Out of equilibrium Stat. Mech.: The Kardar-Parisi-Zhang equation

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Scaling out of thermal equilibrium

$=$ Paradise for RG?

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- Relaxation towards equil. (dyn. expo. z)
- Continuous phase transitions in syst. in a NESS

"Branching and annihilating random walks": (BARW)

Partciles (A) diffusing (rate D) on a lattice and that undergo reactions.

Directed Percolation:

$$\begin{aligned} 2A &\to \emptyset \qquad \text{rate } \lambda \\ A &\to 2A \qquad \text{rate } \sigma \end{aligned}$$

Generalized Voter model:

$$\begin{aligned} 2A &\to \emptyset \qquad \text{rate } \lambda \\ A &\to 3A \qquad \text{rate } \sigma \end{aligned}$$

- Phase transition between active and absorbing phase?
- Universality classes? Exponents? etc.

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What does non equil. mean?

- Relaxation towards equil. (dyn. expo. z)
- Continuous phase transitions in syst. in a NESS
- Systems showing generic scaling (in a NESS)

Scaling out of thermal equilibrium

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What does non equilibrium mean?

• Relaxation towards equil. (dyn. expo. z)
• Continuous phase transitions in syst. in a NESS
• Systems showing generic scaling (in a NESS)
• Scaling away from stationarity: short time critical dynamics, coarsening,...

Criticality out of thermal equilibrium

Systems studied:

particles diffusing and reacting
or
systems coupled to a "stochastic bath"

(deposition of particles on a surface)
$\Downarrow$
Not necessarily an Hamitonian evolution
$\Downarrow$
No Boltzmann weight for the stationary distribution
$\Downarrow$
Difficulty: need to describe the whole dynamics

Models studied: Langevin equations for a N-body system

Questions: long-time and large-scale properties of the system?

Langevin equations

$$\partial_t \varphi(\vec{x}, t) = -F[\varphi] + N[\varphi]\zeta(\vec{x}, t),$$

where ζ is a gaussian (white) noise:

$$\langle \zeta(\vec{x}, t) \rangle = 0,$$
$$\langle \zeta(\vec{x}, t)\zeta(\vec{x}', t') \rangle = 2\delta^{(d)}(\vec{x} - \vec{x}')\delta(t - t').$$

For BARW: $\varphi(\vec{x}, t) =$ density (of particles) field.

In general (when detailed balance is violated) the probability distribution of the stationary states is not known.

Averages of $\mathcal{O}(\phi(t,\vec{x}))$ over the noise distribution:

$$\langle \mathcal{O}(\phi) \rangle = \int D\zeta \, P(\zeta) \, \mathcal{O}(\phi_\zeta)$$

$$\begin{aligned}
\langle \mathcal{O}(\phi) \rangle &= \int D\zeta \, P(\zeta) \, \int D\phi \, \delta(\phi - \phi_\zeta) \, \mathcal{O}(\phi) \\
&= \int D\zeta \, P(\zeta) \, \int D\phi \, \delta(\partial_t \phi + F(\phi) - N(\phi)\zeta) \, \mathcal{J}(\phi) \, \mathcal{O}(\phi) \\
&= \int D\zeta \, P(\zeta) \, \int D\phi D[i\tilde{\phi}] \, e^{\int_{t,\vec{x}} -\tilde{\phi}(\partial_t \phi + F(\phi) - N(\phi)\zeta)} \, \mathcal{J}(\phi) \, \mathcal{O}(\phi)
\end{aligned}$$

$$\mathcal{J}(\phi) = \left| \det \left(\partial_\mathrm{t} + \frac{\delta \mathrm{F}(\phi)}{\delta \phi} - \frac{\delta \mathrm{N}(\phi)}{\delta \phi} \zeta \right) \right| \; .$$

If $\mathcal{J} = 1$ (Ito's prescription)

$$\mathcal{Z}[j, \tilde{j}] = \int D\phi \, D[i\tilde{\phi}] \, e^{-\mathcal{S}[\phi,\tilde{\phi}] + \int_{t,\vec{x}} j\phi + \tilde{j}\tilde{\phi}}$$

with

$$\mathcal{S}[\phi, \tilde{\phi}] = \int_{t,\vec{x}} \tilde{\phi} \, (\partial_t \phi + F(\phi)) - N^2(\phi) \tilde{\phi}^2 .$$

