Scalar theory pressure beyond the local potential approximation

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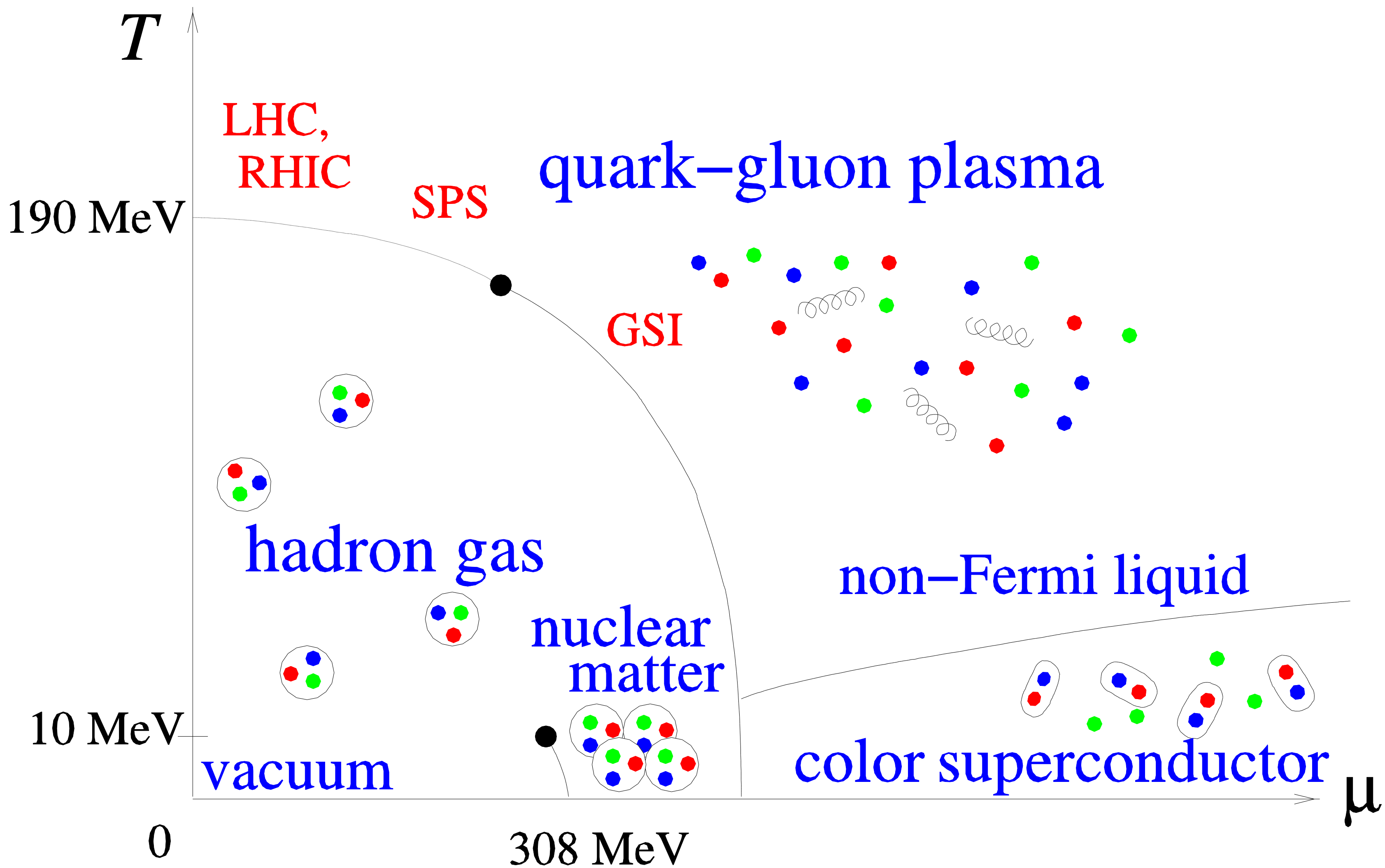
Jean-Paul Blaizot, AI, Nicolás Wschebor,
Nucl.Phys.A849:165,2011

