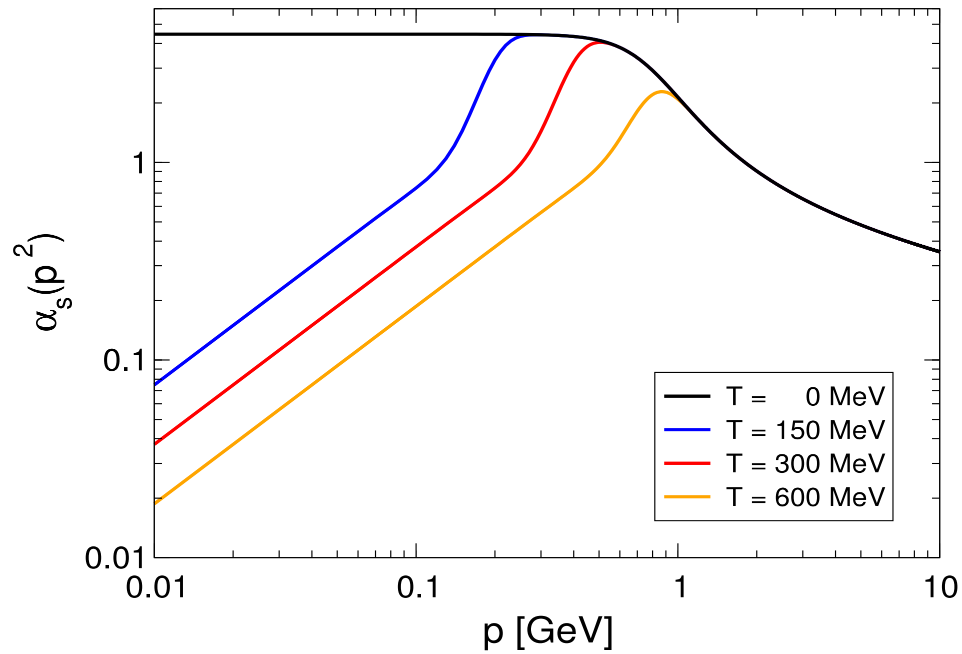


Figure 8: The running gauge coupling constant α_s for temperatures $T = 0, 150, 300, 600$ MeV [reprinted from [F. Marhauser and J.M. Pawlowski, 0812.1144[hep-ph].]

(2) by a qualitative argument of the Landau-Ginzburg type from the coupled RG flow equations for coefficients of the effective potential.
[Kondo, arXiv:1005.0314[hep-th] $\to$ Phys.Rev. D82, 065024 (2010)]

For SU(2) gluodynamics, from the center Z(2) invariance and $L = L^*$:

$$V_{\text{glue}}(L) = C_0 + \frac{C_2}{2}L^2 + \frac{C_4}{4}L^4 + O(L^6). \tag{12}$$

$C_4 > 0$, $C_2 = 0$ is the transition point of a 2nd order transition.

We confirm the correctness of the Landau-Ginzburg description of the deconfinement/confinement transition by making use of the flow equation.

Expand $\hat{V}_{T,\hat{k}}$ in powers of $\tilde{\varphi} = \varphi - \pi$, ($\tilde{\varphi} = 0 \Longleftrightarrow L = 0$)

$$\hat{V}_{T,\hat{k}} = A_{0,k} + \frac{A_{2,k}}{2}\tilde{\varphi}^2 + \frac{A_{4,k}}{4!}\tilde{\varphi}^4 + O(\tilde{\varphi}^6), \tag{13}$$

where odd terms ($\tilde{\varphi}$ and $\tilde{\varphi}^3$) do not appear due to center invariance.
Suppose that $\Delta\hat{V}_{\hat{k}}$ is of the form:

$$\Delta\hat{V}_{\hat{k}} = a_{0,k} + \frac{a_{2,k}}{2}\tilde{\varphi}^2 + \frac{a_{4,k}}{4!}\tilde{\varphi}^4 + O(\tilde{\varphi}^6), \tag{14}$$

The flow equation for the effective potential is reduced to the flow equations for coefficients, i.e., coupled differential equations for coefficients:

$$\partial_{\hat{k}} a_{2,k} = -\frac{(1+\frac{1}{5}\eta_k)\hat{k}^2}{6\pi^2} \frac{\frac{4\pi\alpha_k}{\hat{k}^2}(A_{4,k}+a_{4,k})}{[1+\frac{4\pi\alpha_k}{\hat{k}^2}(A_{2,k}+a_{2,k})]^2} < 0, \tag{15}$$

$$\partial_{\hat{k}} a_{4,k} = +\frac{(1+\frac{1}{5}\eta_k)\hat{k}^2}{6\pi^2} \frac{6[\frac{4\pi\alpha_k}{\hat{k}^2}(A_{4,k}+a_{4,k})]^2}{[1+\frac{4\pi\alpha_k}{\hat{k}^2}(A_{2,k}+a_{2,k})]^3} > 0, \cdots . \tag{16}$$