Langevin equation for Directed Percolation:

$$
\begin{aligned}
2A &\to \emptyset \qquad \text{rate } \lambda \\
A &\to 2A \qquad \text{rate } \sigma
\end{aligned}
$$

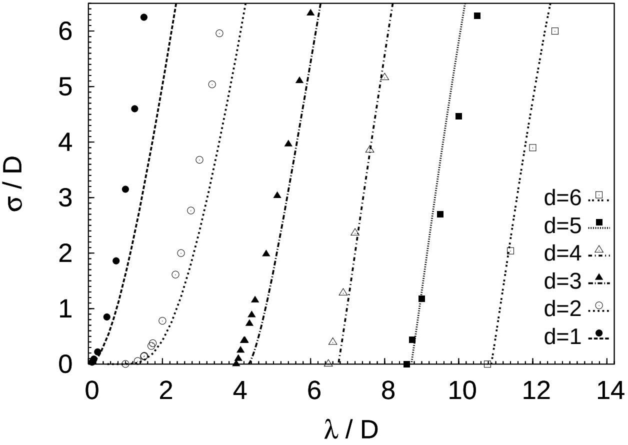
$$\partial_t \varphi(\vec{x}, t) = D\nabla^2\varphi + \sigma\varphi - \lambda\varphi^2 + \sqrt{\sigma\varphi - \lambda\varphi^2}\,\zeta(\vec{x}, t)$$

Mean-field (mass action law)

$$\partial_t \varphi(t) = \sigma\varphi - \lambda\varphi^2$$

Only one stable stationary state: $\varphi = \sigma/\lambda \Rightarrow$ active phase.

$\Rightarrow$ no phase transition... WRONG



σ / D
λ / D
0
1
2
3
4
5
6
0
2
4
6
8
10
12
14
d=6
d=5
d=4
d=3
d=2
d=1

Kardar-Parisi-Zhang equation

M. Kardar, G. Parisi, Y.-C. Zhang, PRL (1986)

$$\partial_t h(\vec{x}, t) = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \sigma \zeta(\vec{x}, t)$$

- describes:
  - surface growth through ballistic deposition of particles
  - disordered systems at equilibrium (directed polymers in random media),
  - Burgers equation ($\vec{v} = \vec{\nabla} h =$ velocity field),
  - magnetic flux lines in superconductors, etc...
- shows generic scaling : $\langle [h(\vec{x}, t) - h(\vec{0}, 0)]^2 \rangle \sim x^{2\chi} f(t/x^z)$. with $z + \chi = 2$.

$$\frac{\partial h(\vec{x},t)}{\partial t} = \nu\, \nabla^2 h(\vec{x},t) + \frac{\lambda}{2} \left(\nabla h(\vec{x},t)\right)^2 + \zeta(\vec{x},t)$$

- becomes a multiplicative noise equation via the Cole-Hopf transformation $w(\vec{x},t) = \exp\left(\frac{\lambda}{2\nu} h(\vec{x},t)\right)$:

$$\partial_t w(\vec{x},t) = \nabla^2 w(\vec{x},t) + \frac{\lambda\, D^{1/2}}{2\nu^{3/2}} w(\vec{x},t)\, \zeta(\vec{x},t)$$

- shows two phases for $d > 2$ ($d+1$ space-time dimenions):
  - smooth phase for small non-linearities (gaussian fluctuations)
  - rough phase for large non-linearities.
- is underlied by symmetries : (gauged) Galilean symmetry, (gauged) shift of h, time reversal symmetry in d=1, and other discrete symmetries (non-linearly realized).
- is perturbatively trivial in the Cole-Hopf representation, but...
- BUT... the rough phase is unreachable perturbatively.

Field theory associated with KPZ:

$$\mathcal{Z}[j,\tilde{j}] = \int \mathcal{D}[h, i\tilde{h}] \exp\left(-\mathcal{S}[h,\tilde{h}] + \int_{\mathbf{x}} (j\, h + \tilde{j}\tilde{h})\right)$$
$$\mathcal{S}[h,\tilde{h}] = \int_{\mathbf{x}} \left\{ \tilde{h}\left(\partial_t h - \nu\, \Delta h - \frac{\lambda}{2}(\nabla h)^2\right) - D\,\tilde{h}^2 \right\}$$

Symmetries:

- (i) "invariance" of $\mathcal{S}$ under the gauged Galilean transformation
  $h(\mathbf{x}) \to h(\vec{x} + \lambda\vec{v}(t), t) + \vec{x}.\partial_t\vec{v}(t)$
  $\tilde{h}(\mathbf{x}) \to \tilde{h}(\vec{x} + \lambda\vec{v}(t), t)$
- (ii) "invariance" of $\mathcal{S}$ under the gauged shift symmetry
  $h(\mathbf{x}) \to h(\mathbf{x}) + f(t)$ where $f(t)$ is arbitrary;
- (iii) in $d = 1$, additional time-reversal invariance
  $h(t) \to h(-t), \tilde{h}(t) \to \tilde{h}(-t) + \frac{\nu}{2D}\Delta h(-t)$.
- and nonlinearly realized discrete symmetries...

Aim: compute the long-time and large-distance physics $\Rightarrow$ derivative expansion?

BUT

Problem: interaction=derivative term $\Rightarrow$ necessary to take into account the momentum dependence of the two-point correlation functions $\Rightarrow$ BMW?

BUT

direct implementation is hindered by the symmetries (Ward identities).

Blaizot-Mendez-Wschebor (BMW) approximation in a nutshell:

Exact equation on $\Gamma_k^{(2)}(p,\phi)$ (for uniform field ϕ):

$$\partial_k \Gamma_k^{(2)}(p,\phi) = \int_q \partial_k R_k(q) G(q)^2.$$
$$\left(\Gamma_k^{(3)}(p,q,-p-q)G(p+q)\Gamma_k^{(3)}(-p,-q,p+q) - \tfrac{1}{2}\Gamma_k^{(4)}(p,-p,q,-q)\right)$$

Truncate the q-dependence of $\Gamma_k^{(3)}$, $\Gamma_k^{(4)}$ $\rightarrow$ closed equation on $\Gamma_k^{(2)}$

$$\partial_k \Gamma_k^{(2)}(p,\phi) = \int_q \partial_k R_k(q) G(q)^2.$$
$$\left(\Gamma_k^{(3)}(p,0,-p)G(p+q)\Gamma_k^{(3)}(-p,0,p) - \tfrac{1}{2}\Gamma_k^{(4)}(p,-p,0,0)\right)$$

Final result:

$$\boxed{\partial_k \Gamma_k^{(2)}(p,\phi) = \left(\partial_\phi \Gamma_k^{(2)}\right)^2 J_3(p,\phi) - \frac{1}{2}\left(\partial_\phi^2 \Gamma_k^{(2)}\right) J_2(0,\phi)}$$

where

$$J_n(p,\phi) = \int_q \partial_k R_k(q)\; G(p+q,\phi) G(q,\phi)^{n-1}$$

Back to KPZ

Two difficulties to implement BMW here:

- no frequency-dependent regulator (forbidden by Galilean sym) $\Rightarrow$ is BMW justified?
- Ward identities / BMW approximation $\Rightarrow$ delicate interplay

Define

$$\varphi = \langle h \rangle \ ; \ \ \tilde{\varphi} = \langle \tilde{h} \rangle$$

$$\Gamma_k^{(m,n)}(\{\mathbf{q}_i\}, \varphi_{\rm u}, \tilde{\varphi}_{\rm u}) = \left.\frac{\delta^{m+n}\Gamma_k}{\delta\varphi_{\mathbf{q}_1}\ldots\delta\varphi_{\mathbf{q}_m}\delta\tilde{\varphi}_{\mathbf{q}_{m+1}}\ldots\delta\tilde{\varphi}_{\mathbf{q}_{m+n}}}\right|_{\varphi_{\rm u},\tilde{\varphi}_{\rm u}}$$

An example of Ward identity (gauged-Galilean):

$$i\omega\frac{\partial}{\partial\vec{p}}\Gamma_\kappa^{(2,1)}(\omega,\vec{p}=\vec{0};\omega_1,\vec{p}_1) = \lambda\vec{p}_1 \times \Big(\Gamma_\kappa^{(1,1)}(\omega+\omega_1,\vec{p}_1) - \Gamma_\kappa^{(1,1)}(\omega_1,\vec{p}_1)\Big)\,,$$

Galilean symmetry = root of the problem
but
Galilean symmetry = part of its solution!

$$i\omega \frac{\partial}{\partial \vec{p}} \Gamma_\kappa^{(2,1)}(\omega, \vec{p} = \vec{0}; \omega_1, \vec{p}_1) = \lambda \vec{p}_1 \times \left(\Gamma_\kappa^{(1,1)}(\omega + \omega_1, \vec{p}_1) - \Gamma_\kappa^{(1,1)}(\omega_1, \vec{p}_1) \right),$$

$\Rightarrow \Gamma_\kappa^{(2,1)}$ in terms of $\Gamma_\kappa^{(1,1)}$ doesn't need $\omega = 0$!

