Ginsparg-Wilson realization of gauge symmetry in QED

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Yuji Igarashi

Faculty of Education, Niigata University

Motivation

♢ One of the most important subjects in ERG:

How to realize gauge symmetries which are naively incompatible with regularization scheme with a momentum cutoff

♢ Since pioneering work of Becchi('93) and Ellwanger('94) appeared, a lot of discussion on this subject. (Becchi '93, Ellwanger, Bonini *et al.*. '94, Morris *et al.* '00, Freire *et al.* '01, Pawlowski '05: YI-Itoh-Sonoda '09 for a recent review)

Among those contributions, most convincing method to show the presence of exact symmetry and to describe its properties is to construct

• Ward-Takahashi (WT) identies for Wilson action $S[\Phi]$:

$$\Sigma[\Phi] = \frac{\partial S}{\partial \Phi}\delta\Phi - \frac{\partial}{\partial \Phi}\delta\Phi = 0.$$

• or its extension to Quantum Master Equation (QEM) in antifield formalism:

$$\Sigma[\Phi, \Phi^*] = \frac{\partial S}{\partial \Phi}\frac{\partial S}{\partial \Phi^*} - \frac{\partial}{\partial \Phi}\frac{\partial S}{\partial \Phi^*} = 0.$$

In general, $S[\Phi]$ is not invariant under symmetry tr. $\delta\Phi(= \partial S/\partial\Phi^*)$ but its change is cancled by change induced in functional measure. Note that $\delta\Phi$ depends on $S[\Phi]$.

WT or QME are given:

for QED (Bonini*et al.*. '94, Sonoda '07, YI-Itoh-Sonoda '07, Higashi-Itou-Kugo '07)

for YM (Becchi '93, YI-Itoh-Sonoda '09)

How to find (non-perturvative) solutions ?

♢ The prototype of reg. dependent symmetry is chiral symmetry on the lattice, whose WT known as the Ginsparg-Wilson (GW) relation: $\{\gamma_5, D\} = 2aD\gamma_5 D$.

Here, Dirac action $\bar{\psi}D\psi$ is invariant under chiral transformation which depends on Dirac operator: $\delta\psi = i\gamma_5(1 - aD)\psi,\ \delta\bar{\psi} = i\bar{\psi}(1 - aD)\gamma_5$.

- WT for Wilson action with Yukawa couplings are solved to construct more general Dirac operator with (non-polynomial) Yukawa interactions. (YI-So-Ukita '02, Echigo-YI '11)

Can one construct GW-type action in gauge theory ?

Using WT for QED, we discuss here a GW type solution, where Dirac action has non-polynonial gauge interactions.

♢ For RG flows, Wilson action obeys Polchinski flow eq.

$$\partial_k S_{\mathrm{eff}}[\Phi] \quad = \quad -\frac{1}{2}\frac{\partial^r S_{\mathrm{eff}}}{\partial \Phi^A}(\dot{\mathbf{\Delta}})^{AB}\frac{\partial^l S_{\mathrm{eff}}}{\partial \Phi^B} + \frac{1}{2}(-)^{\epsilon_A}(\dot{\mathbf{\Delta}})^{AB}\frac{\partial^l \partial^r S_{\mathrm{eff}}}{\partial \Phi^B \partial \Phi^A}$$

However, we are interested in its 1PI part Γ_{eff} which obeys Wetterich eq.

$$\partial_k \Gamma_{\mathrm{eff}}[\Phi] \quad = \quad \frac{1}{2}(-)^{\epsilon_A}\dot{\mathbf{\Delta}}^{-1}_{AB}\left[\mathbf{\Delta}^{-1}_{BA} + \frac{\partial^l \partial^r \Gamma_{\mathrm{eff}}[\Phi]}{\partial \Phi^B \partial \Phi^A}\right]^{-1}$$

Based on tree expansion of S_{eff} in terms of Γ_{eff}, we find specific subset of S_{eff} is useful to reduce Polchinski eq. to Wetterich eq. (Ishikake-Ukita-YI '05)

♢ Plan

[1] Derivation of the WT identities for the Wilson action

[2] WT identities for QED

[3] Ginsparg-Wilson type solution in QED

[4] Reduction of Polchinski eq. to Wetterich eq.

