Workshop at Yukawa Institute, Kyoto, Japan
Renormalization Group Approach from Ultra Cold Atoms to the Hot QGP, August 22- September 9,2011

Non-equilibrium time evolution of bosons from the functional renormalization group

Peter Kopietz, Universität Frankfurt

Collaboration: T. Kloss (Grenoble), A. Kreisel, J. Hick (Frankfurt)
A. Serga, B. Hillebrands (Kaiserslautern, experiment)

talk based on:
- Eur. Phys. J. B 71, 59 (2009)
- Phys. Rev. B 81,104308 (2010)
- Phys. Rev. B 83, 205118 (2011)

SFB/Transregio 49
Frankfurt-Kaiserslautern-Mainz
Condensed matter systems with variable many-body interactions

outline:
1. Functional integral formulation of Keldysh technique
2. Non-equilibrium FRG
3. Parametric resonance of magnons in Yttrium-Iron-Garnet
4. FRG for toy model

warning!

D.Mermin, Physics Today,
November 1992, page 9:

"It is absolutely impossible to give too elementary a physics talk. Every talk I have even attended in four decades of lecture going has been too hard.There is therefore no point in advising you to make your talk clear and comprehensible. You should merely strive to place as far as possible from the beginning the grim moment when more than 90% of your audience is able to make sense of less than 10% of anything you say."

1. functional integral formulation of the Keldysh technique

(A. Kamenev, Les Houches, 2004)

- model: magnons coupled to phonons:

$$\mathcal{H} = \sum_{\boldsymbol{k}} \epsilon_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + \sum_{\boldsymbol{q}} \omega_{\boldsymbol{q}} b^\dagger_{\boldsymbol{q}} b_{\boldsymbol{q}} + \frac{1}{\sqrt{V}} \sum_{\boldsymbol{q}} U_{\boldsymbol{q}} \rho_{-\boldsymbol{q}} (b_{\boldsymbol{q}} + b^\dagger_{-\boldsymbol{q}})$$

magnon phonon $U_{\boldsymbol{q}} = U_0 \sqrt{\omega_{\boldsymbol{q}}}$ $\rho_{\boldsymbol{q}} = \sum_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}+\boldsymbol{q}}$

- non-equilibrium Green functions:

$$iG^R_{\boldsymbol{k}}(t,t') = \Theta(t-t')\langle [a_{\boldsymbol{k}}(t), a^\dagger_{\boldsymbol{k}}(t')]\rangle,$$
$$iG^A_{\boldsymbol{k}}(t,t') = -\Theta(t'-t)\langle [a_{\boldsymbol{k}}(t), a^\dagger_{\boldsymbol{k}}(t')]\rangle,$$
$$iG^K_{\boldsymbol{k}}(t,t') = \langle \{a_{\boldsymbol{k}}(t), a^\dagger_{\boldsymbol{k}}(t')\}\rangle,$$

$$iG^K_{\boldsymbol{k}}(t,t) = 1 + 2n_{\boldsymbol{k}}(t)$$

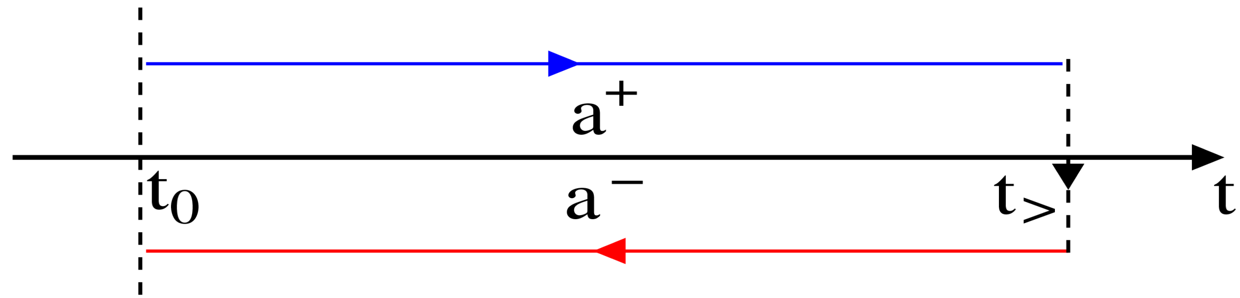
$$X_{\boldsymbol{q}} = \frac{1}{\sqrt{2}}\left(b_{\boldsymbol{q}} + b^\dagger_{-\boldsymbol{q}}\right)$$

$$iF^R_{\boldsymbol{q}}(t,t') = \Theta(t-t')\langle [b_{\boldsymbol{q}}(t), b^\dagger_{\boldsymbol{q}}(t')]\rangle,$$
$$iF^A_{\boldsymbol{k}}(t,t') = -\Theta(t'-t)\langle [b_{\boldsymbol{q}}(t), b^\dagger_{\boldsymbol{q}}(t')]\rangle,$$
$$iF^K_{\boldsymbol{k}}(t,t') = \langle \{b_{\boldsymbol{q}}(t), b^\dagger_{\boldsymbol{q}}(t')\}\rangle.$$

$$iD^R_{\boldsymbol{q}}(t,t') = \Theta(t-t')\langle [X_{\boldsymbol{q}}(t), X_{-\boldsymbol{q}}(t')]\rangle,$$
$$iD^A_{\boldsymbol{q}}(t,t') = -\Theta(t'-t)\langle [X_{\boldsymbol{q}}(t), X_{-\boldsymbol{q}}(t')]\rangle$$
$$iD^K_{\boldsymbol{q}}(t,t') = \langle \{X_{\boldsymbol{q}}(t), X_{-\boldsymbol{q}}(t')\}\rangle.$$

Keldysh contour and classical/quantum components

- Keldysh contour:



- change of basis: from contour labels to classical and quantum labels:

$$a_{\boldsymbol{k}}^{C}(t) = \frac{1}{\sqrt{2}}\left[a_{\boldsymbol{k}}^{+}(t) + a_{\boldsymbol{k}}^{-}(t)\right]$$

$$a_{\boldsymbol{k}}^{Q}(t) = \frac{1}{\sqrt{2}}\left[a_{\boldsymbol{k}}^{+}(t) - a_{\boldsymbol{k}}^{-}(t)\right]$$

- functional integral representation of non-equilibrium Green functions:

$$iG_{\boldsymbol{k}}^{R}(t,t') = \langle a_{\boldsymbol{k}}^{C}(t)\bar{a}_{\boldsymbol{k}}^{Q}(t')\rangle \equiv iG_{\boldsymbol{k}}^{CQ}(t,t')$$

$$iG_{\boldsymbol{k}}^{A}(t,t') = \langle a_{\boldsymbol{k}}^{Q}(t)\bar{a}_{\boldsymbol{k}}^{C}(t')\rangle \equiv iG_{\boldsymbol{k}}^{QC}(t,t')$$

$$iG_{\boldsymbol{k}}^{K}(t,t') = \langle a_{\boldsymbol{k}}^{C}(t)\bar{a}_{\boldsymbol{k}}^{C}(t')\rangle \equiv iG_{\boldsymbol{k}}^{CC}(t,t')$$

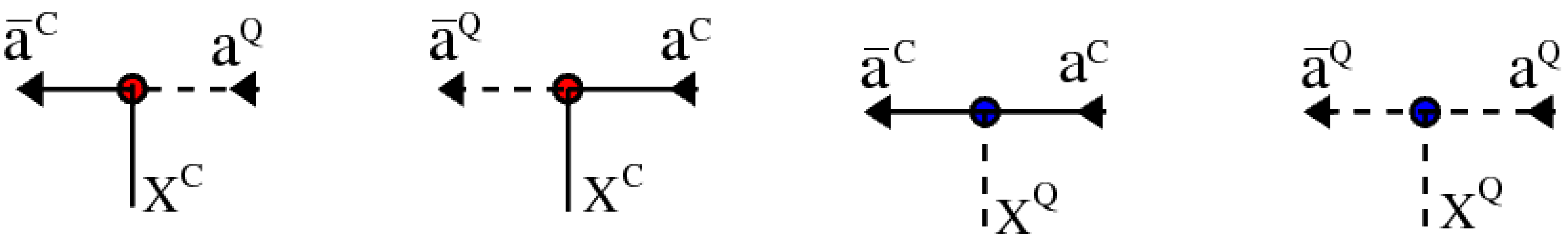
$$\langle a_{\boldsymbol{k}}^{\lambda}(t)\bar{a}_{\boldsymbol{k}}^{\lambda'}(t')\rangle = \int \mathcal{D}[a,\bar{a},b,\bar{b}]e^{iS[\bar{a},a,\bar{b},b]}a_{\boldsymbol{k}}^{\lambda}(t)\bar{a}_{\boldsymbol{k}}^{\lambda'}(t')$$