Nevertheless, difficult to implement directly BMW.

Solution: devise an ansatz close in spirit to BMW but that circumvents the above difficulties.

A priori difficult. But...

Geometric formulation of KPZ

Call scalar (under Galilean transfos) a quantity such that:

$$\delta f(\mathbf{x}) = t\lambda\vec{v}\cdot\nabla f(\mathbf{x}).$$

$\Rightarrow \int d^d\vec{x} f$ is Galilean-invariant.

- $\tilde{h}$, $\nabla_i\nabla_j h$ are scalars, $\nabla(\text{scalar}) = \text{scalar}$.
- h and $\partial_t(\text{scalar})$ are not scalars.

$\Rightarrow$ as in fluid mechanics build a covariant time derivative

$$\tilde{D}_t \equiv \partial_t - \lambda\nabla h(\mathbf{x})\cdot\nabla$$

$\Rightarrow \tilde{D}_t(\text{scalar}) = \text{scalar}.$

Covariant derivative of h: $D_t h(\mathbf{x}) \equiv \partial_t h(\mathbf{x}) - \frac{\lambda}{2}(\nabla h(\mathbf{x}))^2$

Scalars at our disposal: $\tilde{h}$, $\nabla_i\nabla_j h$, $D_t h$, $\nabla(\text{scalar})$, $\tilde{D}_t(\text{scalar})$.

Truncation:

- full momentum dependence (derivative interaction),
- full frequency dependence (comparison with exact results in $d = 1$),
- minimal field content.

full frequency dependence

$\Rightarrow$ non polynomial dependence in $\tilde{D}_t \quad (\tilde{D}_t \equiv \partial_t - \lambda \nabla\varphi \cdot \nabla)$

$\Rightarrow$ non polynomial dependence in φ

$\Rightarrow$ minimal $=$ minimal in $\tilde{\varphi}$, $\nabla^2\varphi$ and $D_t\varphi$.

Our Ansatz:

$$\Gamma_\kappa[\varphi, \tilde{\varphi}] = \int_{\mathbf{x}} \tilde{\varphi}\, f_\kappa^\lambda(-\tilde{D}_t^2, -\nabla^2)\, D_t\varphi - \tilde{\varphi}\, f_\kappa^D(-\tilde{D}_t^2, -\nabla^2)\, \tilde{\varphi}$$

$$-\frac{\nu}{2D}\left[\nabla^2\varphi\, f_\kappa^\nu(-\tilde{D}_t^2, -\nabla^2)\, \tilde{\varphi} + \tilde{\varphi}\, f_\kappa^\nu(-\tilde{D}_t^2, -\nabla^2)\, \nabla^2\varphi\right].$$

- For $d=1$, time reversal sym. $\Rightarrow \begin{cases} f_\kappa^\lambda(\omega^2, p^2) = 1 \\ f_\kappa^\nu(\omega^2, p^2) = f_\kappa^D(\omega^2, p^2) \end{cases}$

$\Rightarrow$ only (!) one function (of ω^2 and p^2).

- For $d > 1$, 3 functions and 3-dimensional integrals $\Rightarrow$ simplify !

Aim: zero frequency sector $\Rightarrow$ neglect all ω-dependence (but the bare one).
Moreover, since $f_\kappa^\lambda(p^2=0)=1 \Rightarrow$ impose $f_\kappa^\lambda(p^2)=1$, $\forall \vec{p}$.
$\Rightarrow$ two functions: $f_\kappa^\nu(p^2)$ and $f_\kappa^D(p^2)$.

Two additional inputs:

- causality has to be preserved (and Ito's prescription),
- $R_\kappa(\vec{q}) = r\left(\frac{q^2}{\kappa^2}\right)\begin{pmatrix} 0 & \nu_\kappa q^2 \\ \nu_\kappa q^2 & -2D_\kappa \end{pmatrix}$, $r(x) = \alpha/(\exp(x)-1)$

And now, turn the crank...