[5] Discussion and outlook

Derivation of WT identities for the Wilson action

◇ Consider generic gauge-fixed theory described by

$$\mathcal{Z}_\phi[J] = \int \mathcal{D}\phi \exp\left(-\mathcal{S}[\phi] + J\cdot\phi\right), \qquad J\cdot\phi = J_A\phi^A$$

$$\mathcal{S}[\phi] = \frac{1}{2}\phi\cdot D\cdot\phi + \mathcal{S}_I[\phi], \qquad \phi\cdot D\cdot\phi = \phi^A D_{AB}\phi^B$$

and introduce momentum cutoff function

$$K(p) \approx \begin{cases} 1 & \text{for } p^2 < \Lambda^2 \\ 0 & \text{for } p^2 > \Lambda^2 \end{cases}$$

to decompose ϕ with propagator $D^{-1}(p)$

$\Rightarrow$ IR fields Φ with $K(p)D^{-1}(p)$ $\oplus$ UV field χ with $(1-K(p))D^{-1}(p)$

- To this end, insert gaussian integral for new fields Φ^A (cf. Wetterich, Bonini *et al.*, Morris)

$$\int \mathcal{D}\Phi \exp -\frac{1}{2}\left[\left(\Phi - K\phi - J(1-K)D^{-1}\right)\cdot\frac{D}{K(1-K)}\cdot\left(\Phi - K\phi - (-)^{\epsilon(J)}D^{-1}(1-K)J\right)\right]$$
$$= const$$

into r.h.s of $\mathcal{Z}_\phi[J]$. (Source terms in gaussian are introduced to cancel $J\cdot\phi$ for UV theory.)

Exchange order of integration gives path-integral representation of Wilson action:

$$\exp -S_{\text{eff}}[\Phi] \equiv \int \mathcal{D}\phi \exp -\left\{\frac{1}{2}\underbrace{(\phi-\Phi)}_{\chi}\cdot(1-K)^{-1}D\cdot(\phi-\Phi) + \mathcal{S}_I[\phi]\right\}$$

For total action $S[\Phi] = \Phi\cdot K^{-1}D\cdot\Phi/2 + S_{\text{eff}}[\Phi]$, we define

$$Z_\Phi[J] = \int \mathcal{D}\Phi \exp\left(-S[\Phi] + J\cdot K^{-1}\Phi\right)$$

to obtain relation $\mathcal{Z}_\phi[J] = N_J Z_\Phi[J]$, where

$$N_J = \exp -\frac{1}{2}\Big((-)^{\epsilon(J)} J \cdot K^{-1}(1-K)D^{-1} \cdot J\Big)$$

♢ Consider symmetry properties of Wilson action, making a change of variables

$$\phi^A \to \phi'^A = \phi^A + \delta\phi^A \,, \qquad \delta\phi^A = \mathcal{R}^A[\phi]$$

Since $\mathcal{Z}$ is invariant under the change of variables, we obtain

$$\int \mathcal{D}\phi \Big(J \cdot \delta\phi - \Sigma[\phi]\Big) \exp\left(-\mathcal{S}[\phi] + J \cdot \phi\right) = 0 \tag{1}$$

where $\Sigma[\phi]$ is *the WT operator* given as

$$\Sigma[\phi] \equiv \delta\mathcal{S} + \delta \ln \mathcal{D}\phi = \frac{\partial \mathcal{S}}{\partial \phi^A}\delta\phi^A - \frac{\partial}{\partial \phi^A}\delta\phi^A \,.$$

It is given by sum of the change of the action $\mathcal{S}$ and that of the functional measure $\mathcal{D}\phi$.