In fact, $\partial_{\hat{k}} a_{1,k} = 0$, $\partial_{\hat{k}} a_{3,k} = 0$ with no odd terms. With an initial condition, $a_{1,k} = 0 = a_{3,k}$, we have $a_{1,k} \equiv 0$ and $a_{3,k} \equiv 0$.

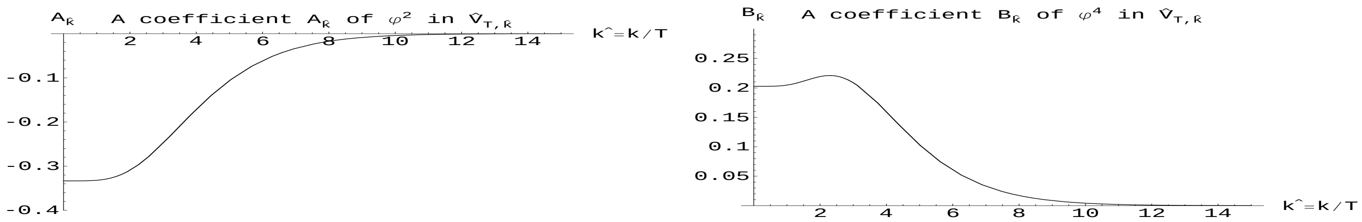


Figure 9: $A_{2,k}$ and $A_{4,k}$

Low-temperature case

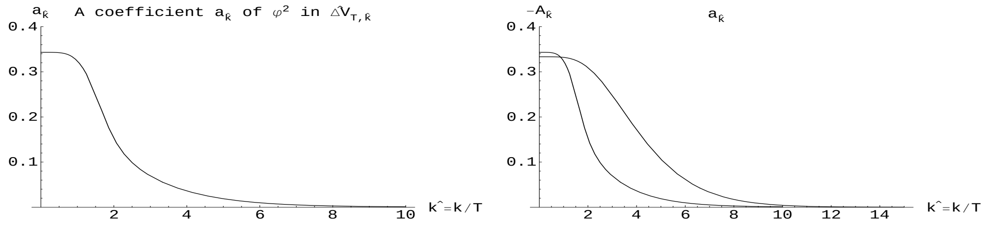


Figure 10: $a_{2,k}$ vs. $-A_{2,k}$ at $T = 0.3$GeV

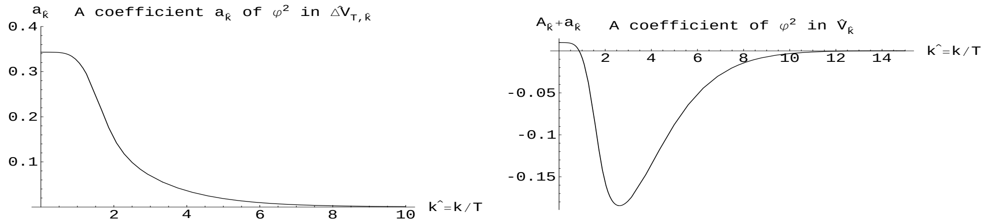


Figure 11: $a_{2,k}$ vs. $A_{2,k} + a_{2,k}$ at $T = 0.3$GeV

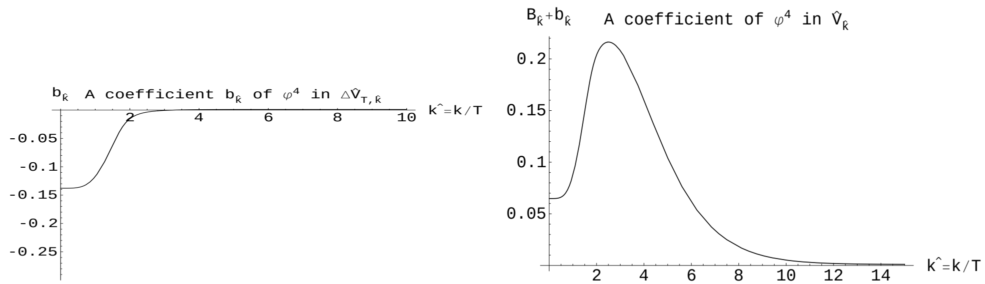


Figure 12: $a_{4,k}$ vs. $A_{4,k} + a_{4,k}$ at $T = 0.3$GeV

Explanation for the existence of a 2nd order transition

$$\partial_{\hat{k}} a_{2,k} = -\frac{(1+\frac{1}{5}\eta_k)}{6\pi^2}\frac{4\pi\alpha_k(A_{4,k}+a_{4,k})}{[1+\frac{4\pi\alpha_k}{\hat{k}^2}(A_{2,k}+a_{2,k})]^2}, . \tag{17}$$