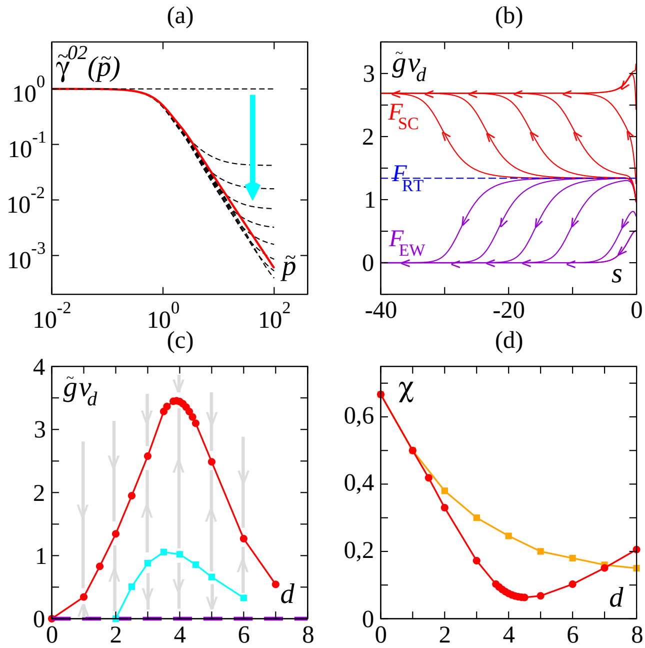
Results with the simplified ansatz ($\forall d$):

L.Canet, H. Chaté, B.D. and N. Wschebor, PRL (2009), Arxiv (2011)

- three fixed points: gaussian F_{EW}, transition F_{TR} and strong coupling F_{SC},
- generic scaling at F_{SC} (no phase transition associated with this FP),
- asymptotic safety (F_{TR}),
- exact results for exponents recovered in $d = 0$ and $d = 1$ at F_{SC},
- exponents not so bad at F_{SC} for $d \leq 3$.

BUT

- exact results for exponents not recovered for F_{TR} (in $d > 2$)
- strange behavior of the critical exponents for $d \gtrsim 3.5 \Rightarrow$ no prediction as for the existence of an upper critical dimension.


(a)
$\tilde{\gamma}^{02}(\tilde{p})$
$\tilde{p}$
(b)
$\tilde{g}v_d$
F_{SC}
F_{RT}
F_{EW}
s
(c)
$\tilde{g}v_d$
d
(d)
χ
d

The $d = 1$ case

Exact results in $d = 1$:
M. Praehofer and H. Spohn, PRL (2000), J. Stat. Phys. (2004)
J. Baik and E.M. Rain, J. Stat. Phys. (2000)

Experimental results: K.A. Takeuchi and M. Sano, PRL (2010)

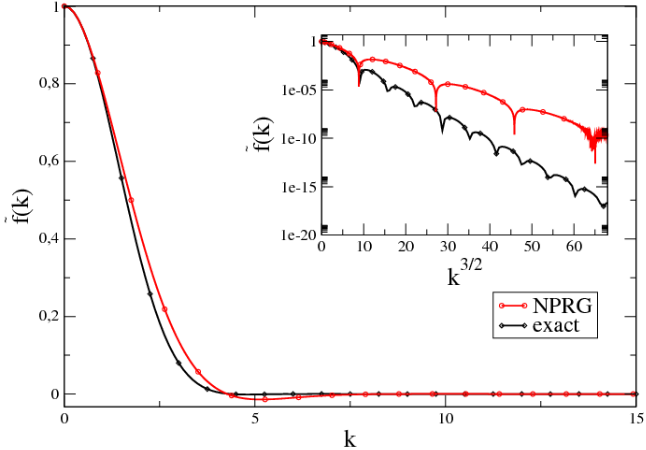
Results with the full ansatz for $d = 1$

L.Canet, H. Chaté, B.D. and N. Wschebor, Arxiv (2011)

$$f_k(\omega^2, p^2) \Rightarrow \text{excellent data collapse at } k \to 0 \Rightarrow f_{k=0}\left(\frac{\omega^2}{p^3}\right)$$

$$\Downarrow$$

existence of scaling



f̃(k)
k
NPRG
exact
k^{3/2}

Conclusion and outlook:

A lot remains to be done for KPZ...

Everything remains to be done for Navier-Stokes!