From (1) we obtain

$$
\begin{aligned}
&\int \mathcal{D}\phi\ \Sigma[\phi] \exp\left(-\mathcal{S}[\phi] + J\cdot\phi\right) =< \Sigma[\phi] >_\phi = \int \mathcal{D}\phi J_A R^A[\phi] \exp\left(-\mathcal{S}[\phi] + J\cdot\phi\right) \\
= \quad & J_A R^A \left[\partial^l/\partial J\right] \mathcal{Z}_\phi[J] = N_J \left\{ N_J^{-1} \left(J \cdot \mathcal{R}\left[\frac{\partial}{\partial J}\right]\ N_J \right) + J\cdot \mathcal{R}\left[\frac{\partial}{\partial J}\right]\ \right\} Z_\Phi[J] \\
= \quad & N_J \int \mathcal{D}\Phi\ \Sigma[\Phi] \exp\left(-S[\Phi] + J\cdot K^{-1}\Phi\right) = N_J < \Sigma[\Phi] >_\Phi
\end{aligned}
$$

We consider below a linear symmetry described by

$$
\begin{aligned}
\Sigma[\phi] \quad &= \quad \delta\mathcal{S} = \frac{\partial \mathcal{S}}{\partial \phi^A}\ \delta\phi^A = 0 \\
\delta\phi^A \quad &= \quad \mathcal{R}^A{}_B \phi^B\,,
\end{aligned}
$$

where $\mathcal{R}^A{}_B$ do not depend on the fields.

Then, *the WT operator* for the Wilson action is given by

$$\begin{aligned}\Sigma[\Phi] &= \frac{\partial S}{\partial \Phi^A}\delta\Phi^A - \frac{\partial}{\partial \Phi^A}\delta\Phi^A \\ \delta\Phi^A &= \mathcal{R}^A{}_B \left\{ \Phi^A - \left[K(1-K)D^{-1}\right]^{AB} \frac{\partial S}{\partial \Phi^B} \right\} .\end{aligned}$$

Note that symmetry transformation $\delta\Phi$ depends on the Wilson action.

Symmetry for the Wilson action is described by the WT identities

$$\Sigma[\Phi] = 0$$

WT identities in QED

◇ Consider QED with UV fields $\phi^A = \{a_\mu, b, c, \bar{c}, \psi, \bar{\psi}\}$ and $J_A = \{J_\mu, J_b, J_c, J_{\bar{c}}, J_\psi, J_{\bar{\psi}}\}$.

The action $\mathcal{S}[\phi] = \phi \cdot D \cdot \phi/2 + \mathcal{S}_I[\phi]$ is given by

$$
\begin{aligned}
\frac{1}{2}\phi^A D_{AB}\phi^B &= \int_k \Big[\frac{1}{2}a_\mu(-k)(k^2\delta_{\mu\nu} - k_\mu k_\nu)a_\nu(k) + \bar{c}(-k)ik^2 c(k) \\
&\qquad - b(-k)\big(ik_\mu a_\mu(k) + \frac{\alpha}{2}b(k)\big)\Big] + \int_p \bar{\psi}(-p)(\not{p} + im)\psi(p) \\
\mathcal{S}_I[\phi] &= -e\int_{p,\ k} \bar{\psi}(-p-k)\not{a}(k)\psi(p)
\end{aligned}
$$

It is inv. under the BRS tr.

$$
\begin{aligned}
\delta a_\mu(k) &= -ik_\mu\ c(k), \qquad \delta\bar{c}(k) = ib(k), \qquad \delta c(k) = \delta b(k) = 0 \\
\delta\psi(p) &= -ie\int_k \psi(p-k)\ c(k), \qquad \delta\bar{\psi}(-p) = ie\int_k \bar{\psi}(-p-k)\ c(k)
\end{aligned}
$$