Keldysh action in continuum notation

$$S[\bar{a},a,\bar{b},b] = \int dt \int dt' \Bigg\{ \sum_{\boldsymbol{k}} (\bar{a}^C_{\boldsymbol{k}}(t), \bar{a}^Q_{\boldsymbol{k}}(t)) \begin{pmatrix} 0 & (\hat{G}^A_0)^{-1} \\ (\hat{G}^R_0)^{-1} & -(\hat{G}^R_0)^{-1}\hat{G}^K_0(\hat{G}^A_0)^{-1} \end{pmatrix}_{tt'} \begin{pmatrix} a^C_{\boldsymbol{k}}(t') \\ a^Q_{\boldsymbol{k}}(t') \end{pmatrix}$$

$$+ \sum_{\boldsymbol{q}} (\bar{b}^C_{\boldsymbol{q}}(t), \bar{b}^Q_{\boldsymbol{q}}(t)) \begin{pmatrix} 0 & (\hat{F}^A_0)^{-1} \\ (\hat{F}^R_0)^{-1} & -(\hat{F}^R_0)^{-1}\hat{F}^K_0(\hat{F}^A_0)^{-1} \end{pmatrix}_{tt'} \begin{pmatrix} b^C_{\boldsymbol{q}}(t') \\ b^Q_{\boldsymbol{q}}(t') \end{pmatrix} \Bigg\}$$

$$+ \int dt \frac{1}{\sqrt{V}} \sum_{\boldsymbol{k},\boldsymbol{q}} U_{\boldsymbol{q}} \left[\left(\bar{a}^C_{\boldsymbol{k}+\boldsymbol{q}} a^Q_{\boldsymbol{k}} + \bar{a}^Q_{\boldsymbol{k}+\boldsymbol{q}} a^C_{\boldsymbol{k}} \right) X^C_{\boldsymbol{q}} + \left(\bar{a}^C_{\boldsymbol{k}+\boldsymbol{q}} a^C_{\boldsymbol{k}} + \bar{a}^Q_{\boldsymbol{k}+\boldsymbol{q}} a^Q_{\boldsymbol{k}} \right) X^Q_{\boldsymbol{q}} \right].$$



- QQ-blocks of Gaussian propagators are infinitesimal regularization:

$$-(\hat{G}^R_0)^{-1}\hat{G}^K_0(\hat{G}^A_0)^{-1} = 2i\eta\hat{g}_0$$

$$-(\hat{F}^R_0)^{-1}\hat{F}^K_0(\hat{F}^A_0)^{-1} = 2i\eta\hat{f}_0$$

$$[\hat{g}_0]_{tt'} = \delta(t-t')g_0 = \delta(t-t')[1 + 2\langle a^\dagger a\rangle_0]$$

$$[\hat{f}_0]_{tt'} = \delta(t-t')f_0 = \delta(t-t')[1 + 2\langle b^\dagger b\rangle_0]$$

non-equilibrium time-evolution: quantum kinetic equations

- Keldysh component of non-equilibrium Dyson equation gives kinetic equation for distribution function:

- Green function matrix: $\mathbf{G} = \begin{pmatrix} [\mathbf{G}]^{CC} & [\mathbf{G}]^{CQ} \\ [\mathbf{G}]^{QC} & 0 \end{pmatrix} = \begin{pmatrix} \hat{G}^K & \hat{G}^R \\ \hat{G}^A & 0 \end{pmatrix}$

matrices in momentum and time

- self-energy matrix: $\mathbf{\Sigma} = \begin{pmatrix} 0 & [\mathbf{\Sigma}]^{CQ} \\ [\mathbf{\Sigma}]^{QC} & [\mathbf{\Sigma}]^{QQ} \end{pmatrix} = \begin{pmatrix} 0 & \hat{\Sigma}^A \\ \hat{\Sigma}^R & \hat{\Sigma}^K \end{pmatrix}$

$$\hat{\Sigma}^K = -[\mathbf{G}^{-1}]^{QQ} = (\hat{G}^R)^{-1}\hat{G}^K(\hat{G}^A)^{-1}$$

- subtracting left/right Dyson eqns. gives kinetic eq.:

$$\left(\mathbf{G}_0^{-1} - \mathbf{\Sigma}\right)\mathbf{G} = \mathbf{I} \qquad \left[\hat{M}_0, \hat{G}^K\right] = \hat{\Sigma}^K\hat{G}^A - \hat{G}^R\hat{\Sigma}^K + \hat{\Sigma}^R\hat{G}^K - \hat{G}^K\hat{\Sigma}^A$$

$$\mathbf{G}\left(\mathbf{G}_0^{-1} - \mathbf{\Sigma}\right) = \mathbf{I} \qquad [\hat{M}_0]_{\boldsymbol{k}t,\boldsymbol{k}'t'} = \delta_{\boldsymbol{k},\boldsymbol{k}'}\left[i\partial_t - \epsilon_{\boldsymbol{k}}\right]\delta(t-t')$$

different forms of the kinetic equation

1.) time-domain:

$$\begin{aligned}(i\partial_t + i\partial_{t'})G^K(t,t') &= \int_{t_0}^{t} dt_1 [\Sigma^R(t,t_1)G^K(t_1,t') - G^R(t,t_1)\Sigma^K(t_1,t')] \\ &+ \int_{t_0}^{t'} dt_1 [\Sigma^K(t,t_1)G^A(t_1,t') - G^K(t,t_1)\Sigma^A(t_1,t')]\end{aligned}$$

2.) with subtractions to identify collision integrals:

$$\hat{\Sigma}^M = \frac{1}{2}[\hat{\Sigma}^R + \hat{\Sigma}^A] \qquad \hat{G}^M = \frac{1}{2}[\hat{G}^R + \hat{G}^A] \qquad \hat{M} = \hat{M}_0 - \hat{\Sigma}^M$$

$$\hat{\Sigma}^I = i[\hat{\Sigma}^R - \hat{\Sigma}^A] \qquad \hat{G}^I = i[\hat{G}^R - \hat{G}^A]$$

$$\boxed{[\hat{M}, \hat{G}^K] - [\hat{\Sigma}^K, \hat{G}^M] = \hat{C}^{\rm in} - \hat{C}^{\rm out}}$$

$$\hat{C}^{\rm in} = \frac{i}{2}\{\hat{\Sigma}^K, \hat{G}^I\}$$

$$\hat{C}^{\rm out} = \frac{i}{2}\{\hat{\Sigma}^I, \hat{G}^K\}$$

....different forms of the kinetic equation

3.) for distribution function: $\hat{G}^K = \hat{G}^R \hat{g}^\dagger - \hat{g}\hat{G}^A$

$$-i(\hat{M}\hat{g} - \hat{g}^\dagger \hat{M}) = \hat{\Sigma}^{\mathrm{in}} - \hat{\Sigma}^{\mathrm{out}}$$

$$\hat{\Sigma}^{\mathrm{in}} = i\hat{\Sigma}^K$$

$$\hat{\Sigma}^{\mathrm{out}} = \frac{1}{2}\left(\hat{\Sigma}^I \hat{g} + \hat{g}^\dagger \hat{\Sigma}^I\right)$$

4.) for distribution function, Wigner transformed:

$$A(\tau;\omega) = \int_{-\infty}^{\infty} ds e^{i\omega s}[\hat{A}]_{\tau+\frac{s}{2},\tau-\frac{s}{2}}$$

$$\partial_\tau \mathrm{Re} g(\tau;\omega) + 2(\omega - \epsilon_{\boldsymbol{k}})\mathrm{Im} g(\tau;\omega) \ + i(\hat{\Sigma}^M \hat{g} - \hat{g}^\dagger \hat{\Sigma}^M)_{(\tau;\omega)}$$

$$= \Sigma^{\mathrm{in}}(\tau;\omega) - \Sigma^{\mathrm{out}}(\tau;\omega)$$

2. Non-equilibrium renormalization group

- several numerical implementations in condensed matter:

PHYSICAL REVIEW B **70**, 121302(R) (2004)

1.) **Nonequilibrium electron transport using the density matrix renormalization group method**

Peter Schmitteckert
Institut für Theorie der Kondensierten Materie, Universität Karlsruhe, 76128 Karlsruhe, Germany
(Received 22 March 2004; revised manuscript received 10 May 2004; published 8 September 2004)

We extended the density matrix renormalization group method to study the real time dynamics of interacting one-dimensional spinless Fermi systems by applying the full time evolution operator to an initial state. As an

2.) PRL **95,** 196801 (2005) PHYSICAL REVIEW LETTERS week ending 4 NOVEMBER 2005

Real-Time Dynamics in Quantum-Impurity Systems: A Time-Dependent Numerical Renormalization-Group Approach

Frithjof B. Anders[1] and Avraham Schiller[2]
[1]*Department of Physics, Universität Bremen, P.O. Box 330 440, D-28334 Bremen, Germany*
[2]*Racah Institute of Physics, The Hebrew University, Jerusalem 91904, Israel*
(Received 29 April 2005; published 31 October 2005)

We develop a general approach to the nonequilibrium dynamics of quantum-impurity systems for arbitrary coupling strength. The numerical renormalization group is used to generate a complete basis set

different versions of numerical non-equilibrum RG

3.)