The flow starts from $k = \Lambda \gg 1$ where $a_{2,k} = 0$ and $A_{2,k} + a_{2,k} < 0$. Assume $(A_{4,k} + a_{4,k}) > 0$ for $0 < k < \Lambda$, which guarantees a possible 2nd order transition. Then the right-hand side is negative. Consequently, $a_{2,k}$ is positive and increases (monotonically) as k decreases.

The difference between high and low temperature phases attributes to the behavior of the running coupling constant.

In the high-temperature region $T \gg 1$, $k = T\hat{k}$ becomes large for a wide range of $\hat{k}$ and the $\alpha_{2,k}$ stays small in this perturbative region. This leads to slow increase of $a_{2,k}$ and $A_{2,k} + a_{2,k} < 0$ or $a_{2,k} < -A_{2,k}$ at $k = 0$. breaking of the center symmetry.

In the low-temperature region $T \ll 1$, $k = T\hat{k}$ becomes small for a wide range of $\hat{k}$ and the $\alpha_{2,k}$ gets into the deep IR non-perturbative region where $\alpha_{2,k}$ takes large value. This leads to rapid increase of $a_{2,k}$ and $A_{2,k} + a_{2,k} > 0$ or $a_{2,k} > -A_{2,k}$ at $k = 0$. recovery of the center symmetry.

In the flow equation, the temperature-dependence comes from the running coupling constant. At $T = 0$, the running coupling is well parameterized as [Fischer and Alkofer, hep-ph/0202202]

$$\alpha_k = \frac{4\pi \times 0.709/N_c}{\ln[e + a_1(k^2)^{a_2} + b_1(k^2)^{b_2}]}, \tag{18}$$

where in units of GeV, $a_1 = 5.292,\ a_2 = 2.324,\ b_1 = 0.034,\ b_2 = 3.169.$
For $k \gg T$, we adopt this form: $k^2 = T^2\hat{k}^2$

$$\alpha_k = \frac{g_k^2}{4\pi} = \frac{4\pi \times 0.709/N_c}{\ln[e + a_1(T^2\hat{k}^2)^{a_2} + b_1(T^2\hat{k}^2)^{b_2}]}, \tag{19}$$

For $k \ll T$, we adopt the running coupling which is governed by an infrared fixed point:

$$\alpha_k = \alpha^*_{3d}\frac{k}{T} + c_1\left(\frac{k}{T}\right)^2 + c_2\left(\frac{k}{T}\right)^3 + \cdots = \alpha^*_{3d}\hat{k} + c_1\hat{k}^2 + c_2\hat{k}^3 + \cdots, \tag{20}$$

where we determine coefficients $c_1, c_2, ...$ such that the coupling at zero temperature and its derivative with respect to k are connected continuously with this ansatz at the scale set by the lowest non-vanishing bosonic Matsubara-mode $\omega = 2\pi T$.
[J. Braun and H. Gies, hep-ph/0512085, Phys. Lett. B 645, 53 (2007).]
[J. Braun and H. Gies, hep-ph/0602226, J. High Energy Phys. 06, 024 (2006).]

§ Entanglement NJL coupling

In our model, G and $\tilde{C}$ are determined in conjunction with the Polyakov loop. This leads to the temperature-dependence of the NJL coupling of nonlocal NJL model

$$G(T) = 2\tilde{\mathcal{G}}(p=0) = \frac{g^2}{(T\varphi)^2 + M_0^2} \Longrightarrow \frac{G(T)}{G(0)} = \frac{M_0^2}{(T\varphi)^2 + M_0^2}. \qquad (1)$$

Below the deconfinement temperature T_d^*, $L \simeq 0$, i.e., $\varphi \simeq \pi$ $(T < T_d^*)$

$$\frac{G(T)}{G(0)} \simeq \frac{M_0^2}{\pi^2 T^2 + M_0^2} \quad (T < T_d^*). \qquad (2)$$

The chiral phase transition is understood, since $G(T)$ is monotonicaly decreasing in T.