This fixes $J_A R^A \left[\partial^l/\partial J\right] \equiv J\cdot R$ and the factor N_J:

$$\ln N_J = \int_p \Big(\frac{1-K}{K}\Big)(p) J_\psi(-p)\frac{1}{\not p + im} J_{\bar\psi}(p) + (\text{terms with } J_\mu, J_b, J_c, J_{\bar c})$$

♢ Now compute $(J\cdot R)\ \mathcal{Z}_\phi[J] = \ (J\cdot R)\ N_J Z_\Phi[J] \quad =< \Sigma[\Phi] >_\Phi = 0$

to find $\Sigma[\Phi]$ for IR fields: $\Phi^A = \{A_\mu, B, C, \bar C, \Psi, \bar\Psi\}$.

Note that bilinear source terms given above generate

$$J_\psi \cdots J_{\bar\psi} \to \exp(S)\frac{\partial}{\partial\bar\Psi}\cdots\frac{\partial}{\partial\Psi}\exp(-S) \to \left\langle \frac{\partial S}{\partial\bar\Psi}\cdots\frac{\partial S}{\partial\Psi} - \frac{\partial}{\partial\bar\Psi}\cdots\frac{\partial}{\partial\Psi}S \right\rangle_\Phi$$

Using matrix U (regularized both in IR and UV regions)

$$U(-p, p-k) = \frac{1-K(p-k)}{\not p - \not k + im}K(p) - \frac{1-K(p)}{\not p + im}K(p-k)$$

We obtain WT in QED (Sonoda '07)

$$
\begin{aligned}
\Sigma[\Phi] \quad = \quad & \int_k \Big\{ \frac{\partial S}{\partial A_\mu(k)}(-ik_\mu)C(k) + \frac{\partial^r S}{\partial \bar{c}(k)} iB(k) \Big\} \\
& -ie \int_{p,\ k} \Big\{ \frac{\partial^r S}{\partial \Psi(p)} \frac{K(p)}{K(p-k)} \Psi(p-k) - \frac{K(p)}{K(p+k)} \bar{\Psi}(-p-k) \frac{\partial^l S}{\partial \bar{\Psi}(-p)} \Big\} C(k) \\
& -ie \int_{p,\ k} \Big\{ \frac{\partial^l S}{\partial \bar{\Psi}(-p+k)} \frac{\partial^r S}{\partial \Psi(p)} - \frac{\partial^l \partial^r S}{\partial \bar{\Psi}(-p+k) \partial \Psi(p)} \Big\} U(-p, p-k) C(k) = 0
\end{aligned}
$$

$\diamondsuit$ We may put the quadratic functional derivative term $(\partial S/\partial \Psi)(\partial S/\partial \bar{\Psi})$ as

$$
\left[\frac{\partial^r S}{\partial \Psi(p)}\ C(k) \Big\{ \frac{K(p)}{K(p-k)} \Psi(p-k) - U(-p, p-k) \frac{\partial^l S}{\partial \bar{\Psi}(-p+k)} \Big\} \right]
$$

to define BRS tr. for the fields Φ^A:

$$
\delta A_\mu(k) \quad = \quad -ik_\mu\ C(k), \quad \delta \bar{C}(k) = iB(k), \quad \delta C(k) = \delta B(k) = 0
$$

$$
\begin{aligned}
\delta\Psi(p) &= ie\int_k C(k)\Big\{\frac{K(p)}{K(p-k)}\Psi(p-k) - U(-p,p-k)\frac{\partial^l S}{\partial\bar\Psi(-p+k)}\Big\} \\
\delta\bar\Psi(-p) &= ie\int_k \frac{K(p)}{K(p+k)}\bar\Psi(-p-k)\ C(k)
\end{aligned}
$$

Then, $\Sigma[\Phi]$ takes the form

$$
\Sigma[\Phi] = \frac{\partial^r S}{\partial\Phi^A}\delta\Phi^A + ie\frac{\partial^l\partial^r S}{\partial\bar\Psi\partial\Psi}\ U\ C
$$

The last term can be interpreted as the Jacobian factor associated with $\Psi(p) \to \Psi(p) + \delta\Psi(p)$.