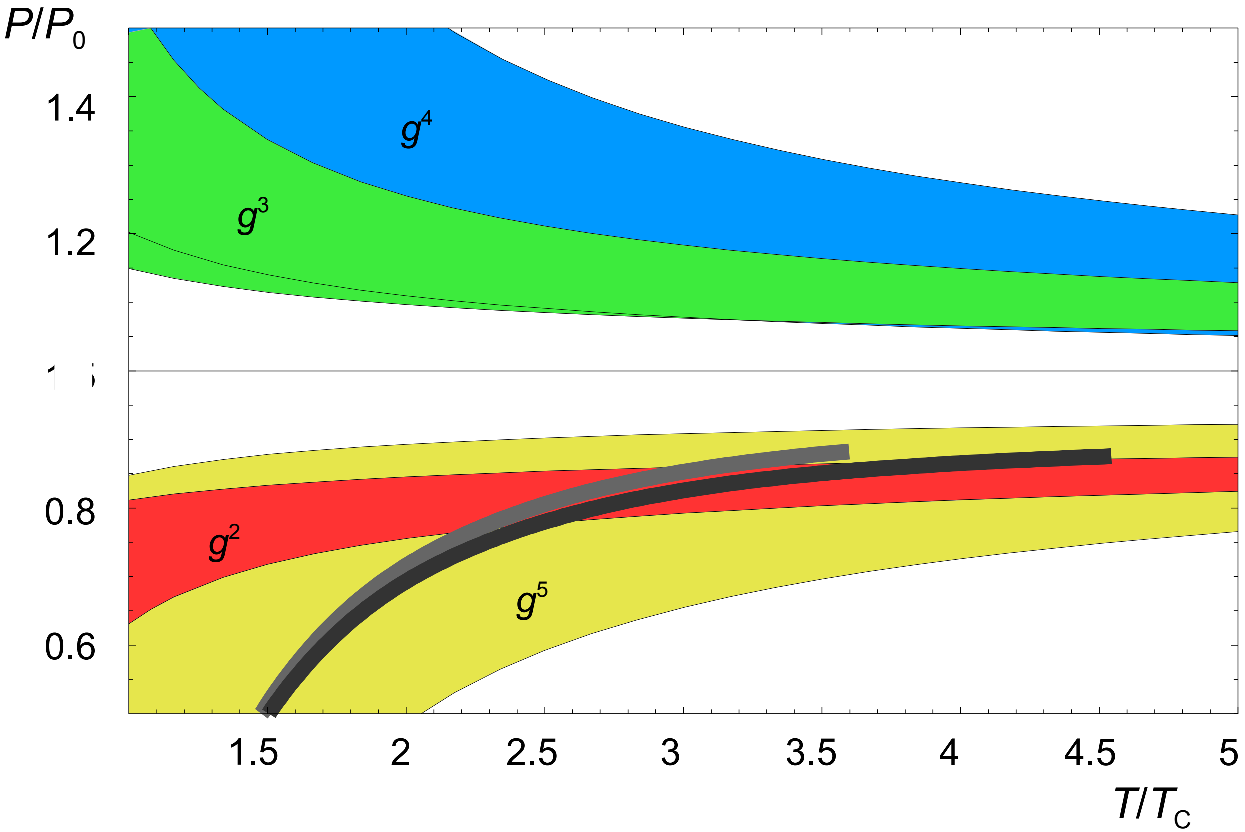
YITP Kyoto, Sept 1, 2011

QCD phase diagram



QCD pressure

Perturbative expansion of QCD pressure converges badly



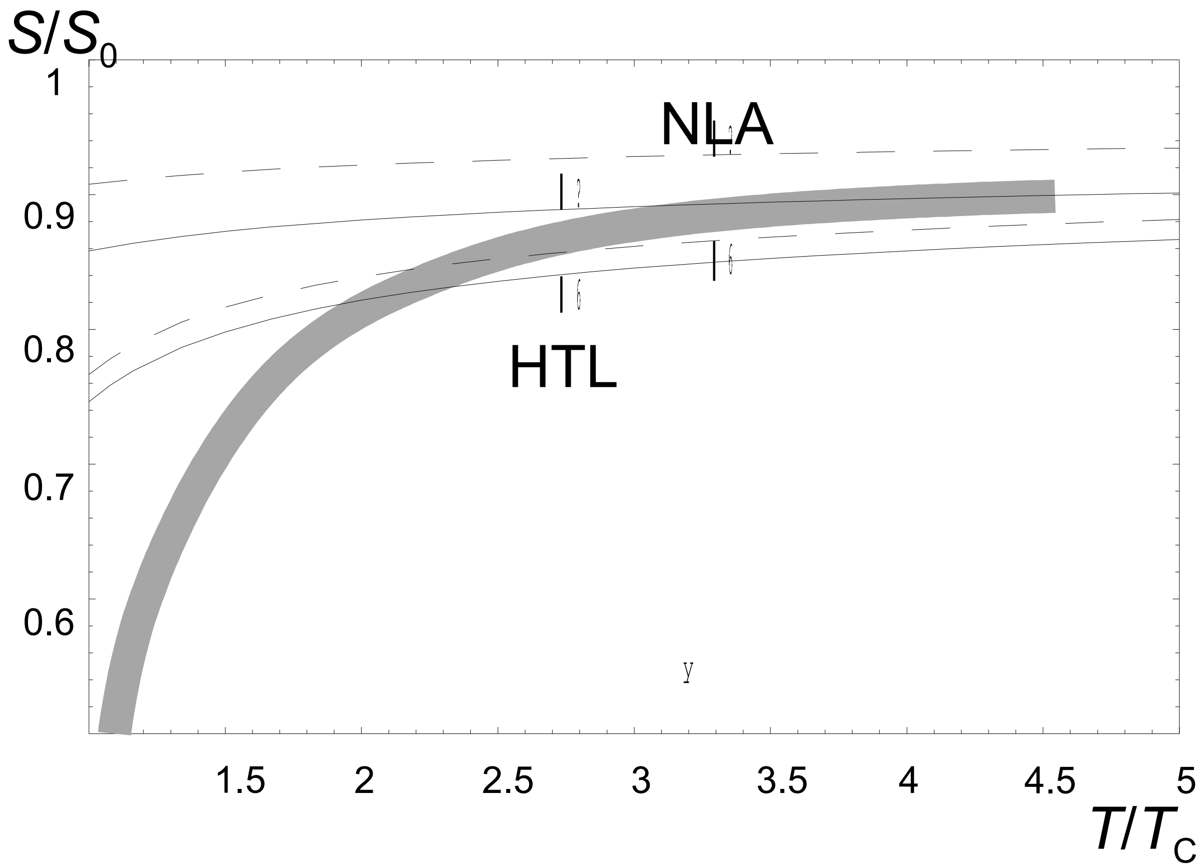
Perturbation theory:

g^2: Shuryak; Chin (1978)
g^3: Kapusta (1979)
g^4 ln g: Toimela (1983)
g^4 : Arnold, Zhai (1994)
g^5 : Zhai, Kastening (1995), Braaten, Nieto (1996)
g^6 ln g: Kajantie, Laine, Rummukainen, Schröder (2002)
g^6 (partly): Di Renzo, Laine, Miccio, Schröder, Torrero (2006)

Lattice data: G. Boyd et al. (1996); M. Okamoto et al. (1999).

QCD pressure

Self–consistent 2PI resummation works for $T \geqslant 2.5\, T_c$



Lattice data: G. Boyd et al. (1996).

Φ-derivable approximation

Blaizot, Iancu, Rebhan, PRD63 (2001)

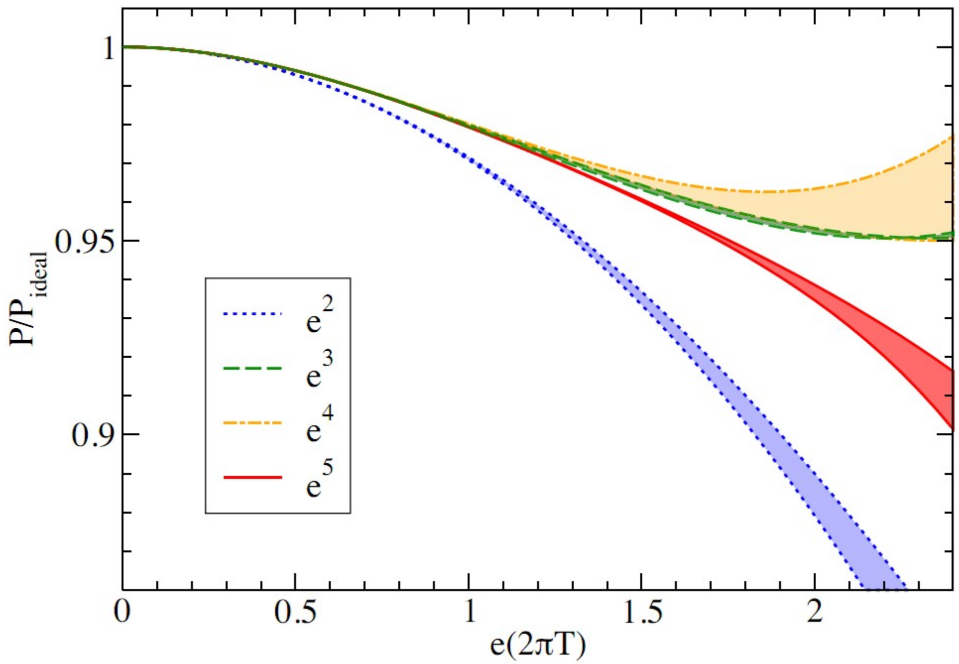
tested at large N_f

Blaizot, AI, Rebhan, Reinosa, PRD72 (2005)

- 2PI: 2 Particle Irreducible diagrams
- HTL: Hard Thermal Loop

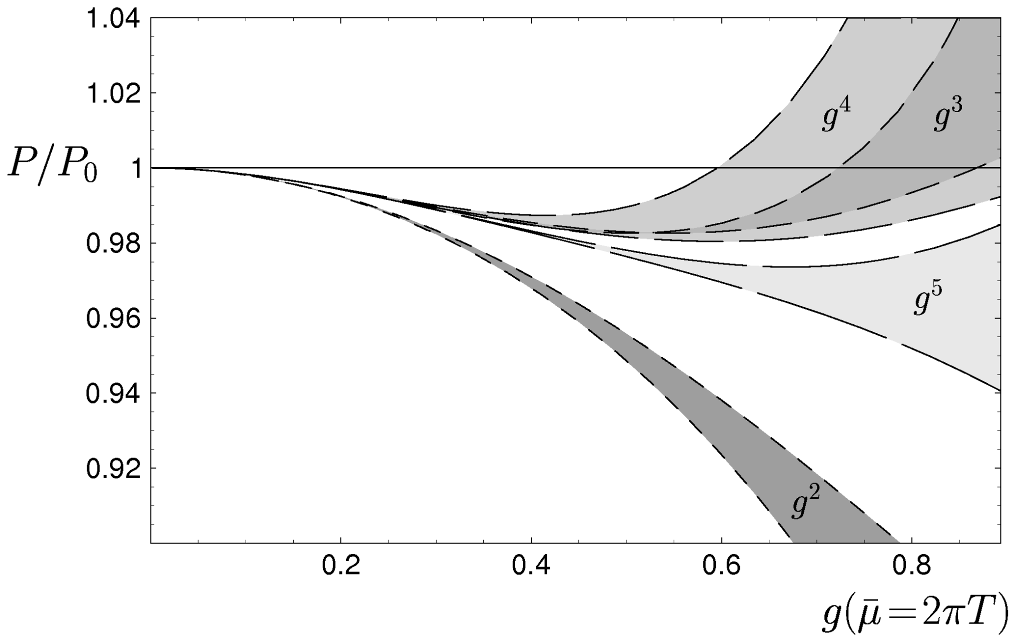
Scalar theory

Poor convergence appears also in other systems



QED

Andersen, Strickland, Su, PRD80, 085015 (2009).



Scalar φ^4 theory

R. Parwani and H. Singh,PRD51, 4518 (1995),

Also: Large N φ^4 theory

Drummond, Horgan, Landshoff, Rebhan, Nucl. Phys. B524, 579 (1998).

Outline

- **Introduction**
  - Thermodynamics at finite temperature
  - Non-perturbative renormalization group
  - BMW approximation (**B**laizot–**M**endez-Galain–**W**schebor)

- **BMW at finite temperature**
  - Role of the regulator
  - Numerical results

- **Summary**

Perturbation theory at finite *T*

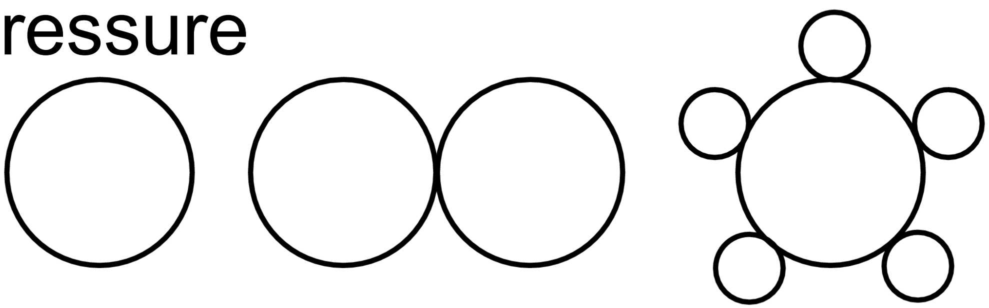
Scalar *O(N)* theory $L=\frac{1}{2}[\partial\varphi(x)]^2-\frac{1}{2}m_0^2\varphi^2(x)-g^2[\varphi^2(x)]^2$

action $S=\int_0^{1/T}d\tau\int d^3x\,L(x) \xrightarrow{F.T.} T\sum_{\omega_n}\int\frac{d^3q}{(2\pi)^3}\cdots$