$$G(0) = \frac{g^2}{M_0^2} > G_c, \quad T \uparrow \infty \Longrightarrow G(T) \downarrow 0 \qquad (3)$$

The chiral transition temperature T_χ will be determined (if $T_\chi \leq T_d$) from

$$G(T_\chi) \equiv G(0) \frac{M_0^2}{T_\chi^2 \pi^2 + M_0^2} = G_c \qquad (4)$$

Recall the first approximation of the effective potenital

$$V^{\rm eff}_{\rm gNJL}(\sigma, L) = -\frac{N_f}{2\pi^2}\Lambda^4_{\rm NJL} - N_f\frac{7\pi^2}{1440}T^4 + \frac{N_f}{24}(T^2 - T_c^2)\sigma^2 + \frac{N_f}{8\pi^2}\sigma^4 \ln\frac{4\Lambda^2_{\rm NJL}e^{-1/2}}{\sigma^2} + \ldots$$
$$- N_f\frac{\pi}{4}T^4 L + \frac{N_f}{2\pi}T^2 L\sigma^2 + \ldots \qquad (5)$$

The temperature and Polyakov-loop dependence of the NJL coupling,

$$\frac{1}{G(T)}\sigma^2 = \frac{M_0^2}{g^2}\sigma^2 + \frac{T^2\varphi^2}{g^2}\sigma^2 = \frac{1}{G(0)}\sigma^2 + \frac{\pi^2}{g^2}T^2\sigma^2 - \frac{4\pi}{g^2}T^2 L\sigma^2 + O(L^2\sigma^2) \qquad (6)$$

where we have used $L = \cos\frac{\varphi}{2}, \quad \varphi = 2\arccos L = \pi - 2L - \frac{1}{3}L^3 + O(L^5)$ reduces the chiral phase transition temperature to the value less than $2T_\chi(L \equiv 1)$:

$$T_\chi(L \equiv 0) = \sqrt{\frac{1}{\frac{N_f}{12} + \frac{\pi^2}{g^2}}\left(\frac{1}{g_c^2} - \frac{1}{g_F^2}\right)} = 2\sqrt{\frac{N_f}{N_f + \frac{12\pi^2}{g^2}}}T_\chi(L \equiv 1) < 2T_\chi(L \equiv 1), \qquad (7)$$

where $T_\chi(L \equiv 1) = \sqrt{\frac{3}{N_f}\left(\frac{1}{g_c^2} - \frac{1}{g_F^2}\right)}$, and $g^2 = G(0)M_0^2 \cong 10 \sim 14$.

Parameters for the local NJL model with $N_c = 3$ and $N_f = 2$:
noncovariant cutoff

$$\Lambda_{\mathrm{NJL}} = 0.65[\mathrm{GeV}],\ G\Lambda_{\mathrm{NJL}}^2 = 4.2,\ (G = 10.1[\mathrm{GeV}^{-2}]),\ m_q = 0.0032[\mathrm{GeV}], \tag{8}$$

covariant cutoff

$$\Lambda_{\mathrm{NJL}} = 1.0[\mathrm{GeV}],\ G\Lambda_{\mathrm{NJL}}^2 = 8.0,\ (G = 8.0[\mathrm{GeV}^{-2}]),\ m_q = 0.24[\mathrm{GeV}], \tag{9}$$

the resulting physical quantities

$$f_\pi = 0.092[\mathrm{GeV}],\ m_\pi = 0.14[\mathrm{GeV}],\ \left\langle \bar{\psi}\psi \right\rangle = -(0.251[\mathrm{GeV}])^3 \tag{10}$$

Parameters for the nonlocal NJL model with $N_c = 3$ and $N_f = 2$:

$$G = 41.1[\mathrm{GeV}^{-2}],\ m_q = 3.3[\mathrm{meV}],\ f_\pi = 0.092[\mathrm{GeV}],\ m_\pi = 0.14[\mathrm{GeV}] \tag{11}$$

Summary for the emergence of the initial NJL interactions from QCD

1. The quark-gluon fundamental interaction in QCD induce a four-quark NJL interaction as a result of integration over high-energy gluon mode $\mathscr{X}$, which is understood as exchange of one $\mathscr{X}$ gluon with the effective mass M_X.

   $\cong$ momentum shell integration over $M_X < p < \Lambda$ in the Wilsonian RG

2. The NJL model is valid below $\Lambda_{\rm NJL}$ which is nearly equal to M_X.

3. The NJL coupling can be nonlocal. $\rightarrow$ nonlocal NJL

   Hell, Rössner, Cristoforetti and Weise, arXiv:0810.1099, Phys.Rev.D**79**, 014022 (2009).
   Sasaki, Friman and Redlich, [hep-ph/0611147], Phys. Rev. D**75**, 074013 (2007).
   Blaschke, Buballa, Radzhabov and Volkov, arXiv:0705.0384 [hep-ph], Phys. Atom. Nucl.**71**, 1981–1987 (2008).