Ginsparg-Wilson type solution in QED

Assume the Wilson action $S[\Phi]$ is bilinear in Dirac fields, and solve WT in QED.

$$
\begin{aligned}
S_{\mathrm{GW}}[\Phi] &= \int_k K^{-1}(k)\Big[\frac{1}{2}A_\mu(-k)(k^2\delta_{\mu\nu} - k_\mu k_\nu)A_\nu(k) + \bar{C}(-k)ik^2C(k) \\
&\qquad -B(-k)\big(ik_\mu B_\mu(k) + \frac{\alpha}{2}B(k)\big)\Big] + S_D + S[A] \\
S_D &= \int_{p,\ q} \bar{\Psi}(-p)\mathcal{D}(p,q)\Psi(q), \qquad \mathcal{D}(p,q) = K^{-1}(p)(\not{p} + im)\delta(p-q) + \Theta(p,q)
\end{aligned}
$$

where $S[A]$ and Θ depends on A_μ. For this form of action, BRS tr. becomes

$$
\begin{aligned}
\delta A_\mu(k) &= -ik_\mu\ C(k), \quad \delta\bar{C}(k) = iB(k), \quad \delta C(k) = \delta B(k) = 0 \\
\delta\bar{\Psi}(-p) &= ie\int_k \frac{K(p)}{K(k)}\bar{\Psi}(-k)\ C(k-p) \\
\delta\Psi(p) &= ie\int_{k,q} C(p-k)\Big[\frac{K(p)}{K(k)}\Psi(k) - \Big\{\Delta_H(k)K(p) - \Delta_H(p)K(k)\Big\}\mathcal{D}(k,q)\Psi(q)\Big]
\end{aligned}
$$

where $\Delta_H(p) = (1 - K(p))/(\not{p} + im)$ is high-ernergy propagator for Dirac fields.

To solve the WT, we impose

$$\delta S_D = \int_{p,\ q} \Big[\bar{\Psi}(-p)\mathcal{D}(p,q)\delta\Psi(q) - \bar{\Psi}(-p)\delta\mathcal{D}(p,q)\Psi(q) - \delta\bar{\Psi}(-p)\mathcal{D}(p,q)\Psi(q) \Big] = 0.$$

This gives a relation something like

$$\alpha + \beta\mathcal{D} + \gamma\mathcal{D}^2 + \delta\Theta = 0$$

which can be interpreted as "Ginsparg-Wilson relation in QED". We obtain

$$\begin{aligned}
\delta\Theta(p,q) &= ieC(p-q)(\not{p} - \not{q}) + ie\int_l \Big[\Theta(p,l)C(l-q) - C(p-l)\Theta(l,q) \Big] \\
&\quad - ie\int_l \Big[\Theta(p,l)\Delta_H(l)C(l-q)(\not{l} - \not{q}) + C(p-l)(\not{p} - \not{l})\Delta_H(l)\Theta(l,q) \Big] \\
&\quad - ie\int_{k,l,r} \Theta(p,q)C(l-r)\Big[\Delta_H(r)K(l) - \Delta_H(l)K(r) \Big]\Theta(r,q)
\end{aligned}$$

Expanding Θ in powers of $\not{A}$, we find a simple solution:

$$\begin{aligned}\Theta(p,q) &= -e\not{A}(p-q) - e^2\int_l \not{A}(p-l)\Delta_H(l)\not{A}(l-q)\\ &\quad -e^3\int_{l,r}\not{A}(p-l)\Delta_H(l)\not{A}(l-r)\Delta_H(r)\not{A}(r-q) - \dots\\ &\equiv -e\int_k \not{A}(p-k)\left[\frac{1}{1-e\Delta_H\not{A}}\right](k,q).\end{aligned}$$