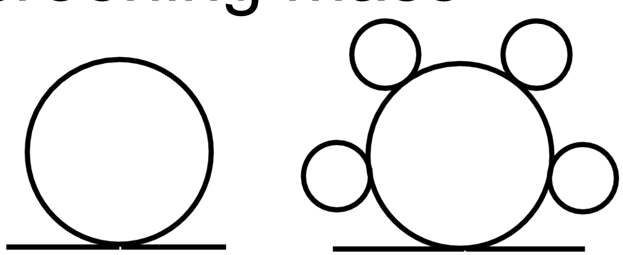
with $\varphi(1/T)=\varphi(0)$

Matsubara frequency $\omega_n=2\pi Tn$

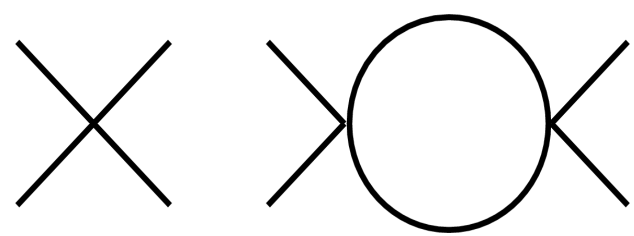
pressure



screening mass

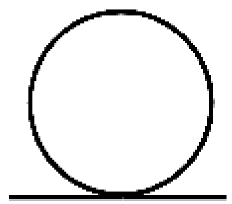


scattering amplitude



Screening mass

Leading order from perturbation theory:



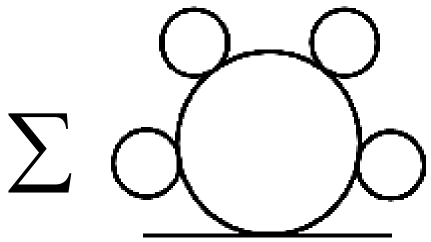
$$m_D^2 = 12 g^2 T \sum_{\omega_n} \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\omega_n^2 + q^2} + \delta m^2$$

$$= \frac{6 g^2}{\pi^2} \int_0^\infty dq\, q\, n(q)$$

$$= g^2 T^2$$

$$n(q) = \frac{1}{e^{q/T} - 1}$$

Resum IR divergent diagrams to obtain g^3 contribution:



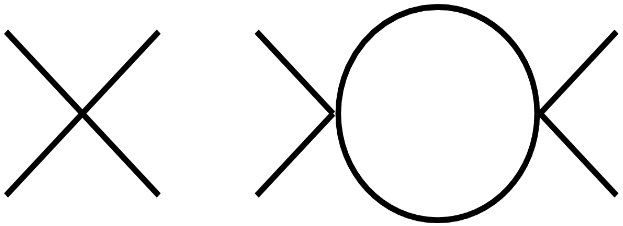
$$m_D^2 = 12 g^2 T \sum_{\omega_n} \int \frac{d^3 q}{(2\pi)^3} \left(\frac{1}{\omega_n^2 + q^2 + m^2} - \frac{1}{\omega_n^2 + q^2} \right)$$

$$= \frac{6 g^2}{\pi^2} T \int_0^\infty dq \frac{-m^2}{q^2 + m^2} \qquad (\text{for } \omega_n = 0)$$

$$= -\frac{3}{\pi} g^2 T m = -\frac{3}{\pi} g^3 T^2$$

Running of the coupling

scattering amplitude



$$\beta(g^2)=\mu\frac{\partial g^2}{\partial\mu}=\frac{9}{2\pi^2}g^4+O(g^6)$$

$$T=0 \qquad g^2 \qquad \frac{9}{2\pi^2}g_\mu^4\ln\frac{\mu}{\mu'}$$

$$\Leftrightarrow\quad \frac{1}{g_\mu^2}=\frac{1}{g_\mu'^2}+\frac{9}{2\pi^2}\log\frac{\mu'}{\mu}$$

$$T>0 \qquad g^2 \qquad -\frac{9}{2\pi}g^3$$

implies Landau pole

$$\Lambda_{\mathrm{Landau}}=\mu\exp\frac{2\pi^2}{9g_\mu^2}$$

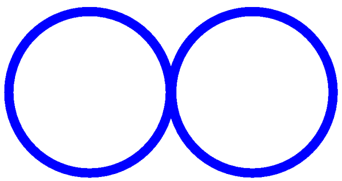
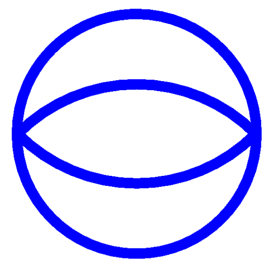
- coupling diverges at Landau pole
- coupling is bound and vanishes at small scales

Self-consistent 2PI resummation

Baym (1962); Luttinger, Ward (1960); Cornwall, Jackiw, Tomboulis (1974)

Pressure expressed through dressed propagators:

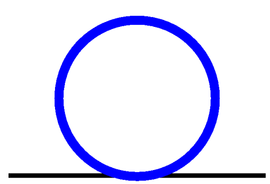
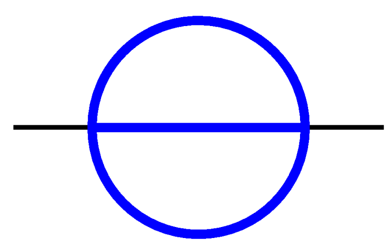
$$-P=\int \frac{d^4k}{(2\pi)^4}\, n(\omega)\left(\Im \log G^{-1}-\Im\, \Pi G\right)+T\,\Phi[G]/V$$

$\Phi[G]$ =  +  + ...

with $G^{-1}=G_0^{-1}+\Pi$

2PI: 2 Particle Irreducible diagrams

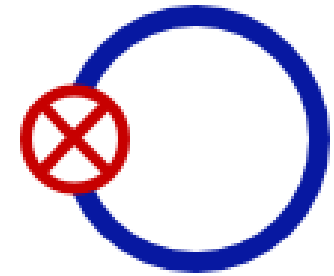
Gap equation:

$\Pi=2\frac{\partial\Phi[G]}{\partial G}=$  +  + ...

Flow equation

effective action

$$\partial_t \Gamma_\kappa[\phi] = \frac{1}{2} T \sum_n \int \frac{d^3\mathbf{q}}{(2\pi)^3} \partial_t R_\kappa(Q) \boxed{G_\kappa(Q;\rho)} =$$



Tetradis, Wetterich 1993

propagator $\boxed{G_\kappa^{-1}(Q;\rho)} \equiv \Gamma_\kappa^{(2)}(\mathbf{Q}, -\mathbf{Q}; \rho) + R_\kappa(Q)$

regulator

four-momentum vectors:

$$\mathbf{Q} = (\omega_n, \mathbf{q})$$

$$Q = |\mathbf{Q}|$$

Matsubara frequency $\omega_n = 2\pi n T$

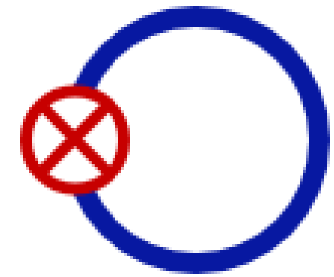
momentum derivative: $\partial_t \equiv \kappa\, \partial_\kappa$

scalar field: $\rho \equiv \frac{1}{2}\phi^2$

Flow equation

effective potential

$$\partial_t V_\kappa(\rho) = \frac{1}{2} T \sum_n \int \frac{d^3\mathbf{q}}{(2\pi)^3} \partial_t R_\kappa(Q) G_\kappa(Q;\rho) \quad =$$

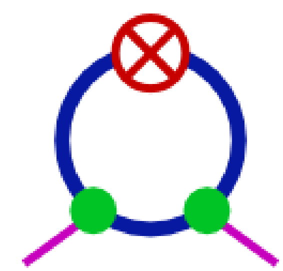


propagator $G_\kappa^{-1}(Q;\rho) \equiv \Gamma_\kappa^{(2)}(\mathbf{Q},-\mathbf{Q};\rho) + R_\kappa(Q)$