4. The NJL coupling can depend on the Polyakov loop. $\rightarrow$ entanglement NJL

Therefore, we expect that the modified NJL model written in terms of collective fields σ, π can be a good low-energy effective model of QCD below $\Lambda_{\rm NJL} \simeq M_X$.

EPNJL model: The idea of Polyakov-loop-dependent NJL couplling was incorporated as the Entanglement PNJL (EPNJL) model for $N_c = 3$:

$$G \to G_{\rm s}(\Phi) := G[1 - \alpha_1 \Phi\Phi^* - \alpha_2(\Phi^3 + \Phi^{*3})] \qquad (12)$$

[Y. Sakai, T. Sasaki, H. Kouno and M. Yahiro, e-Print: arXiv:1006.3648 [hep-ph]. Phys.Rev.D82, 076003 (2010)] $N_f = 2$, $m_q = 5.5$MeV, $(\alpha_1, \alpha_2) = (0.2, 0.2)$,

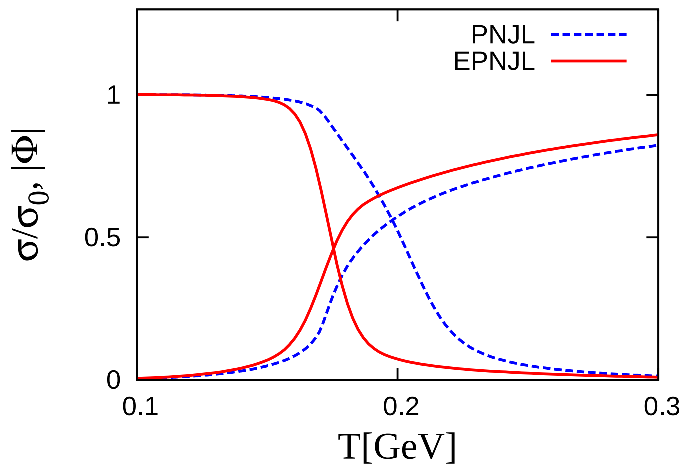


Figure 13: T dependence of the chiral condensate and the Polyakov loop. The curves that decrease (increase) as T increases represent the chiral condensate (Polyakov loop). The solid (dashed) curves are the results of the EPNJL (PNJL) model. Here, the chiral condensate is normalized by the value σ_0 at $T = 0$.