Consider "the functional measure contribution" $J = \partial\delta\Psi/\partial\Psi$ in $\Sigma[\Phi]$

$$\begin{aligned}J &= ie\frac{\partial^l\partial^r S}{\partial\bar{\Psi}\partial\Psi}\,U\,C\\ &= ie\int_{p,q}[K^{-1}(p)(\not{p}+im)\delta(p-q)+\Theta(p,q)]\{\Delta_H(p)K(q)-K(p)\Delta_H(q)\}C(q-p)\\ &= ie\int_{p,q}\Theta(p,q)\{\Delta_H(p)K(q)-K(p)\Delta_H(q)\}C(q-p)\end{aligned}$$

To cancel this contribution, we choose a counter action

$$S[A] = -\mathrm{Tr}\ln(1 - e\Delta_H \not{A})$$

whose BRS variation is shown to cancel functional measure contribution:

$$\delta S[A] = -J$$

In summary, we have constructed a Ginsparg-Wilson type solution to WT:

$$\begin{aligned}
S_{\mathrm{GW}}[\Phi] &= \int_k K^{-1}(k)\Big[\frac{1}{2}A_\mu(-k)(k^2\delta_{\mu\nu} - k_\mu k_\nu)A_\nu(k) + \bar{C}(-k)ik^2C(k) \\
&\qquad -B(-k)\Big(ik_\mu B_\mu(k) + \frac{\alpha}{2}B(k)\Big)\Big] + S_D + S[A] \\
S_D &= \int_{p,\ q} \bar{\Psi}(-p)\mathcal{D}(p,q)\Psi(q), \qquad \mathcal{D}(p,q) = K^{-1}(p)(\not{p} + im)\delta(p-q) + \Theta(p,q) \\
\Theta(p,q) &= -e\int_k \not{A}(p-k)\left[\frac{1}{1 - e\Delta_H \not{A}}\right](k,q) \\
S[A] &= -\mathrm{Tr}\ln[1 - e\Delta_H \not{A}]
\end{aligned}$$

Reduction of Polchinski flow eq. to Wetterich eq.

◇ For RG flow of the Wilson action

$$\exp\{-S_{\rm eff}[\Phi]\} = \int \mathcal{D}\chi \exp -\Big(\frac{1}{2}\chi\cdot\mathbf{\Delta}^{-1}\cdot\chi + \mathcal{S}_I[\Phi+\chi]\Big)$$

where $\mathbf{\Delta} = \Delta_H = (1-K)D^{-1}$, we are interested in its 1PI part, "Legendre effective action" defined by Legendre transformation

$$\Gamma_{\rm eff}[\varphi] = S_{\rm eff}[\Phi] - \frac{1}{2}(\varphi-\Phi)\cdot\mathbf{\Delta}^{-1}\cdot(\varphi-\Phi)$$

where classical UV fields given by

$$\varphi^A = \Phi^A - (\mathbf{\Delta})^{AB}\partial^l S_{\rm eff}[\Phi]/\partial\Phi^B$$

This leads to

$$S_{\rm eff}[\Phi] = \Gamma_{\rm eff}\left[\Phi - \mathbf{\Delta}\cdot\frac{\partial^l S_{\rm eff}[\Phi]}{\partial\Phi}\right] + \frac{1}{2}\frac{\partial^r S_{\rm eff}[\Phi]}{\partial\Phi}\cdot\mathbf{\Delta}\cdot\frac{\partial^l S_{\rm eff}[\Phi]}{\partial\Phi}.$$