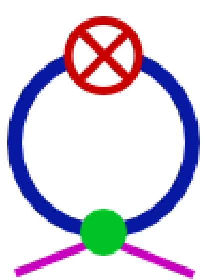
Infinite tower of coupled differential equations: approximation necessary

$$\partial_t \Gamma_\kappa^{(2)}(\mathbf{P},-\mathbf{P};\rho) = T \sum_n \int \frac{d^3\mathbf{q}}{(2\pi)^3} \partial_t R_\kappa(Q) G_\kappa^2(Q;\rho)$$

$$\times \left\{ \Gamma_\kappa^{(3)}(\mathbf{P},\mathbf{Q},-\mathbf{P}-\mathbf{Q};\phi) G_\kappa(\mathbf{Q}+\mathbf{P};\rho) \Gamma_\kappa^{(3)}(-\mathbf{P},\mathbf{P}+\mathbf{Q},-\mathbf{Q};\phi) \right.$$

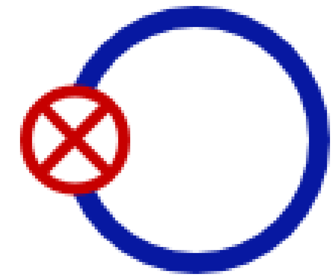


$$\left. -\frac{1}{2} \Gamma_\kappa^{(4)}(\mathbf{P},-\mathbf{P},\mathbf{Q},-\mathbf{Q};\phi) \right\}$$



Local potential approximation

$$\partial_t V_\kappa(\rho) = \frac{1}{2} T \sum_n \int \frac{d^3\mathbf{q}}{(2\pi)^3}\, \partial_t R_\kappa(Q)\, G_\kappa(Q;\rho) \quad =$$



propagator $G_\kappa^{-1}(Q;\rho) \equiv \Gamma_\kappa^{(2)}(\mathbf{Q},-\mathbf{Q};\rho) + R_\kappa(Q)$

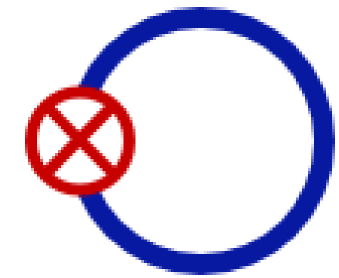
$$\Gamma_\kappa^{(2)}(\mathbf{Q},-\mathbf{Q};\rho) = Q^2 + m_\kappa^2(\rho) \quad \text{with } m_\kappa^2(\rho) \equiv \frac{\partial^2 V_\kappa}{\partial \phi^2}$$

Applying BMW method at the bottom of the hierarchy,
one recovers the **local potential approximation**.

BMW method

effective potential

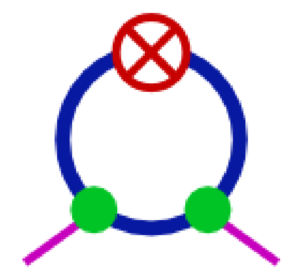
$$\partial_t V_\kappa(\rho) = \frac{1}{2} T \sum_n \int \frac{d^3\mathbf{q}}{(2\pi)^3} \partial_t R_\kappa(Q) \left[\Gamma_\kappa^{(2)}(\mathbf{Q}, -\mathbf{Q}; \rho) + R_\kappa(Q) \right]^{-1}$$

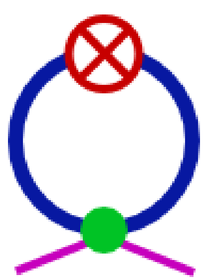


$$\partial_t \Gamma_\kappa^{(2)}(\mathbf{P}, -\mathbf{P}; \rho) = T \sum_n \int \frac{d^3\mathbf{q}}{(2\pi)^3} \partial_t R_\kappa(Q) G_\kappa^2(Q; \rho)$$

$$\times \Big\{ \Gamma_\kappa^{(3)}(\mathbf{P}, \mathbf{0}, -\mathbf{P}-\mathbf{0}; \phi) G_\kappa(\mathbf{Q}+\mathbf{P}; \rho) \Gamma_\kappa^{(3)}(-\mathbf{P}, \mathbf{P}+\mathbf{0}, -\mathbf{0}; \phi)$$

$$-\frac{1}{2} \Gamma_\kappa^{(4)}(\mathbf{P}, -\mathbf{P}, \mathbf{0}, -\mathbf{0}; \phi) \Big\}$$





BMW method:
"set loop momentum Q=0 in the *n*-point functions on the r.h.s of the flow equation"
Blaizot–**M**endez-Galain–**W**schebor, 2006

close equations with:

$$\Gamma_\kappa^{(3)}(\mathbf{P}, -\mathbf{P}, \mathbf{0}; \phi) = \frac{\partial \Gamma_\kappa^{(2)}(P, \rho)}{\partial \phi}, \quad \Gamma_\kappa^{(4)}(\mathbf{P}, -\mathbf{P}, \mathbf{0}, \mathbf{0}; \phi) = \frac{\partial^2 \Gamma_\kappa^{(2)}(P, \rho)}{\partial \phi^2}$$

Technicalities

The second flow equation contains also information about the potential

$$\Gamma_\kappa^{(2)}(\mathbf{P}=0,-\mathbf{P}=0;\phi)=\frac{\partial^2 V_\kappa(\phi)}{\partial\phi^2}$$

In order to respect this relation, it is better to use the set

$$V_\kappa(\rho),$$
$$\Delta_\kappa(P;\rho)=\Gamma_\kappa^{(2)}(P;\rho)-\Gamma_\kappa^{(2)}(P=0;\rho)-P^2$$

It is convenient to define the *Z*-factor in the regulator.

$$Z_\kappa=\left.\frac{\partial\,\Gamma_\kappa^{(2)}(p_0=0,\mathbf{p},\rho)}{\partial\,\mathbf{p}^2}\right|_{P^2=0,\,\rho=\rho_0}=1+\left.\frac{\partial\,\Delta_\kappa(p_0=0,\mathbf{p},\rho)}{\partial\,\mathbf{p}^2}\right|_{P^2=0,\,\rho=\rho_0}$$

3D vs. 4D regulator

- **LPA**: 3D regulator possible.

$$R_\kappa(\omega_n, \mathbf{q}) = (\kappa^2 - \mathbf{q}^2)\,\theta(\kappa^2 - \mathbf{q}^2)$$

Litim regulator

- analytical treatment of Matsubara sums possible

$$T\sum_n \frac{1}{-(i\omega_n)^2 + \omega_\kappa^2} = \frac{1 + 2\mathrm{n}(\omega_\kappa)}{2\,\omega_\kappa}$$

- But: does not respect Euclidean invariance; frequency is not regulated

- **BMW**: Euclidean method.
- 4D regulator more appropriate

$$R_\kappa(Q) = Z_\kappa \kappa^2 r(Q/\kappa)$$

$$r(q) = \frac{\alpha q^2}{e^{q^2} - 1} \quad \text{or} \quad r(q) = \alpha\, e^{-\beta q^2 - \gamma q^4}$$

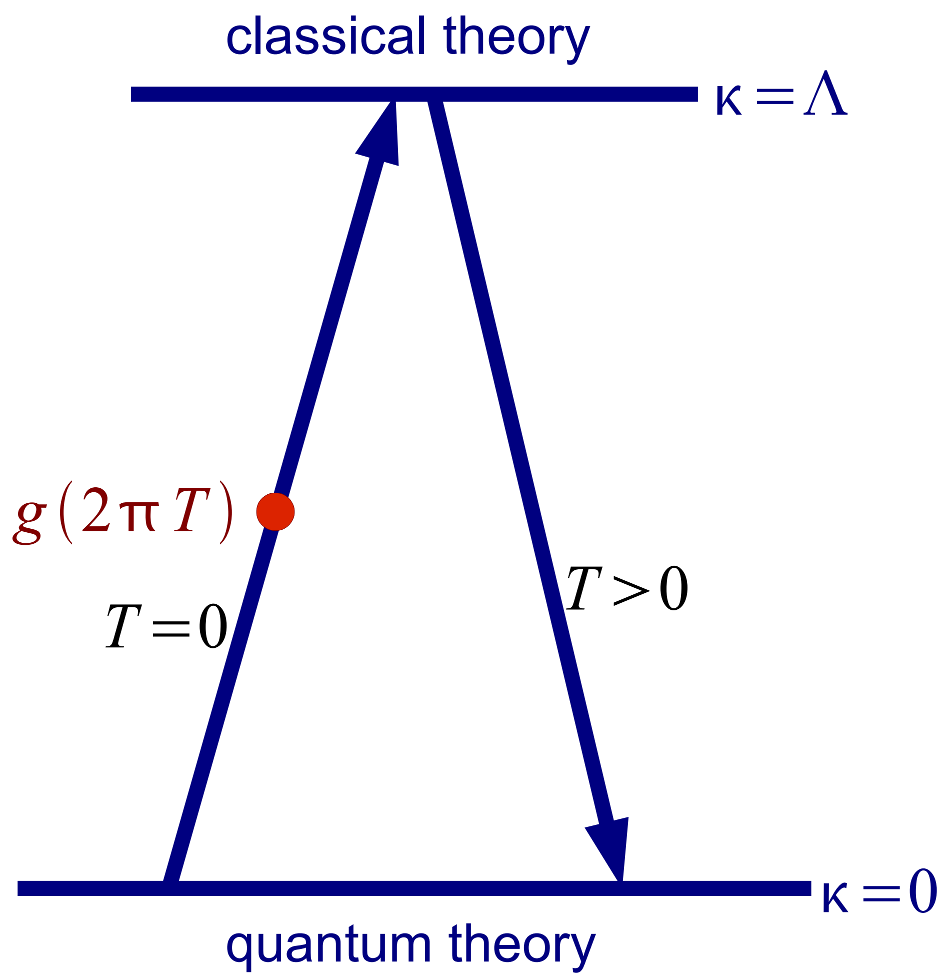
$$\text{with}\ \beta = 1/2,\ \gamma = 1/24$$

Exponential regulator

- Need to sum over Matsubara frequencies.