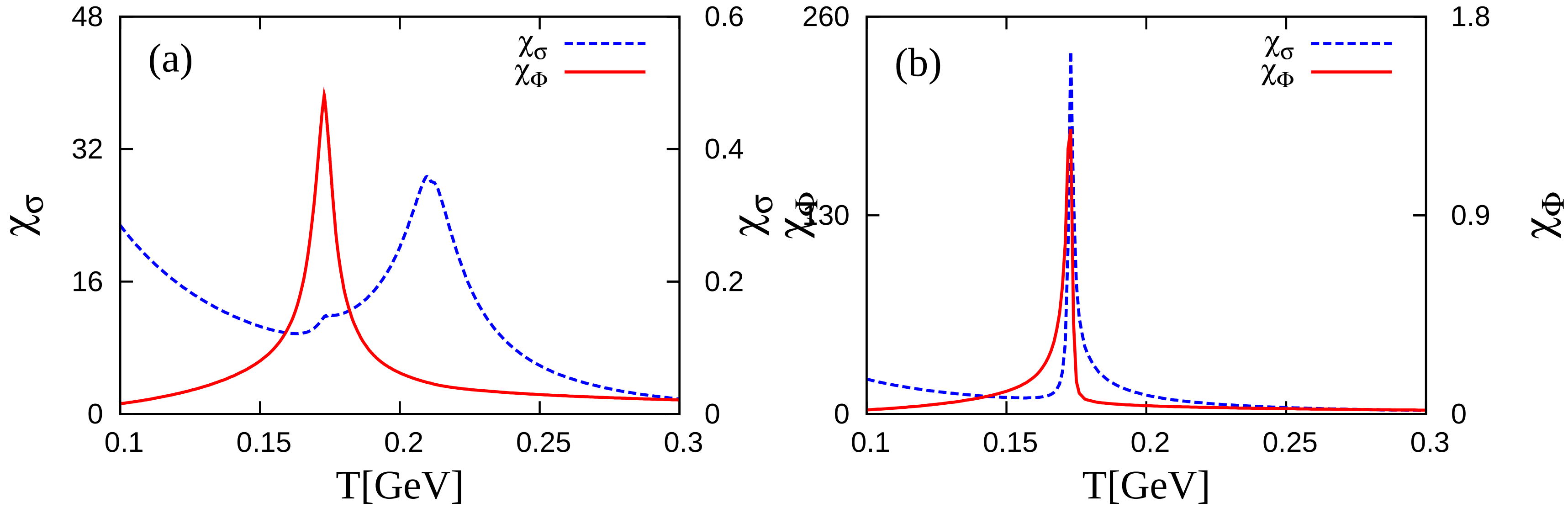


Figure 14: T dependence of the susceptibilities of the chiral condensate (dashed curve) and the Polyakov loop (solid curve). Panels (a) and (b) correspond to the PNJL and EPNJL models, respectively.

In the EPNJL model, the mesonic potential is modified into

$$G\sigma^2 \to G_{\rm s}(\Phi)\sigma^2 = G[1 - \alpha_1\Phi\Phi^* - \alpha_2(\Phi^3 + \Phi^{*3})]\sigma^2 \tag{13}$$

The entanglement vertex $G_{\rm s}(\Phi)$ makes the correlation between the chiral restoration and the deconfinement transition stronger, as expected.

§ Step 4(b): Integrating out σ

In our strategy, a full effective potential $V_{\mathrm{eff},k}^{\mathrm{QCD}}(\sigma, \varphi)$ of QCD is given by

$$V_{\mathrm{eff},k}^{\mathrm{QCD}}(\sigma, \varphi) = V_k^{\mathrm{glue}}(\varphi) + V_k^{\mathrm{quark}}(\sigma, \varphi) + \Delta V_k^{\mathrm{QCD}}(\sigma, \varphi), \tag{1}$$

with the pure gluon part $V_k^{\mathrm{glue}}(\varphi) = V_{T,k}$

$$V_k^{\mathrm{glue}}(\varphi) = \mathrm{Tr}\ln[-\partial^2\delta^{ab} + (T\varphi)^2\delta^{ab} + 2\epsilon^{ab3}T\varphi\partial_0 + \delta^{ab}R_k], \tag{2}$$

the quark part

$$V_k^{\mathrm{quark}}(\sigma, \varphi) = \frac{1}{2G}\sigma^2 - \mathrm{Tr}\ln\{i\gamma^\mu\partial_\mu + m_q + \mathcal{C}\sigma - T\varphi(\sigma^3/2)\gamma^4 + i\mu_q\gamma^4 + R_k^{\mathrm{quark}}\}, \tag{3}$$

and a non-perturbative part $\Delta V_k^{\mathrm{QCD}}(\sigma, \varphi)$ induced in the RG evolution.
After integrating out the fields other than σ and φ, Γ_k is assumed to be

$$\Gamma_k = \int_0^{1/T} dx_4 \int d^3x \Big\{\frac{1}{2}Z_0[\partial_j\mathscr{V}_0(\boldsymbol{x})]^2 + \frac{1}{2}Z_\sigma[\partial_j\sigma(x)]^2 + V_{\mathrm{eff},k}^{\mathrm{QCD}}(\sigma, \varphi)\Big\}, \tag{4}$$

and obeys the flow equation:

$$\partial_t \Gamma_k = \frac{1}{2}\mathrm{Tr}\left\{\left[\frac{\overrightarrow{\delta}}{\delta\sigma^T}\Gamma_k\frac{\overleftarrow{\delta}}{\delta\sigma} + R_k\right]^{-1}\cdot\partial_t R_k\right\} + \frac{1}{2}\mathrm{Tr}\left\{\left[\frac{\overrightarrow{\delta}}{\delta\mathscr{V}^T}\Gamma_k\frac{\overleftarrow{\delta}}{\delta\mathscr{V}} + R_k\right]^{-1}\cdot\partial_t R_k\right\}.$$