Using it, we expand $S_{\rm eff}[\Phi]$ in terms of its 1PI part $\Gamma_{\rm eff}[\Phi]$ and the cutoff propagator $\mathbf{\Delta}$. According to the number (n) of $\Gamma_{\rm eff}[\Phi]$ and its derivatives such as $\partial^l \Gamma_{\rm eff}[\Phi]/\partial\Phi^A = \overrightarrow{\partial}_A \Gamma_{\rm eff}$, $S_{\rm eff}[\Phi]$ is decomposed as

$$S_{\rm eff}[\Phi] = \sum_{n=1} S_{\rm eff}^{(n)}[\Phi]$$

where

$$\begin{aligned}
S_{\rm eff}^{(1)}[\Phi] &= \Gamma_{\rm eff}[\Phi] \\
S_{\rm eff}^{(2)}[\Phi] &= -\frac{1}{2}(\Gamma_{\rm eff}\overleftarrow{\partial}_A)\mathbf{\Delta}^{AB}(\overrightarrow{\partial}_B\Gamma_{\rm eff}) \\
S_{\rm eff}^{(3)}[\Phi] &= +\frac{1}{2}(\Gamma_{\rm eff}\overleftarrow{\partial}_A)\mathbf{\Delta}^{AB}(\overrightarrow{\partial}_B\Gamma_{\rm eff}\overleftarrow{\partial}_C)\mathbf{\Delta}^{CD}(\overrightarrow{\partial}_D\Gamma_{\rm eff}) \\
S_{\rm eff}^{(4)}[\Phi] &= -\frac{1}{2}(\Gamma_{\rm eff}\overleftarrow{\partial}_A)\mathbf{\Delta}^{AB}(\overrightarrow{\partial}_B\Gamma_{\rm eff}\overleftarrow{\partial}_C)\mathbf{\Delta}^{CD}(\overrightarrow{\partial}_D\Gamma_{\rm eff}\overleftarrow{\partial}_E)\mathbf{\Delta}^{EF}(\overrightarrow{\partial}_F\Gamma_{\rm eff}) \\
&\quad -\frac{1}{3!}(\Gamma_{\rm eff}\overleftarrow{\partial}_A\overleftarrow{\partial}_B\overleftarrow{\partial}_C)(\mathbf{\Delta}^{CD}\overrightarrow{\partial}_D\Gamma_{\rm eff})(\mathbf{\Delta}^{BE}\overrightarrow{\partial}_E\Gamma_{\rm eff})(\mathbf{\Delta}^{AF}\overrightarrow{\partial}_F\Gamma_{\rm eff})
\end{aligned}$$

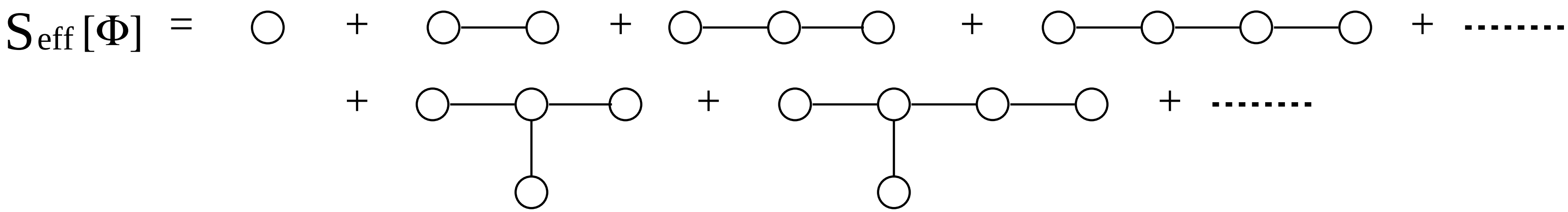

Figure 1: A graphical representaion of the tree expansion. Each circle and solid line denote $\Gamma_{\rm eff}$ and cutoff propagator $\mathbf{\Delta}$, respectively. Summation of the diagrams in the first line of rhs gives $S^L_{\rm eff}$. All the remainings form $S^X_{\rm eff}$.