PHYSICAL REVIEW B **80**, 045117 (2009)

Real-time renormalization group in frequency space: A two-loop analysis of the nonequilibrium anisotropic Kondo model at finite magnetic field

Herbert Schoeller and Frank Reininghaus
Institut für Theoretische Physik, Lehrstuhl A, RWTH Aachen, 52056 Aachen, Germany and JARA-Fundamentals of Future Information Technology
(Received 9 February 2009; revised manuscript received 25 May 2009; published 22 July 2009)

We apply a recently developed nonequilibrium real-time renormalization group (RG) method in frequency space to describe nonlinear quantum transport through a small fermionic quantum system coupled weakly to

4.)

Annals of Physics 324 (2009) 2146–2178

Real-time evolution for weak interaction quenches in quantum systems

Michael Moeckel *, Stefan Kehrein

Arnold-Sommerfeld-Center for Theoretical Physics, Center for NanoSciences and Department für Physik, Ludwig-Maximilians-Universität München, Theresienstraße 37, 80333 München, Germany

(flow equations for continuous unitary transfomations)

A R T I C L E   I N F O

Article history:
Received 13 March 2009
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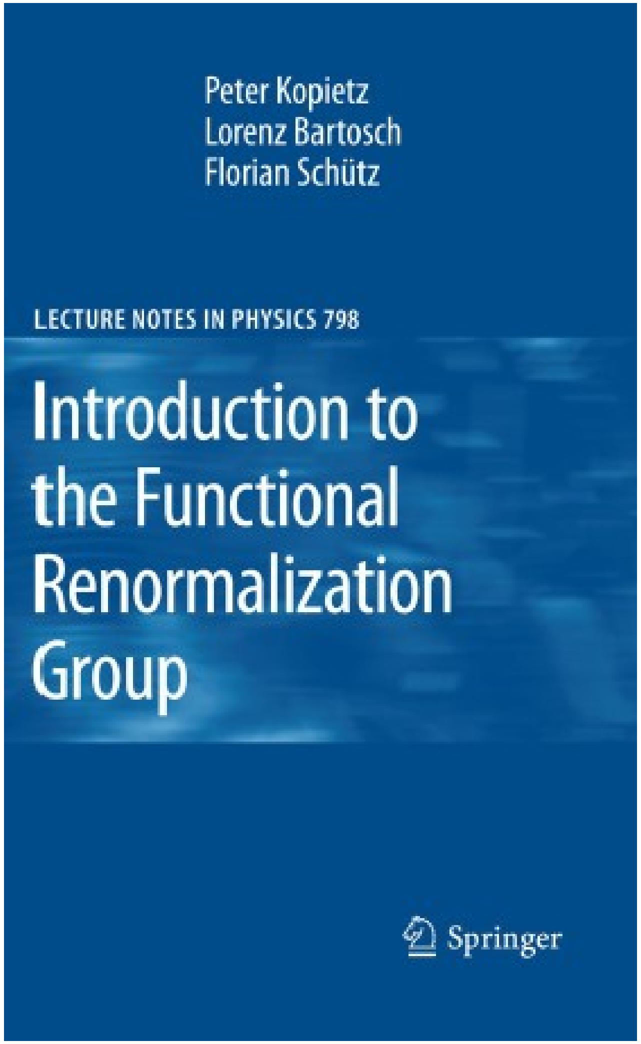
A B S T R A C T

Motivated by recent experiments in ultracold atomic gases that explore the nonequilibrium dynamics of interacting quantum many-body systems, we investigate the nonequilibrium properties

functional renormalization group

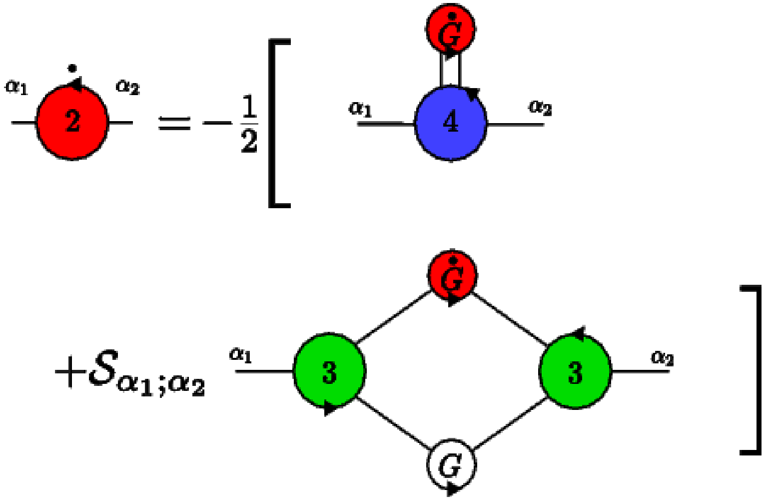


2010. XII, 380 p. (Lecture Notes in Physics, Vol. 798) Hardcover

- exact equation for change of generating functional of irreducible vertices as IR cutoff is reduced (Wetterich 1993)

$$\partial_\Lambda \Gamma_\Lambda[\Phi] = \frac{1}{2} \mathrm{Tr} \left[(\partial_\Lambda \boldsymbol{R}_\Lambda) \left(\frac{\delta}{\delta \Phi} \otimes \frac{\delta}{\delta \Phi} \Gamma_\Lambda[\Phi] + \boldsymbol{R}_\Lambda \right)^{-1} \right]$$

- exact RG flow equations for all vertices
- flow of self-energy:



non-equilibrium time-evolution from the FRG
the basic idea

- introduce cutoff parameter Λ which somehow simplifies time evolution
- write down suitably truncated FRG flow equations for the self-energies
- structure of resulting equations:

$$\partial_\tau f_\Lambda(\boldsymbol{k},\omega,\tau) = C_\Lambda[\boldsymbol{k},\omega,\tau,f,\Sigma^K,\Sigma^R,\Sigma^A]$$
$$\partial_\Lambda \Sigma_\Lambda^R(\boldsymbol{k},\omega,\tau) = I_\Lambda^R[\boldsymbol{k},\omega,\tau,f,\Sigma^K,\Sigma^R,\Sigma^A]$$
$$\partial_\Lambda \Sigma_\Lambda^A(\boldsymbol{k},\omega,\tau) = I_\Lambda^A[\boldsymbol{k},\omega,\tau,f,\Sigma^K,\Sigma^R,\Sigma^A],$$
$$\partial_\Lambda \Sigma_\Lambda^K(\boldsymbol{k},\omega,\tau) = I_\Lambda^K[\boldsymbol{k},\omega,\tau,f,\Sigma^K,\Sigma^R,\Sigma^A]$$

coupled system of partial integro-differential equations in two variables

- make standard approximations to simplify system (e.g. reduction to Fokker-Planck eq) or solve by brute force numerically.