In the neighborhood of the transition point,

$$\begin{aligned} V_{\mathrm{eff}}^{\mathrm{QCD}}(\sigma,\varphi) =& V^1(\tilde{\varphi}) + V^2(\sigma) + V^{\mathrm{mix}}(\sigma,\tilde{\varphi}), \\ V^1(\tilde{\varphi}) =& c_{0,k} + c_{1,k}\tilde{\varphi} + \frac{c_{2,k}}{2}\tilde{\varphi}^2 + \frac{c_{3,k}}{3}\tilde{\varphi}^3 + \frac{c_{4,k}}{4}\tilde{\varphi}^4 + O(\tilde{\varphi}^6), \\ V^2(\sigma) =& \frac{E_{2,k}}{2}\sigma^2 + \frac{E_{4,k}}{4}\sigma^4 + O(\sigma^6), \\ V^{\mathrm{mix}}(\sigma,\tilde{\varphi}) =& f_{1,k}\sigma^2\tilde{\varphi} + \cdots, \end{aligned} \tag{5}$$

where $V^1(\tilde{\varphi})$ denotes a part written in terms of $\tilde{\varphi}$ alone, and $V^2(\sigma)$ a part written in terms of σ alone, while $V^{\mathrm{mix}}(\sigma,\tilde{\varphi})$ denotes the cross term between σ and $\tilde{\varphi}$.

• $c_1\tilde{\varphi}$: center-symmetry breaking term by dynamical quark

Once dynamical quarks are introduced, the center symmetry is explicitly broken. $c_1\tilde{\varphi}$ is the center symmetry breaking term which is absent in the gluon part.

•$f_1\sigma^2\tilde{\varphi}$: induce an entanglement between center symmetry and chiral symmetry.

By solving the flow equation from $k = \Lambda$ to $k = 0$, we have at $k = 0$

$$\begin{aligned} V_{\mathrm{eff}}^{\mathrm{QCD}}(\sigma, L) =& V^{\mathrm{glue}}(L) + V^{\mathrm{quark}}(\sigma) + V^{\mathrm{mix}}(\sigma, L), \\ V^{\mathrm{glue}}(L) =& C_1 L + \frac{C_2}{2} L^2 + \frac{C_3}{3} L^3 + \frac{C_4}{4} L^4 + O(L^5), \\ V^{\mathrm{quark}}(\sigma) =& \frac{E_2}{2} \sigma^2 + \frac{E_4}{4} \sigma^4 + O(\sigma^6), \\ V^{\mathrm{mix}}(\sigma, L) =& F_1 \sigma^2 L + \cdots , \end{aligned} \tag{6}$$

The extremum of the effective potential is given at

$$\begin{aligned} 0 =& \frac{\partial V_{\mathrm{eff}}^{\mathrm{QCD}}(\sigma, L)}{\partial \sigma} = E_2 \sigma + E_4 \sigma^3 + 2 F_1 L \sigma + ... \\ 0 =& \frac{\partial V_{\mathrm{eff}}^{\mathrm{QCD}}(\sigma, L)}{\partial L} = C_1 + C_2 L \ (+ C_3 L^2 + C_4 L^3) + F_1 \sigma^2 + ..., \end{aligned} \tag{7}$$

At low temperature where $\sigma \neq 0$ and $0 < L \ll 1$,

$$\sigma^2 = -E_2/E_4 - 2F_1/E_4L + ..., \quad E_2 < 0, \quad E_4 > 0$$
$$L = -C_1/C_2 - F_1/C_2\sigma^2 + ..., \quad C_2 > 0, \quad C_4 > 0, \quad C_1 < 0, \tag{8}$$

As the chiral condensate starts to decrease $\sigma : -E_2/E_4 \searrow 0$ towards the chiral-symmetry-restoration T^*_χ, the Polyakov loop average starts to increase $L : 0 \nearrow -C_1/C_2$ signaling the onset of deconfinement around T^*_d, provided that $F_1 > 0$ (and $C_1 < 0$). L behaves oppositely to σ. $\Longrightarrow T^*_d < T^*_\chi$

In the first approximation of the effective potenital, the Polyakov-loop dependence of the NJL coupling leads to

$$\frac{N_f}{2\pi}T^2L\sigma^2 - \frac{4\pi}{g^2}T^2L\sigma^2 = F_1L\sigma^2 \Longrightarrow F_1 = 4\pi\left(\frac{N_f}{8\pi^2} - \frac{1}{G(0)M_0^2}\right)T^2. \tag{9}$$

This result is consistent with the observation that color deconfines when chiral symmetry is restored in hot gauge theories.
[Mocsy, Sannino & Tuominen, Phys.Rev.Lett.**92**, 182302 (2004), hep-ph/0308135.]

§ Conclusion and discussion

⊙ In this talk we discussed how to understand the entanglement betweeen chiral-symmetry restoration/breaking and deconfinement/confinement crossover transitions at finite temperature.

The basic ingredients are a reformulation of QCD in terms of new variables, a non-Abelian Stokes theorem for the Polyakov loop operator, and the flow equation of the Wetterich type based on the Wilsonian renormalization group.