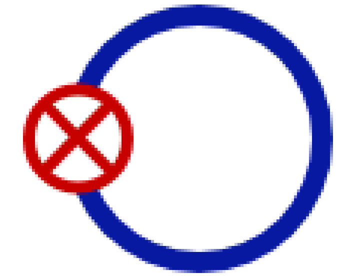
Applying the flow equation

- Effective action links classical and quantum theory
- Coupling is obtained from zero-temperature flow at momentum
  $\kappa = 2\pi T$



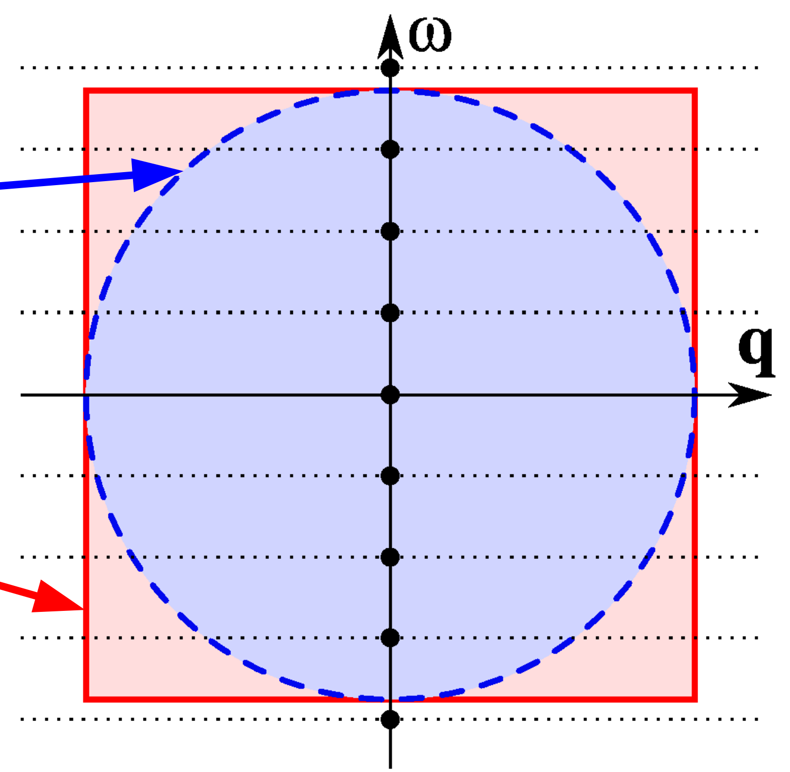
Integration domains

$$\partial_t V_\kappa^T(\rho) - \partial_t V_\kappa^{T=0}(\rho) =$$



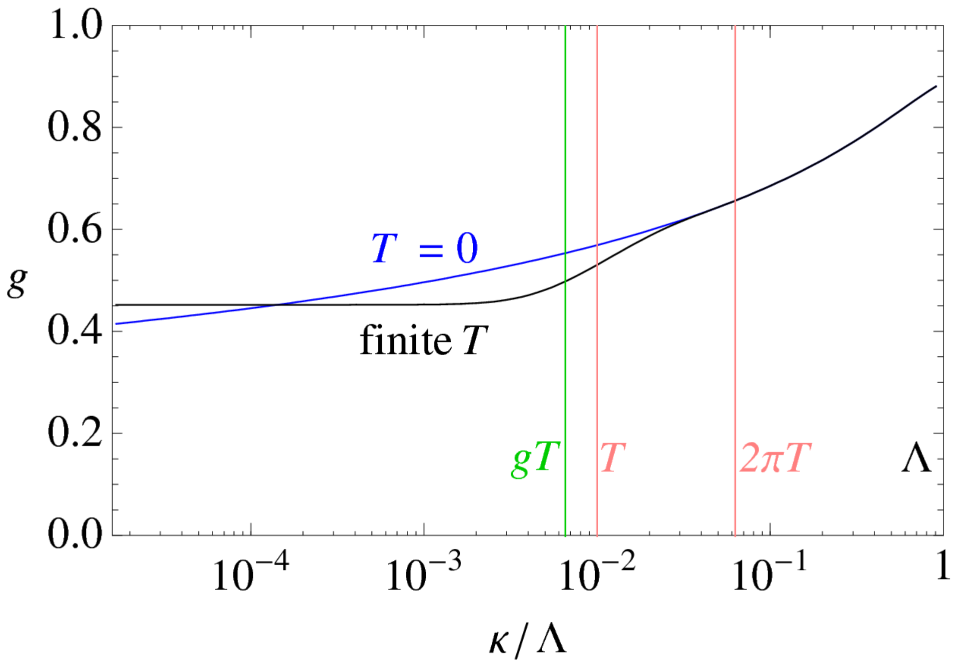
$$= \frac{1}{2} T \sum_{n=-n_{max}}^{n_{max}} \int^{\Lambda} \frac{d^3\mathbf{q}}{(2\pi)^3} \partial_t R_\kappa(Q) G_\kappa^T(Q;\rho) - \frac{1}{2} \int^{\Lambda} \frac{d^4 q}{(2\pi)^4} \partial_t R_\kappa(Q) G_\kappa^{T=0}(Q;\rho)$$

- Euclidean invariance imposes four-dimensional sphere
- Summation over Matsubara frequencies implies a four-dimensional cylinder
- Mismatch can give rise to spurious pressure contributions