FRG cutoff schemes to calculate non-equilibrium time-evolution

- requirments: cutoff procedure should respect:

1) causality, and 2) in equilibrium: fluctuation-dissipation theorem

- possibilies:

1) long-time cutoff (Gasenzer, Pawlowski, 2008)

2) hybridization cutoff (Jakobs, Pletyukhov, Schoeller, 2010)

$$\mathbf{G}_0^{-1} = \begin{pmatrix} 0 & (\hat{G}_0^A)^{-1} \\ (\hat{G}_0^R)^{-1} & -(\hat{G}_0^R)^{-1}\hat{G}_0^K(\hat{G}_0^A)^{-1} \end{pmatrix} = \begin{pmatrix} 0 & \hat{M}_0 - i\eta \\ \hat{M}_0 + i\eta & 2i\eta\hat{g}_0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & \hat{M}_0 - i\Lambda \\ \hat{M}_0 + i\Lambda & 2i\Lambda\hat{g}_0 \end{pmatrix}$$

3) out-scattering rate cutoff (Kloss, P.K., 2011)

$$\mathbf{G}_0^{-1} \rightarrow \mathbf{G}_{0,\Lambda}^{-1} = \begin{pmatrix} 0 & \hat{M}_0 - i\Lambda \\ \hat{M}_0 + i\Lambda & 2i\eta\hat{g}_{0,\Lambda} \end{pmatrix}$$

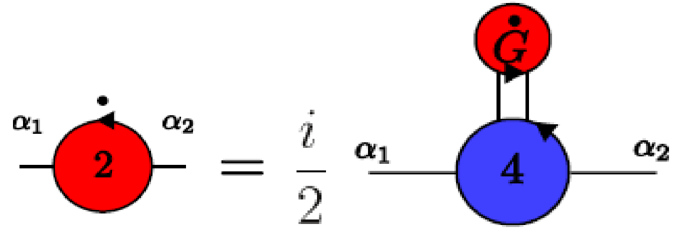
non-equilibrium FRG vertex expansion

(Gezzi et al, 2007; Gasenzer+Pawlowski, 2008; Kloss+PK 2010)

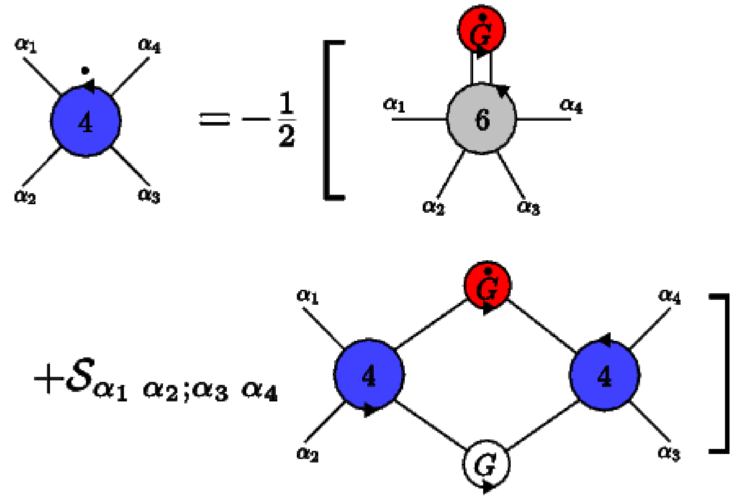
- simple generalization of the equilibrium vertex expansion:

$$\Gamma^{(n)}_{\Lambda,\alpha_1\ldots\alpha_n} \to i\Gamma^{(n)}_{\Lambda,\alpha_1\ldots\alpha_n}; \qquad \mathbf{G}_\Lambda \to -i\mathbf{G}_\Lambda \ , \ \dot{\mathbf{G}}_\Lambda \to -i\dot{\mathbf{G}}_\Lambda;$$

$$\partial_\Lambda \Gamma^{(2)}_{\Lambda,\alpha_1\alpha_2} = \frac{i}{2}\int_{\beta_1}\int_{\beta_2} [\dot{\mathbf{G}}_\Lambda]_{\beta_1\beta_2} \Gamma^{(4)}_{\Lambda,\beta_2\beta_1\alpha_1\alpha_2}$$



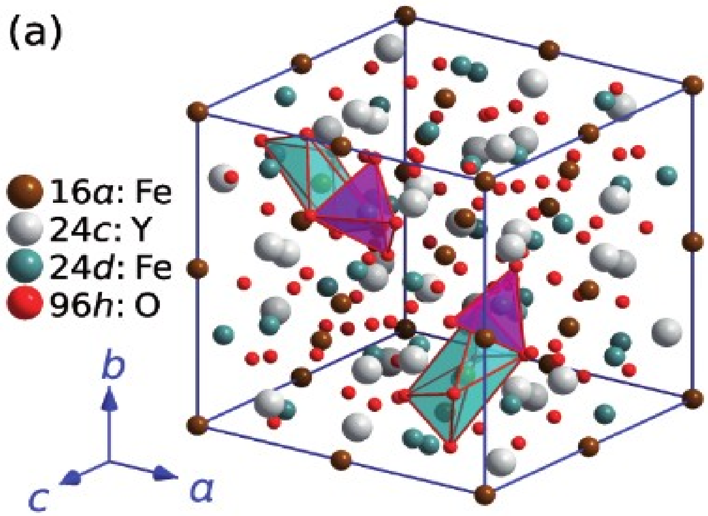
$$\partial_\Lambda \Gamma^{(4)}_{\Lambda,\alpha_1\alpha_2\alpha_3\alpha_4} = \frac{i}{2}\int_{\beta_1}\int_{\beta_2} [\dot{\mathbf{G}}_\Lambda]_{\beta_1\beta_2} \Gamma^{(6)}_{\Lambda,\beta_2\beta_1\alpha_1\alpha_2\alpha_3\alpha_4}$$
$$+\frac{i}{2}\int_{\beta_1}\int_{\beta_2}\int_{\beta_3}\int_{\beta_4} [\dot{\mathbf{G}}_\Lambda]_{\beta_1\beta_2} [\mathbf{G}_\Lambda]_{\beta_3\beta_4}$$
$$\times\Big[\Gamma^{(4)}_{\Lambda,\beta_2\beta_3\alpha_3\alpha_4}\Gamma^{(4)}_{\Lambda,\beta_4\beta_1\alpha_1\alpha_2} + \Gamma^{(4)}_{\Lambda,\beta_2\beta_3\alpha_1\alpha_2}\Gamma^{(4)}_{\Lambda,\beta_4\beta_3\alpha_3\alpha_4}$$
$$+(\alpha_1 \leftrightarrow \alpha_2) + (\alpha_1 \leftrightarrow \alpha_4)\Big]$$

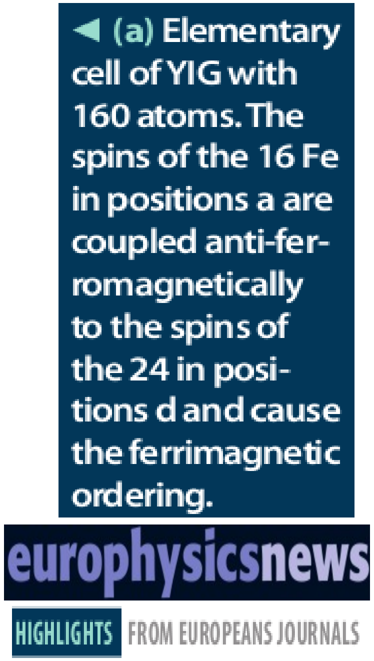


3. Parametric resonance of magnons in yttrium-iron-garnet (YIG)

- what is YIG?
  ferromagnetic
  insulator

- at the first sight:
  too complicated!





A. Kreisel, F. Sauli, L. Bartosch, PK, 2009

- effective quantum spin model for relevant magnon band:

$$\hat{H} = -\frac{1}{2}\sum_{ij} J_{ij}\boldsymbol{S}_i \cdot \boldsymbol{S}_j - \mu \boldsymbol{H}_e \cdot \sum_i \boldsymbol{S}_i - \frac{1}{2}\sum_{ij,i\neq j}\frac{\mu^2}{|\boldsymbol{R}_{ij}|^3}\left[3(\boldsymbol{S}_i\cdot\hat{\boldsymbol{R}}_{ij})(\boldsymbol{S}_j\cdot\hat{\boldsymbol{R}}_{ij}) - \boldsymbol{S}_i\cdot\boldsymbol{S}_j\right]$$

exchange interaction: $J = 1.29$ K.

lattice spacing: $a = 12.376$ Å

saturation magnetization: $4\pi M_S = 1750$ G

effective spin: $S = M_s a^3/\mu \approx 14.2$

experiments on YIG: probing the non-equilibrium dynamics of magnons

- <u>motivation</u>:

  collaboration with experimental group of B. Hillebrands (Kaiserslautern)

  non-equilibrium dynamics of interacting magnons in YIG


nature Vol 443|28 September 2006|doi:10.1038/nature0511

LETTERS

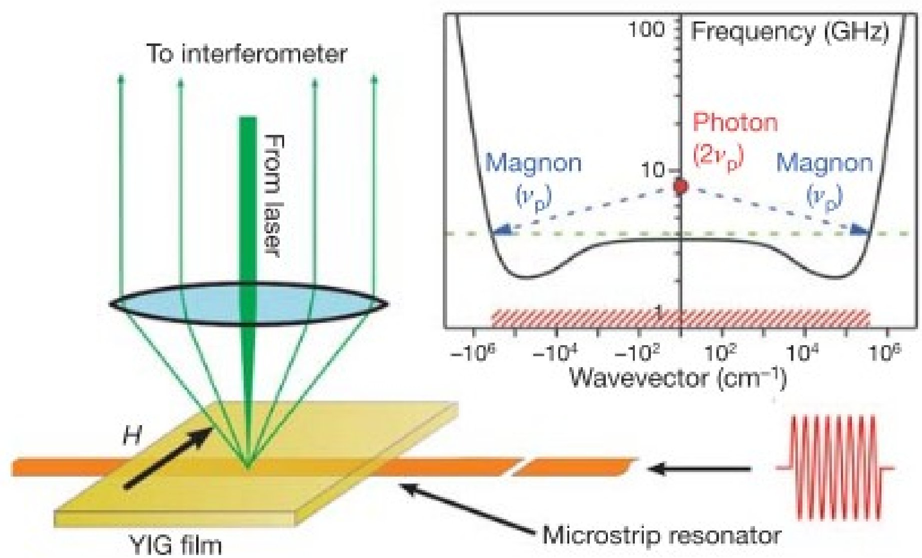
Bose–Einstein condensation of quasi-equilibrium magnons at room temperature under pumping