As shown in Fig.1, $S_{\rm eff}$ is decomposed as $S_{\rm eff}[\Phi] = S^L_{\rm eff}[\Phi] + S^X_{\rm eff}[\Phi]$, where summation of contributions in the first line gives

$$S^L_{\rm eff}[\Phi] \quad = \quad \Gamma_{\rm eff}[\Phi] - \frac{1}{2}(\Gamma_{\rm eff} \overleftarrow{\partial}_A)([1 + \mathbf{\Delta}(\overrightarrow{\partial}\, \Gamma_{\rm eff} \overleftarrow{\partial})]^{-1})^A{}_C \mathbf{\Delta}^{CD}(\overrightarrow{\partial}_D \Gamma_{\rm eff})$$

$S^X_{\rm eff}[\Phi]$ denotes the remaining subsets of $S_{\rm eff}[\Phi]$.

♢ Extract 1PI part of the Polchinski flow eq.

$$\partial_k S_{\rm eff}[\Phi] \quad = \quad -\frac{1}{2}\frac{\partial^r S_{\rm eff}}{\partial \Phi^A}(\dot{\mathbf{\Delta}})^{AB}\frac{\partial^l S_{\rm eff}}{\partial \Phi^B} + \frac{1}{2}(-)^{\epsilon_A}(\dot{\mathbf{\Delta}})^{AB}\frac{\partial^l \partial^r S_{\rm eff}}{\partial \Phi^B \partial \Phi^A}$$

- 1PI part of l.h.s is $\partial\Gamma_{\rm eff}$.

- First term of r.h.s, "dumbbell term", gives no 1PI contributions.

- 1PI part of second term in r.h.s is only generated via $S^L_{\rm eff}$:

$$
\begin{aligned}
\partial_k S^L_{\rm eff}[\Phi]\big|_{\rm 1PI} &= \partial_k \Gamma_{\rm eff}[\Phi] = \frac{1}{2}(-)^{\epsilon_A}\,\dot{\mathbf{\Delta}}^{AB}\frac{\partial^l\partial^r S^L_{\rm eff}}{\partial\Phi^B\partial\Phi^A}\bigg|_{\rm 1PI} \\
&= \frac{1}{2}(-)^{\epsilon_A}\dot{\mathbf{\Delta}}^{-1}_{AB}\left[\mathbf{\Delta}^{-1}_{BA} + \frac{\partial^l\partial^r\Gamma_{\rm eff}[\Phi]}{\partial\Phi^B\partial\Phi^A}\right]^{-1}
\end{aligned}
$$

Therefore, $S^L_{\rm eff}$ is pricisely the subset of the Wilson action which generates 1PI reduction of the Polchinski eq. to Wetterich eq.

$\diamondsuit$ For QED with

$$\Gamma_{\rm eff} = -e \int_{p,q} \bar{\Psi}(-p) \not{A}(p-q) \Psi(q).$$

our Dirac action in GW-type solution is a subset of $S^L_{\rm eff}$:

$$S^L_{\rm eff} \supset \int_{p,q} \bar{\Psi}(-p) \Theta(p-q) \Psi(q)$$

Discussion and outlook

♢ GW-type solution to WT is related to perfect action = Wilson action obtained by blocking (integration) of UV Dirac fields ? :

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp -\left[\bar{\psi}\Delta_H^{-1}\psi - e(\bar{\psi}+\bar{\Psi})\not{A}(\psi+\Psi)\right] = \exp -\Big(S_D[\bar{\Psi},\Psi,A]+S[A]\Big) \quad ?$$

where no-blocking is performed for gauge field $a_\mu = A_\mu$.

Our construction of GW-type solution shows that the blocked action solves WT.

♢ $S[A]$ is UV divergent ?

- introduce a UV cutoff or subtraction

♢ GW realization: first step to find solutions to WT.

How to get more general solutions ?

- blocking of gauge field ?
- including a certain subset in $S_{\rm eff}^L$?