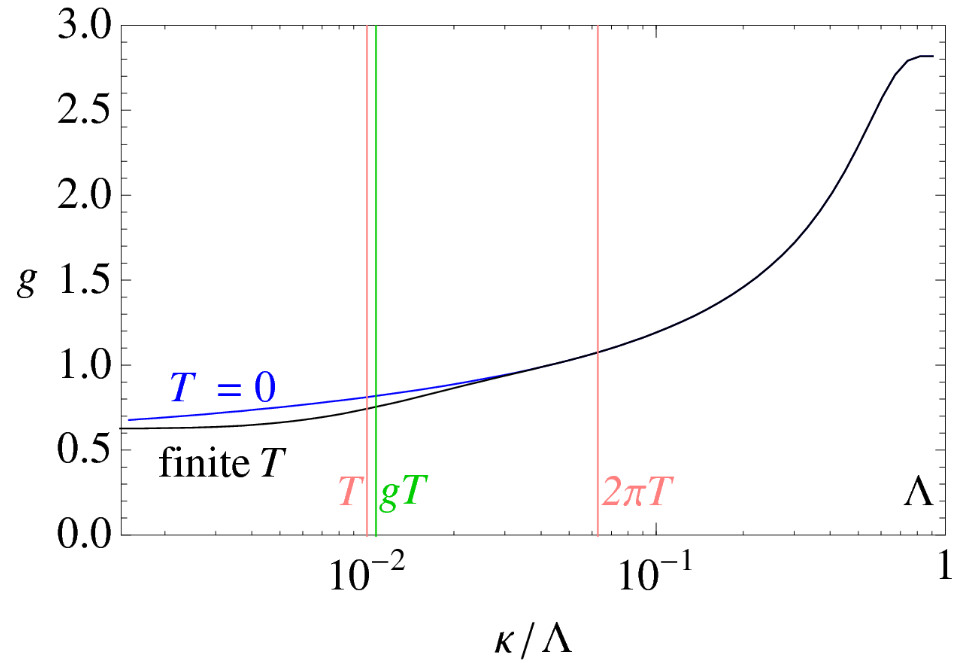


Flow of couping

weak coupling



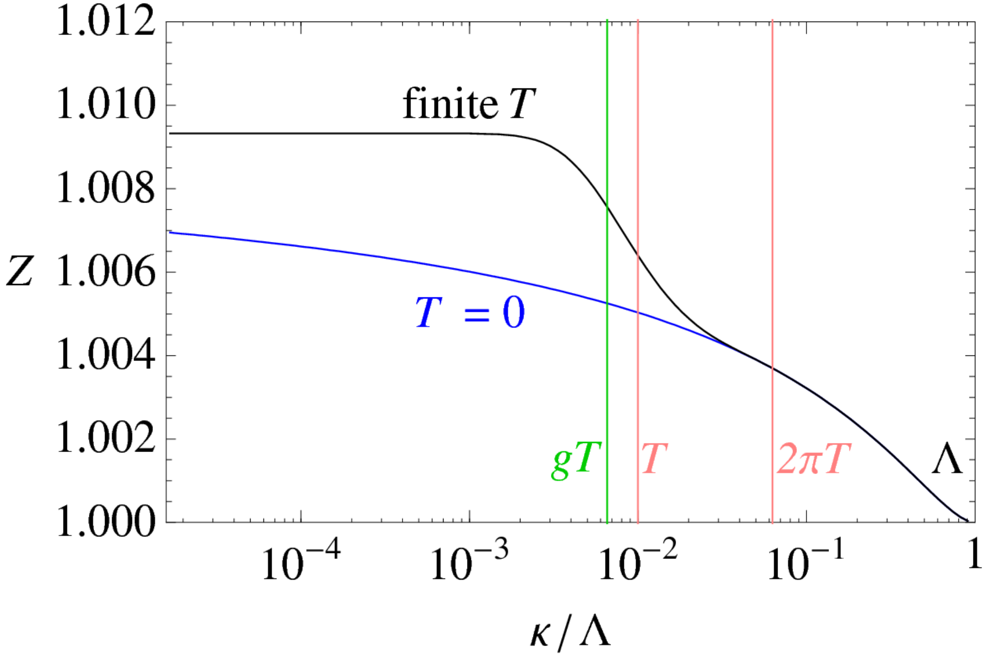
strong coupling

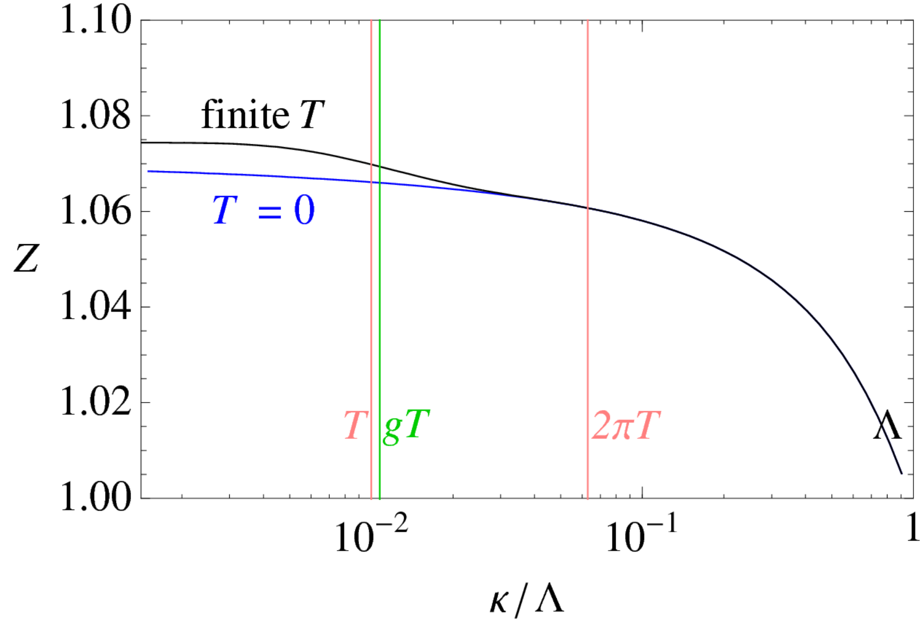


$$g_\kappa^2 = \frac{1}{8} V''(\rho)\Big|_{\rho=0}$$

- Region of flow shrinks but no qualitative difference

Flow of Z-factor





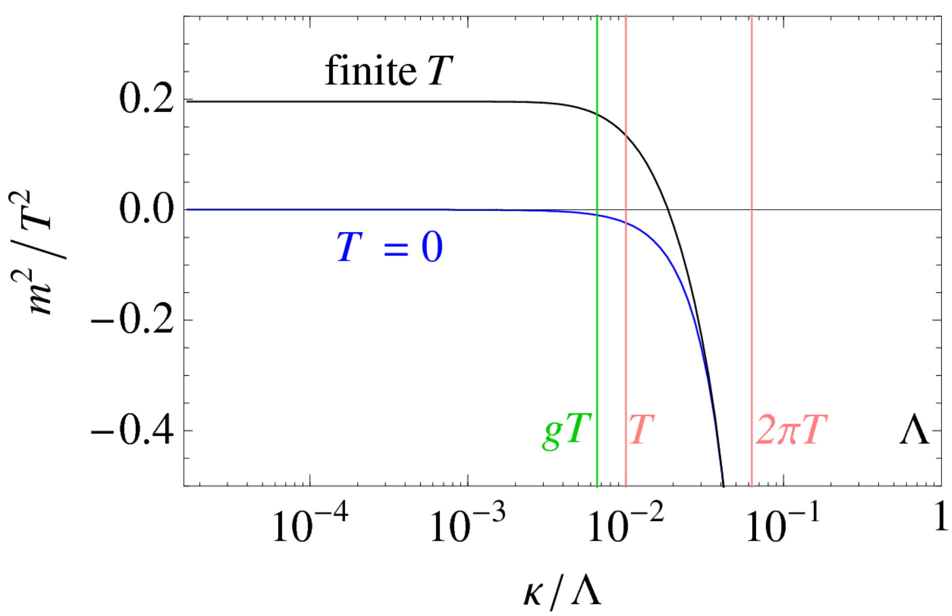
$$Z_\kappa = \left. \frac{\partial \Gamma_\kappa^{(2)}(p_0 = 0, \mathbf{p}, \rho)}{\partial \mathbf{p}^2} \right|_{P^2 = 0,\, \rho = \rho_0}$$