S. O. Demokritov[1], V. E. Demidov[1], O. Dzyapko[1], G. A. Melkov[2], A. A. Serga[3], B. Hillebrands[3] & A. N. Slavin[4]

- <u>experiment:</u>

  microwave-pumping of magnons in YIG

  measurement of magnon distriubution via Brillouin light scattering



from spin operators to bosons: Holstein-Primakoff transformation

•problem: spin-algebra is very complicated: $[S_i^\alpha, S_j^\beta] = i\delta_{ij}\epsilon^{\alpha\beta\gamma}S_i^\gamma \quad \boldsymbol{S}_i^2 = S(S+1)$

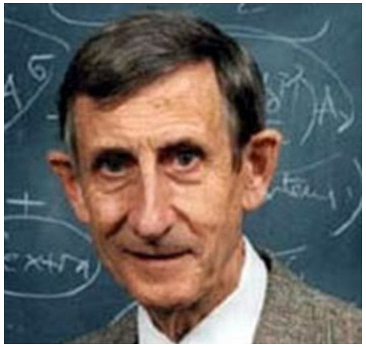
•solution: for ordered magnets: bosonization of spins (Holstein,Primakoff 1940)

$$S_i^+ = S_i^x + iS_i^y = \sqrt{2S}\sqrt{1-\frac{b_i^\dagger b_i}{2S}}\, b_i = \sqrt{2S}\left[b_i - \frac{b_i^\dagger b_i b_i}{4S} + \ldots\right]$$

$$S_i^z = S - b_i^\dagger b_i$$

•spin algebra indeed satisfied if $[b_i, b_j^\dagger] = \delta_{ij}$

•proof that different dimension of Hilbert spaces does not matter by Dyson 1956:



PHYSICAL REVIEW VOLUME 102, NUMBER 5 JUNE 1, 1956

General Theory of Spin-Wave Interactions*

FREEMAN J. DYSON
Department of Physics, University of California, Berkeley, California, and Institute for Advanced Study, Princeton, New Jersey
(Received February 2, 1956)

some history: magnon dynamics in YIG

H. Suhl, 1957, E. Schlömann et al, 1960s,
V. E. Zakharov, V. S. L'vov, S. S. Starobinets, 1970s

- minimal model:

$$\begin{aligned}\hat{H}_{\mathrm{res}}(t) &= \sum_{\boldsymbol{k}} \epsilon_{\boldsymbol{k}} a^{\dagger}_{\boldsymbol{k}} a_{\boldsymbol{k}} + \frac{1}{2}\sum_{\boldsymbol{k}} \left[\gamma_{\boldsymbol{k}} e^{-i\omega_0 t} a^{\dagger}_{\boldsymbol{k}} a^{\dagger}_{-\boldsymbol{k}} + \gamma^{*}_{\boldsymbol{k}} e^{i\omega_0 t} a_{-\boldsymbol{k}} a_{\boldsymbol{k}}\right] \\ &+ \frac{1}{2}\sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{q}} u(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{q}) a^{\dagger}_{\boldsymbol{k}+\boldsymbol{q}} a^{\dagger}_{\boldsymbol{k}'-\boldsymbol{q}} a_{\boldsymbol{k}'} a_{\boldsymbol{k}}\end{aligned}$$

- "S-theory": time-dependent self-consistent Hartree-Fock approximation for magnon distributions functions

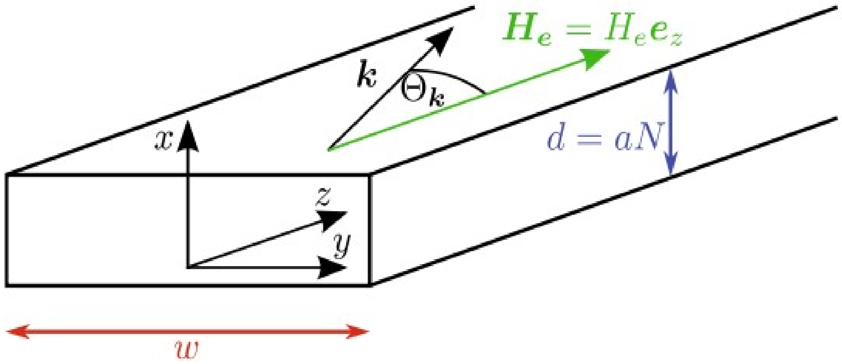
$$n_{\boldsymbol{k}}(t) = \langle a^{\dagger}_{\boldsymbol{k}}(t) a_{\boldsymbol{k}}(t)\rangle$$

$$p_{\boldsymbol{k}}(t) = \langle a_{-\boldsymbol{k}}(t) a_{\boldsymbol{k}}(t)\rangle$$

weak points:
- no microscopic description of dissipation and damping
- possibility of BEC not included!

- goals:
  - consistent quantum kinetic theory for magnons in YIG beyond Hartree-Fock
  - include time-evolution of Bose-condendsate
  - develop functional renormalization group for non-equilibrium

magnon dispersion of finite YIG films



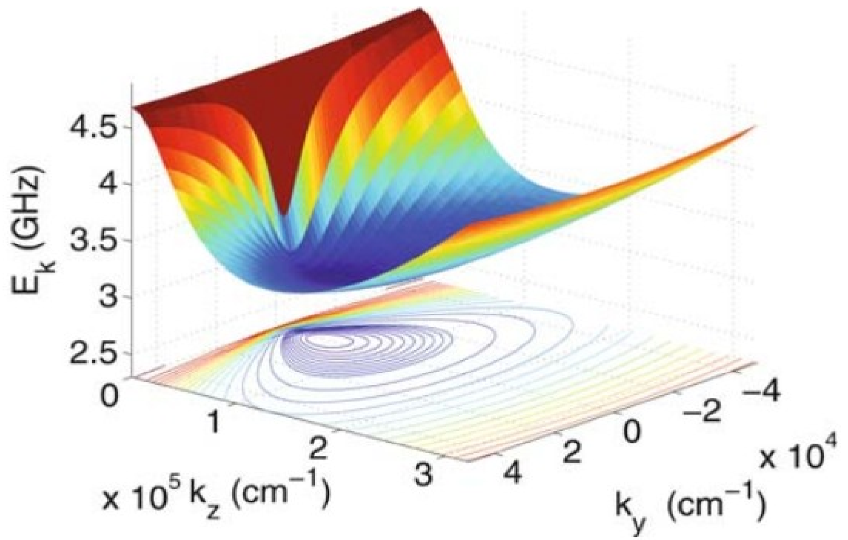
$H_e \approx 1000$ Oe

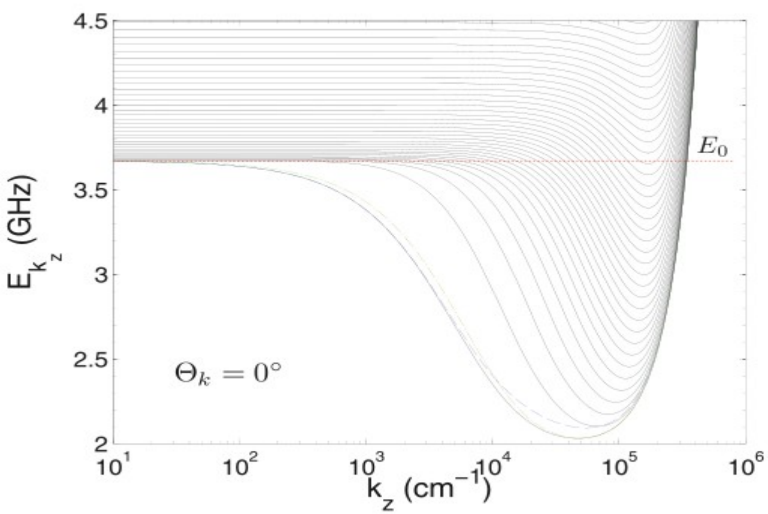
$d \approx 5\mu m \approx 4000a$

$w \gg d$

- dispersion of lowest magnon mode has minimum at finite k

due to interplay between: 1. exchange interaction
2. dipole-dipole interaction
3. finite width of films





toy model for dynamics of lowest magnon mode

T. Kloss, A. Kreisel, PK, PRB 2010

- keep only lowest magnon mode

→ anharmonic oscillator with off-diagonal pumping:

$$\hat{H}(t) = \epsilon_0 a^\dagger a + \frac{\gamma_0}{2} e^{-i\omega_0 t} a^\dagger a^\dagger + \frac{\gamma_0^*}{2} e^{i\omega_0 t} aa + \frac{u}{2} a^\dagger a^\dagger a a$$