⊙ For the quark sector: By a suitable identification of the gluon field $\mathscr{X}$ as a high-energy mode, the functional integration over the high-energy modes in QCD yields the NJL four-quark interaction.

An important observation for supporting the emergence of the NJL interaction is

$$\Lambda_{\rm NJL} \cong M_{\mathscr{X}} \simeq 1.0\text{GeV}$$

This ensures the validity of the NJL model with a (covariant) UV cutoff $\Lambda_{\rm NJL}$ as a low-energy effective model of QCD below the momentum-scale $M_{\mathscr{X}}$, i.e., $\sqrt{p^2} \leq M_{\mathscr{X}}$.

The resulting NJL interacton at finite temperature becomes nonlocal and Polyakov-loop dependent. This supports the phenomenological studies based on the nonlocal PNJL and EPNJL models.

⊙ The perturbative one-loop calculation with the regulator function $R_k(p)$ is the first approximation to solve the flow equation of the Wetterich type, which can be a good initial step to be improved to solve the flow equation for obtaining a non-perturbative result.

The chiral-symmetry breaking/restoration transition can be already understood by the first approximation of the flow equation, although this result is improved by fullly solving the flow equation.

But, this is not the case for the quark confinement/deconfinement transition.

⊙ For the gluon sector: The high temperature deconfinement phase can be treated by the perturbative method. Then the existence of low temperature confinement phase has been shown by solving the flow equation:

[F. Marhauser and J.M. Pawlowski, 0812.1144[hep-ph].] for SU(2)
[J. Braun, H. Gies and J.M. Pawlowski, 0708.2413[hep-th], Phys.Lett. B684, 262–267 (2010)] for SU(2) and SU(3)

space-like fluctuations $\Longrightarrow$ deconfinement; time-like fluctuations $\Longrightarrow$ confinement

For $N_c = 2$, 2nd order For $N_c = 3$, 1st order

The input for solving the flow equation was just a running gauge coupling constant. From the viewpoint of a first-principle derivation, this is superior to phenomenological models with many input parameters.

⊙ In order to understand the entanglement, we have given a microscopic derivation of the Landau-Ginzburg description for the (crossover) phase transition by analysing the flow equations for coefficients of the effective potential.

⊙ The quantum back-reaction of the matter sector to the gluonic sector in PQM model was taken into account as a quark flavor and quark chemical potential dependent transitin temperature $T(N_f, \mu)$ in the Polyakov-loop potential:
Bernd-Jochen Schaefer, Jan M. Pawlowski, Jochen Wambach, The Phase Structure of the Polyakov–Quark-Meson Model, arXiv:0704.3234[hep-ph], Phys.Rev.D76:074023,2007.
The dynamics of the quark-meson sector of the PQM model in the presence of a nontrivial Polyakov-loop average was also included:
Tina Katharina Herbst, Jan M. Pawlowski, Bernd-Jochen Schaefer, The phase structure of the Polyakov–quark-meson model beyond mean field, arXiv:1008.0081[hep-ph], Phys.Lett.B696:58-67,2011.

⊙ Still, however, we must overcome technical issues to achieve the goal of deriving the crossover transition and critical endpoint, etc. from the first principle of QCD. We need more hard work in order to understand $SU(N_c = 3)$, $m_q \neq 0$, $N_f = 2+1$, $\mu \neq 0$, chiral anomaly, Kobayashi-Maskawa-'tHooft determinant, ...

• For the chiral phase transition in one-flavor QCD based on the FRG,
J. Braun, The QCD phase boundary from quark-gluon dynamics, arXiv:0810.1727[hep-ph], Eur.Phys.J. C.64, 459–482 (2009).

• For Yang-Mills theories with other gauge groups:
Jens Braun, Astrid Eichhorn, Holger Gies, Jan M. Pawlowski, On the Nature of the Phase Transition in SU(N), Sp(2) and E(7) Yang-Mills theory, arXiv:1007.2619[hep-ph], Eur.Phys.J. C70, 689-702 (2010).

• For QCD phase diagram from FRG:
Jens Braun, Lisa M. Haas, Florian Marhauser, Jan M. Pawlowski, Phase Structure of Two-Flavor QCD at Finite Chemical Potential, arXiv:0908.0008[hep-ph], Phys.Rev.Lett. 106, 022002 (2011).

• For overview and review:
Jan M. Pawlowski, The QCD phase diagram: Results and challenges, arXiv:1012.5075 [hep-ph], AIP Conf.Proc.1343, 75-80 (2011).

Jens Braun, Fermion Interactions and Universal Behavior in Strongly Interacting Theories, arXiv:1108.4449 [hep-ph]

Thank you very much for your attention!