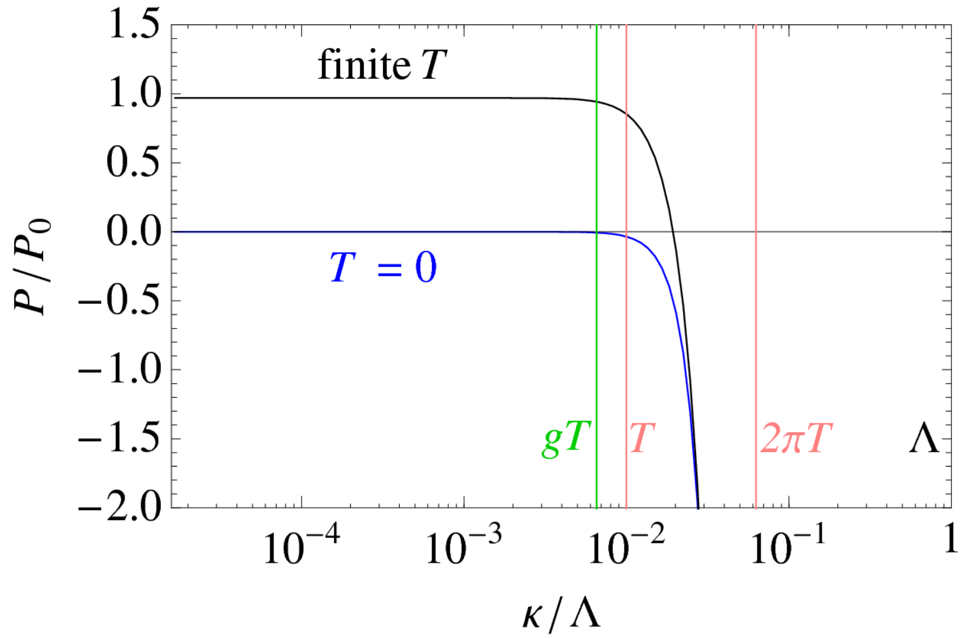
- small anomalous dimension at *d*=4

Flow of mass and pressure

$$m_\kappa^2 = V'(\rho)\big|_{\rho=0}$$

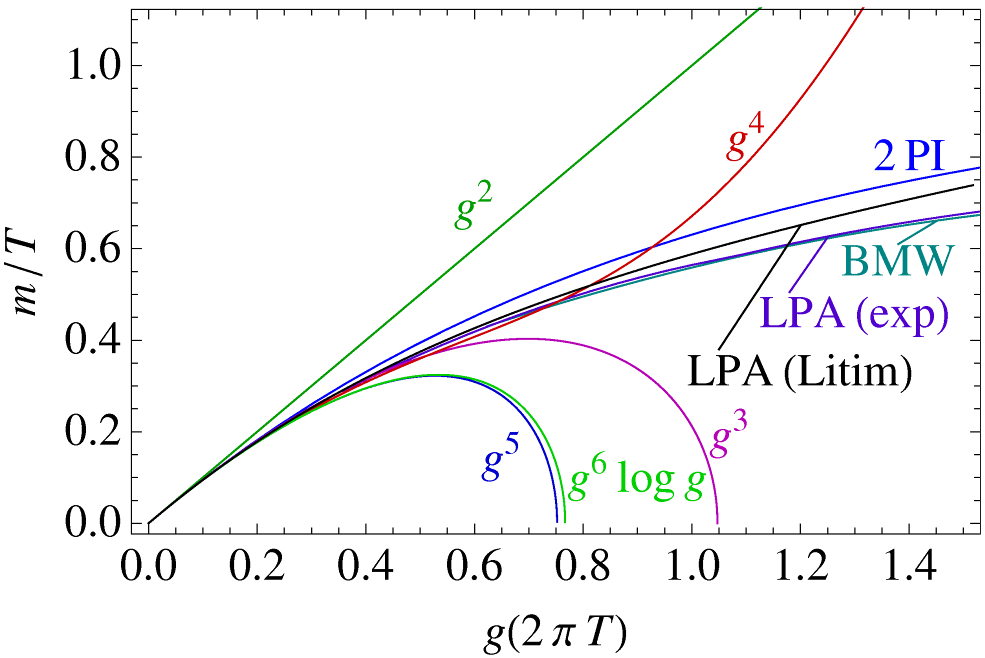
$$P_\kappa = -V(0)$$

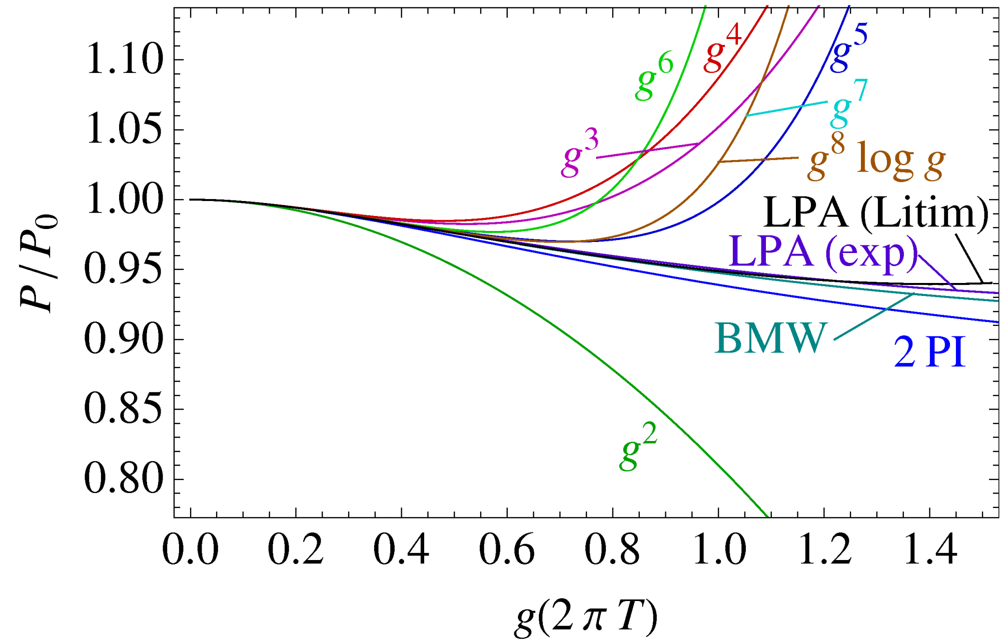




- numerically challenging

Mass and pressure at finite T



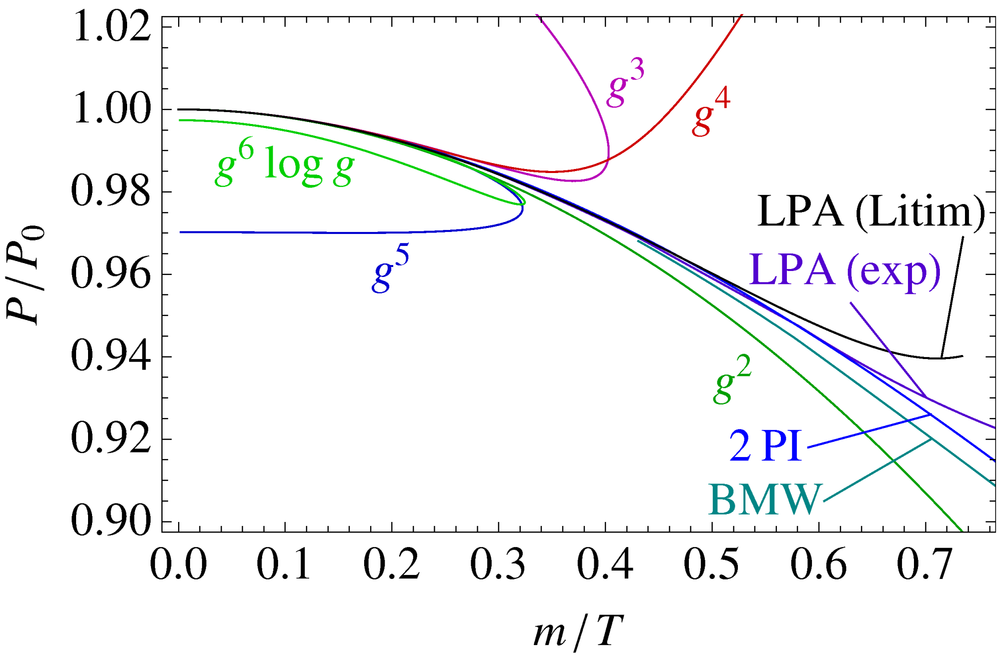


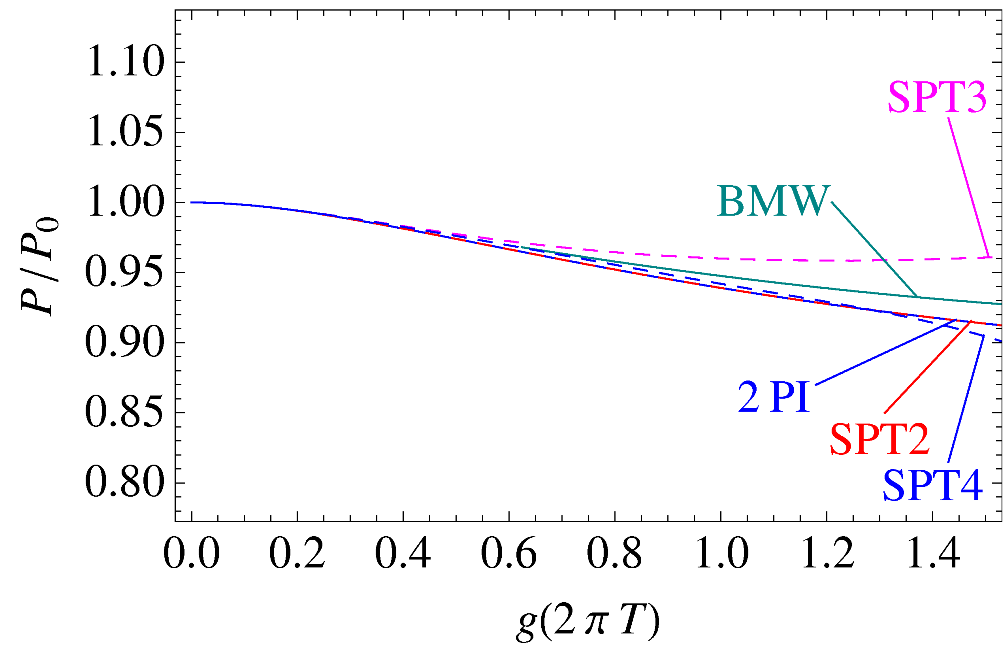
- 2PI and the FRG methods LPA and BMW agree well
- much better behaved than perturbation theory

- 2PI: 2 particle irreducible
- LPA: local potential approximation
- BMW: Blaizot, Mendez-Galain, Wschebor

LPA (Litim): Blaizot, AI, Mendez-Galain, Wschebor, NPA (2007)
BMW: Blaizot, AI,Wschebor, NPA (2011)