- rotating reference frame: $\tilde{a} = e^{\frac{i}{2}\omega_0 t} a$

$$\tilde{H} = \tilde{\epsilon}_0 \tilde{a}^\dagger \tilde{a} + \frac{\gamma_0}{2}\tilde{a}^\dagger \tilde{a}^\dagger + \frac{\gamma_0^*}{2}\tilde{a}\tilde{a} + \frac{u}{2}\tilde{a}^\dagger \tilde{a}^\dagger \tilde{a}\tilde{a} \qquad \tilde{\epsilon}_0 = \epsilon_0 - \frac{\omega_0}{2}$$

- instability of non-interacting system for large pumping:

$$\tilde{a} = \frac{\hat{X} + i\hat{P}}{\sqrt{2}} \qquad \tilde{a}^\dagger = \frac{\hat{X} - i\hat{P}}{\sqrt{2}}$$

$$\tilde{\epsilon}_0 \tilde{a}^\dagger \tilde{a} + \frac{\gamma_0}{2}[\tilde{a}^\dagger \tilde{a}^\dagger + \tilde{a}\tilde{a}] = \frac{\tilde{\epsilon}_0 - \gamma_0}{2}\hat{P}^2 + \frac{\tilde{\epsilon}_0 + \gamma_0}{2}\hat{X}^2.$$

  - for $\gamma_0 > |\tilde{\epsilon}_0|$ oscillator has negative mass
  - non-interacting hamiltonian not positive definite

→ parametric resonance

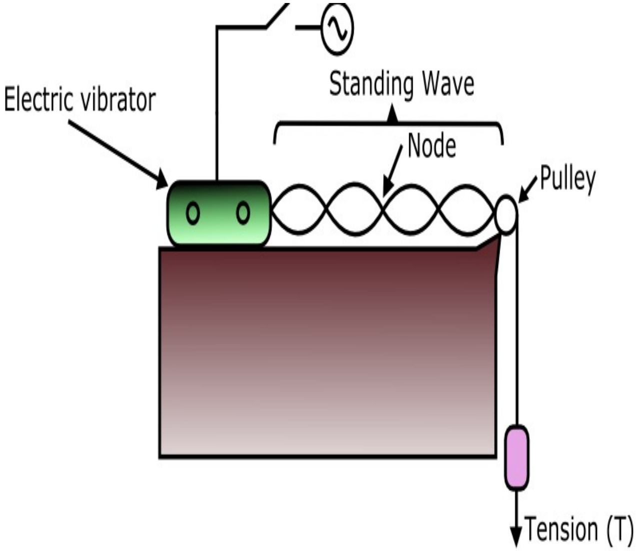
parametric resonance

- what is parametric resonance?
  - classical harmonic oscillator with harmonic frequency modulation:

$$\frac{d^2x(t)}{dt^2} + \Omega^2(t)x(t) = 0 \qquad \Omega(t) = \Omega_0 + \Omega_1 \cos(\omega_0 t)$$

  - resonance condition: $\boxed{\omega_0 \approx 2\Omega_0}$

$\Rightarrow$ oscillator absorbs energy at a rate proportional to the energy it already has!



- history:
  - discovered: Melde experiment, 1859

    excite oscillations of string by periodically varying its tension at twice its resonance frequency

  - theoretically explained: Rayleigh 1883

time-dependent Hartree-Fock approximation ("S-theory")

- order parameter $\phi(t) \equiv \langle \tilde{a}(t) \rangle$ Gross-Pitaevskii equation:

$$i\partial_t \phi = \tilde{\epsilon}_c(t)\phi + \gamma_c(t)\phi^* + u|\phi|^2\phi \qquad \tilde{\epsilon}_c(t) = \tilde{\epsilon}_0 + 2un_c(t)$$

$$\gamma_c(t) = \gamma_0 + u\tilde{p}_c(t)$$

- connected correlation functions: $n_c(t) = \langle \delta\tilde{a}^\dagger(t)\delta\tilde{a}(t) \rangle \quad \tilde{p}_c(t) = \langle \delta\tilde{a}(t)\delta\tilde{a}(t) \rangle$

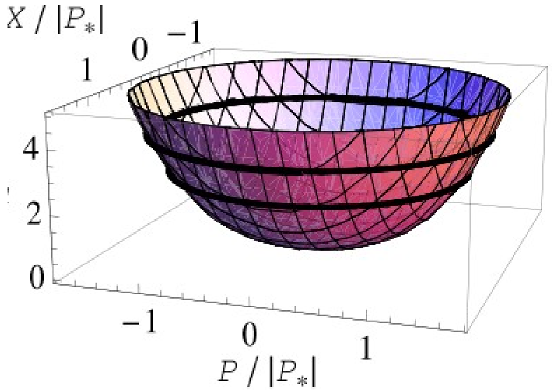
$$\delta\tilde{a}(t) = \tilde{a}(t) - \langle \tilde{a}(t) \rangle$$

kinetic equations:

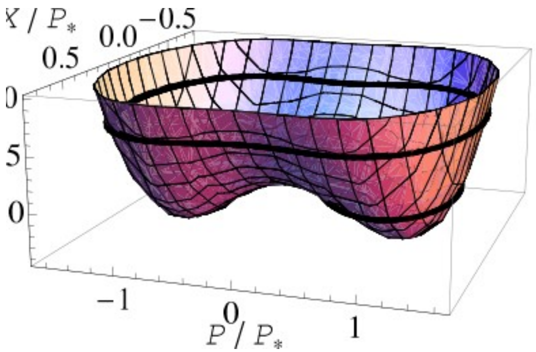
$$\begin{aligned} i\partial_t n_c(t) &= \gamma(t)\tilde{p}_c^*(t) - \gamma^*(t)\tilde{p}_c(t), \\ i\partial_t \tilde{p}_c(t) &= 2\tilde{\epsilon}(t)\tilde{p}_c(t) + \gamma(t)[2n_c(t)+1] \end{aligned} \qquad \begin{aligned} \tilde{\epsilon}(t) &= \tilde{\epsilon}_0 + 2u[n_c(t) + |\phi(t)|^2] \\ \gamma(t) &= \gamma_0 + u[\tilde{p}_c(t) + \phi^2(t)] \end{aligned}$$

- order parameter:
Hamiltonian dynamics in effective potential (Hartree-Fock)

$|\gamma_0| < |\tilde{\epsilon}_0|$



$|\gamma_0| > |\tilde{\epsilon}_0|$



4. Time-evolution of the toy model from the FRG

(T. Kloss, P.K., Phys. Rev. B, 2011)

$$\mathcal{H}(t) = \epsilon a^\dagger a + \frac{1}{2}\left[\gamma e^{-i\omega_0 t} a^\dagger a^\dagger + \gamma^* e^{i\omega_0 t} aa\right]$$

- toy model can be solved numerically exactly by solving time-dependent Schrödinger equation

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} \psi_n(t)|n\rangle$$

$$i\hbar\partial_t \psi_n(t) = \left[\epsilon_0 n + \frac{u_0}{2} n(n-1)\right]\psi_n(t) + \frac{\gamma_0}{2} e^{-i\omega_0 t}\sqrt{n(n-1)}\psi_{n-2}(t) + \frac{\gamma_0^*}{2} e^{i\omega_0 t}\sqrt{(n+2)(n+1)}\psi_{n+2}(t)$$

- generalize FRG to include also off-diagonal Green functions:

$$\begin{aligned} g^R(t,t') &= -i\Theta(t-t')\langle[a(t),a^\dagger(t')]\rangle \\ g^A(t,t') &= i\Theta(t'-t)\langle[a(t),a^\dagger(t')]\rangle \\ g^K(t,t') &= -i\langle\{a(t),a^\dagger(t')\}\rangle \end{aligned}$$

$$\begin{aligned} p^R(t,t') &= -i\Theta(t-t')\langle[a(t),a(t')]\rangle \\ p^A(t,t') &= i\Theta(t'-t)\langle[a(t),a(t')]\rangle \\ p^K(t,t') &= -i\langle\{a(t),a(t')\}\rangle \end{aligned}$$

- Keldysh component at equal times gives distribution functions:

$$G^K(t,t) = \begin{pmatrix} p^K(t,t) & g^K(t,t) \\ g^K(t,t) & p^K(t,t)^* \end{pmatrix} = -2i\begin{pmatrix} p(t) & n(t)+\frac{1}{2} \\ n(t)+\frac{1}{2} & p^*(t) \end{pmatrix}$$

time-dependent Hartree-Fock approximation

- coupled kinetic equations for diagonal/off-diagonal distribution:

$$i\partial_t n(t) = -\gamma^*(t)p(t) + \gamma(t)p^*(t),$$
$$i\partial_t p(t) = 2\epsilon(t)p(t) + \gamma(t)[2n(t)+1]$$

$$\epsilon(t) = \epsilon + 2un(t),$$
$$\gamma(t) = |\gamma| + up(t)$$

(“S-theory, V. E. Zakharov, V. S. L’vov, S. Starobionets, 1970s)

- matrix notation: $F(t) = iZG^K(t,t)Z = \begin{pmatrix} -2p^*(t) & 2n(t)+1 \\ 2n(t)+1 & -2p(t) \end{pmatrix}$

$$i\partial_t F(t) = -M^T(t)F(t) - F(t)M(t)$$

$$M(t) = M + Z\Sigma_1(t) \qquad M = \begin{pmatrix} \epsilon - u & |\gamma| \\ -|\gamma| & -(\epsilon - u) \end{pmatrix}$$

$$Z = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \Sigma_1(t) = u\begin{pmatrix} p^*(t) & 2n(t)+1 \\ 2n(t)+1 & p(t) \end{pmatrix}$$

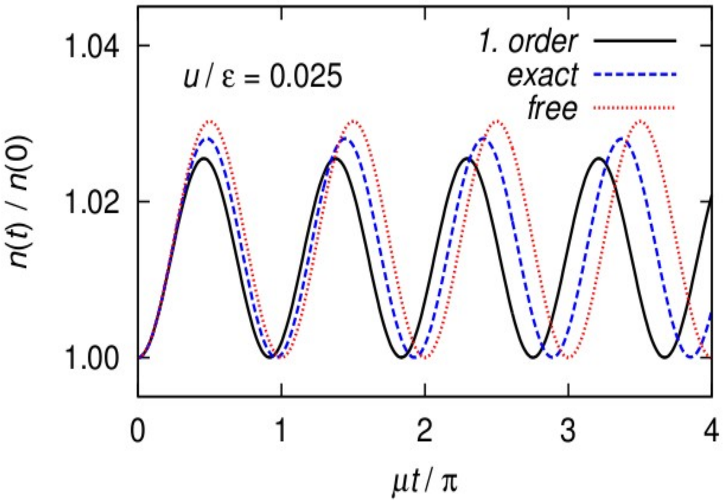
- how good is time-dependent Hartree-Fock?

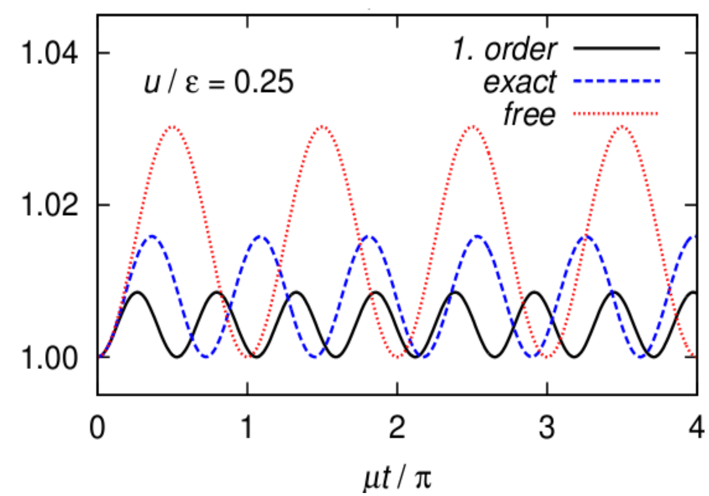
time-dependent Hartree-Fock: comparison with exact non-equilibrium dynamics

small interaction:

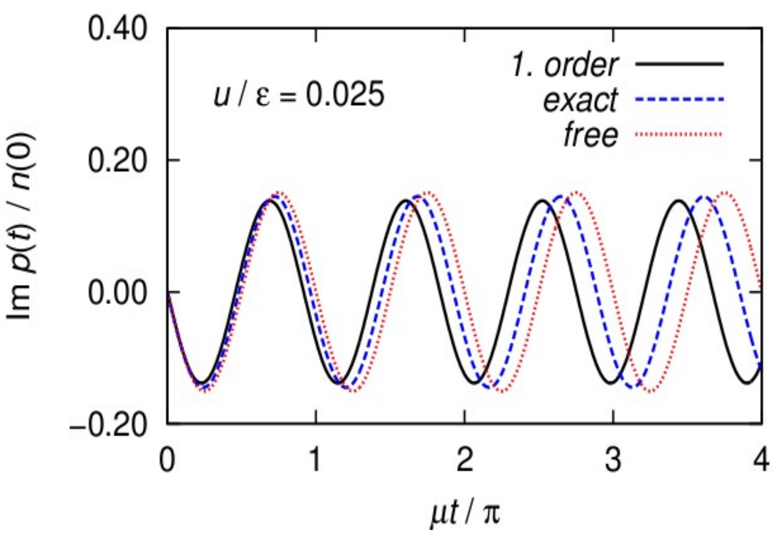
larger interaction:

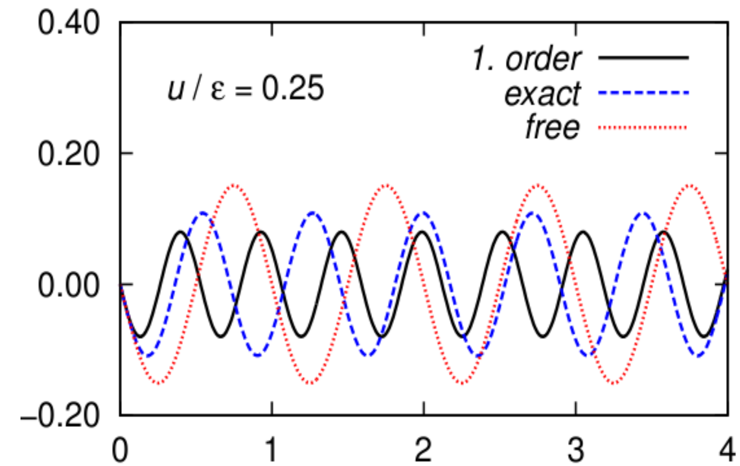
diagonal correlator:





off-diagonal correlator:





cutoff scheme for non-equilibrium FRG

- toy model has no intrinsic dissipation (oscillatory dynamics, no relaxation to equilibrium)
- out-scattering cutoff scheme gives best results

$$(\hat{G}_0^R)^{-1} \to (\hat{G}_{0,\Lambda}^R)^{-1} = (\hat{G}_0^R)^{-1} - i\Lambda\hat{Z}$$

$$(\hat{G}_0^A)^{-1} \to (\hat{G}_{0,\Lambda}^A)^{-1} = (\hat{G}_0^A)^{-1} + i\Lambda\hat{Z}$$

$$[\boldsymbol{G}_0^{-1}]^{QQ} \to [\boldsymbol{G}_{0,\Lambda}^{-1}]^{QQ} = 2i\eta\hat{F}_{0,\Lambda}$$ still infinitesimal regularization

$$\hat{D}\hat{Z}\hat{F}_{0,\Lambda} - \hat{F}_{0,\Lambda}\hat{Z}\hat{D} + 2i\Lambda\hat{F}_{0,\Lambda} = 0$$

- for FRG need single-scale propagators:

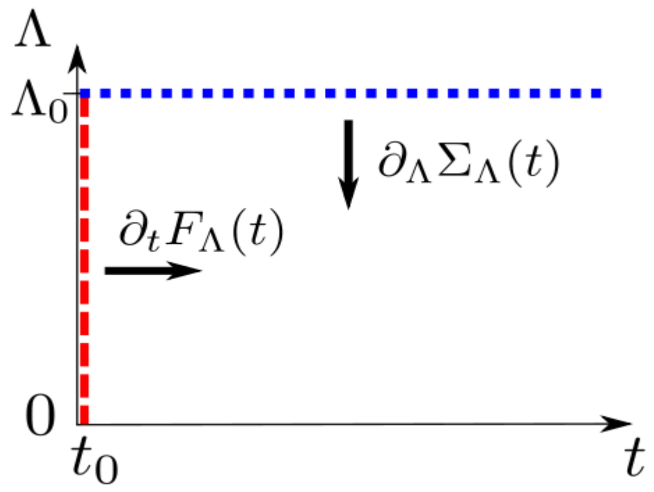
$$\dot{\hat{G}}_\Lambda^R = i\hat{G}_\Lambda^R\hat{Z}\hat{G}_\Lambda^R \qquad \dot{\hat{G}}_\Lambda^K = i\left[\hat{G}_\Lambda^R\hat{Z}\hat{G}_\Lambda^K - \hat{G}_\Lambda^K\hat{Z}\hat{G}_\Lambda^A\right]$$

$$\dot{\hat{G}}_\Lambda^A = -i\hat{G}_\Lambda^A\hat{Z}\hat{G}_\Lambda^A$$

first order truncation of the FRG hierarchy

$$i\partial_t F_\Lambda(t) = -M_\Lambda^T(t) F_\Lambda(t) - F_\Lambda(t) M_\Lambda(t)$$

$$\partial_\Lambda \Sigma_\Lambda(t) = iu \begin{pmatrix} \frac{1}{2}\dot{G}^K_{\Lambda,\bar{a}\bar{a}}(t,t) & \dot{G}^K_{\Lambda,a\bar{a}}(t,t) \\ \dot{G}^K_{\Lambda,\bar{a}a}(t,t) & \frac{1}{2}\dot{G}^K_{\Lambda,aa}(t,t) \end{pmatrix}$$



$$F_\Lambda(t) = \begin{pmatrix} -2p^*_\Lambda(t) & 2n_\Lambda(t)+1 \\ 2n_\Lambda(t)+1 & -2p_\Lambda(t) \end{pmatrix}$$

$$\dot{G}^K_\Lambda(t,t) \approx 2i \int_{t_0}^{t} dt_1 G^R_{0,\Lambda}(t,t_1) F_\Lambda(t) G^A_{0,\Lambda}(t_1,t)$$

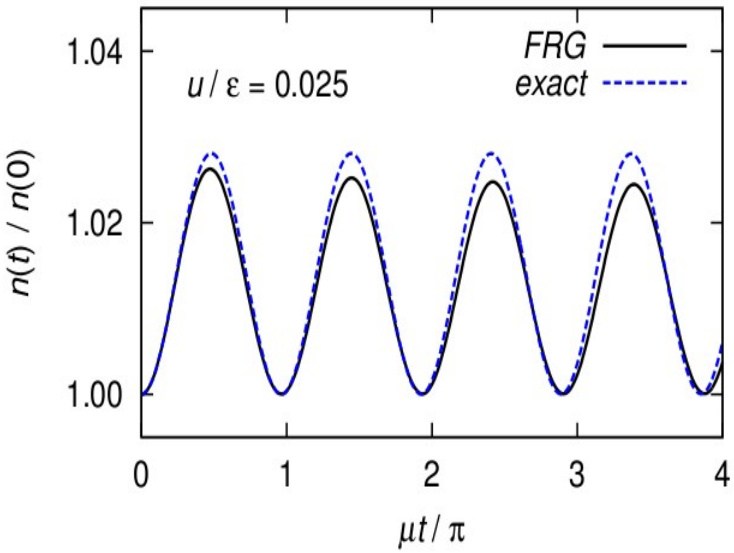
$$M_\Lambda(t) = M - i\Lambda I + Z\Sigma_\Lambda(t) = \begin{pmatrix} \epsilon - u - i\Lambda + \Sigma_{\Lambda,\bar{a}a}(t) & |\gamma| + \Sigma_{\Lambda,\bar{a}\bar{a}}(t) \\ -|\gamma| - \Sigma_{\Lambda,aa}(t) & -[\epsilon - u + i\Lambda + \Sigma_{\Lambda,a\bar{a}}(t)] \end{pmatrix}$$

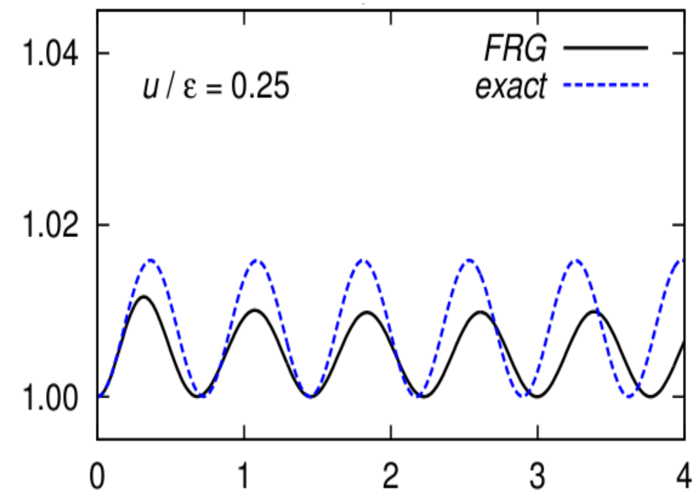
comparison of first order FRG with exact non-equilibrium dynamics

small interaction:

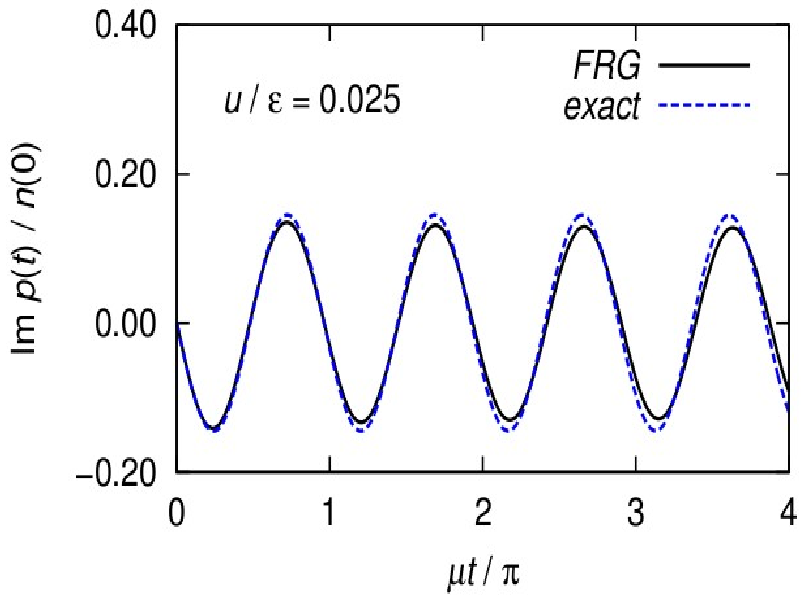
larger interaction:

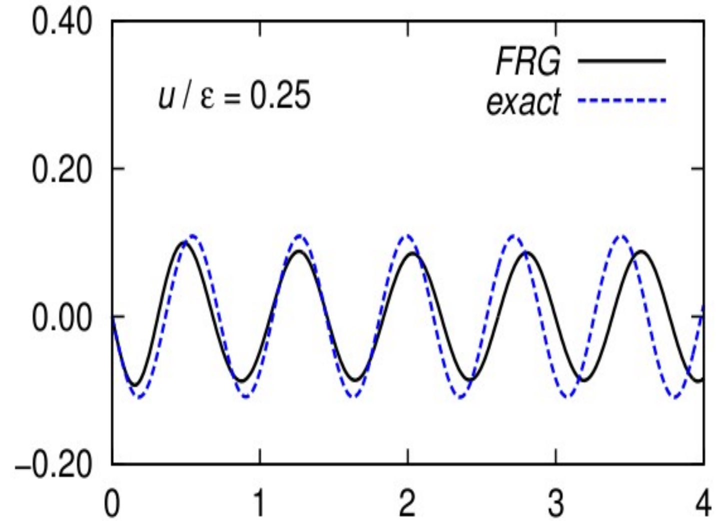
diagonal correlator:





off-diagonal correlator:





Summary+Outlook

- FRG approach to non-equilibrium dynamics of bosons
- cutoff procedures: out-scattering versus hybridization
- exactly solvable toy model for magnon dynamics in YIG:

  first order truncation of FRG quite good!

Outlook:

- relaxation towards equilibrium: rate equations with scale dependent transition rates
- quantum kinetics of BEC from the FRG
- parametric resonance in YIG: beyond S-theory.