Pressure as function of mass





- Comparison of renormalization-scale independent quantities

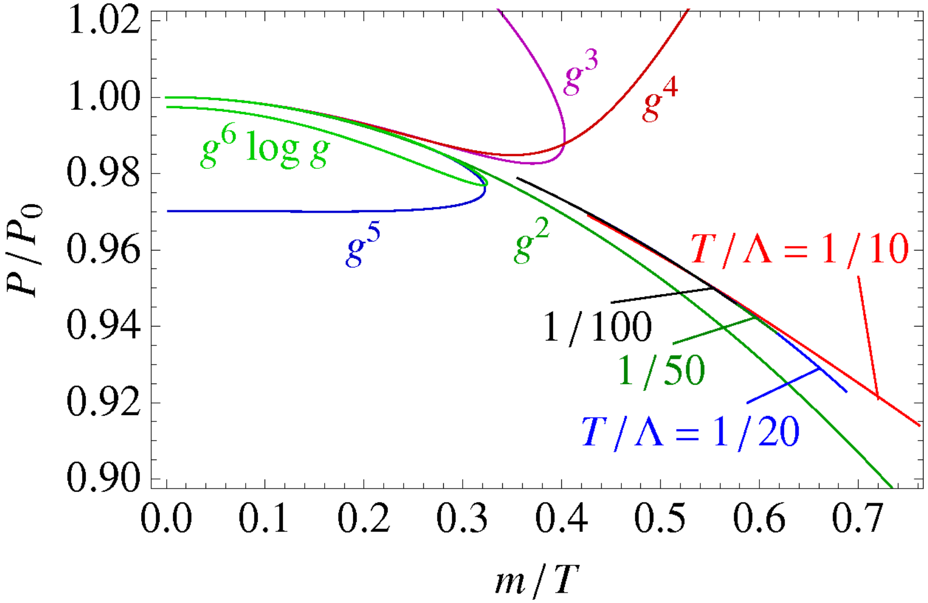
- 2PI: 2 particle irreducible
- LPA: local potential approximation
- BMW: Blaizot, Mendez-Galain, Wschebor
- SPT: Screened perturbation theory

SPT: Andersen and Kyllingstad, Phys. Rev. **D78**, 076008 (2008)

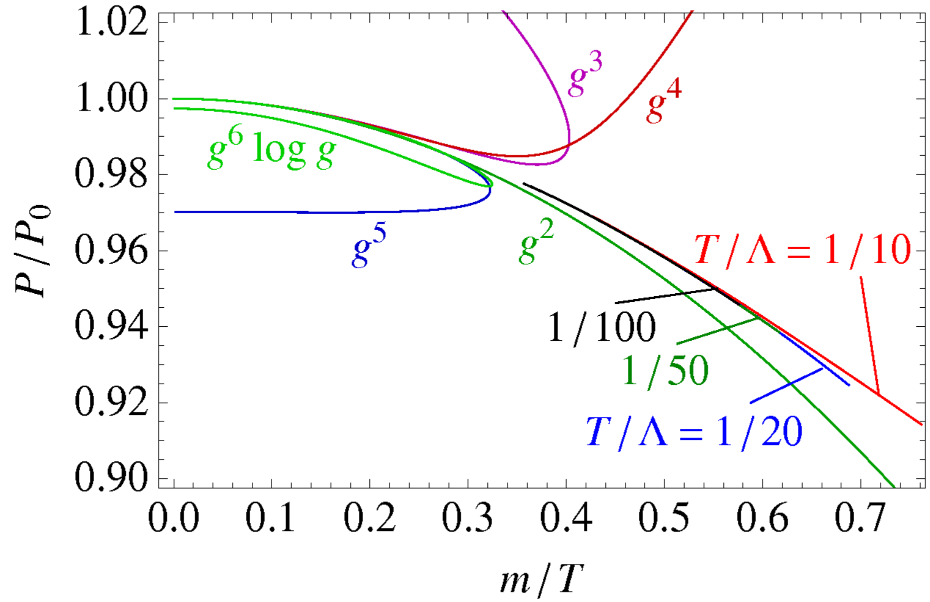
Dependence on temperature

LPA

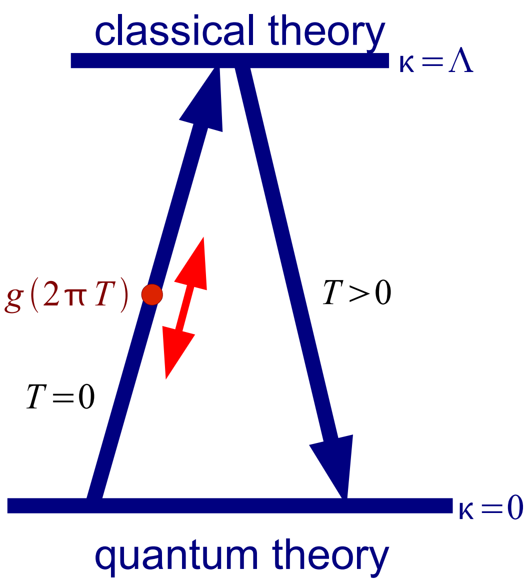
$$r(q) = \alpha q^2 / (e^{q^2} - 1)$$

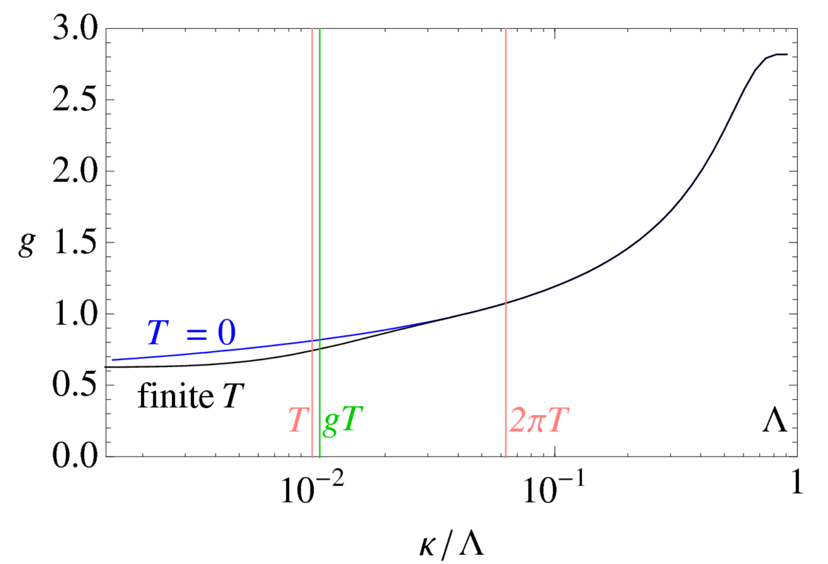


$$r(q) = \alpha e^{-\beta q^2 - \gamma q^4}$$



Flow of coupling

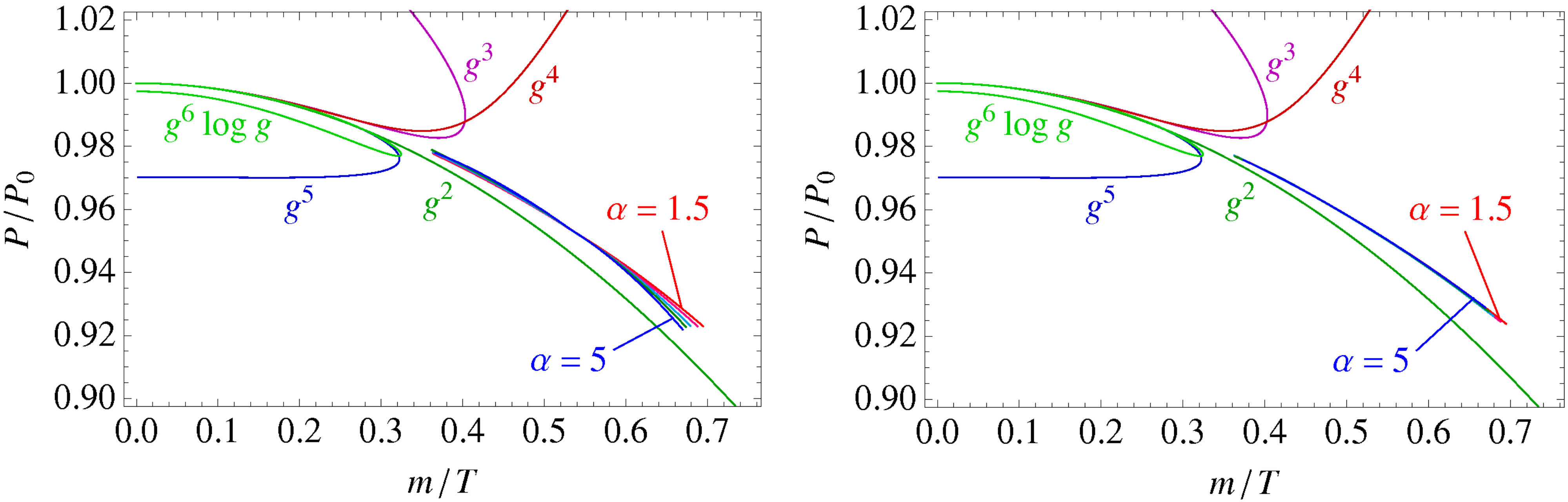




Regulator dependence

LPA

$r(q) = \alpha q^2 / (e^{q^2} - 1)$ $\qquad$ $r(q) = \alpha e^{-\beta q^2 - \gamma q^4}$

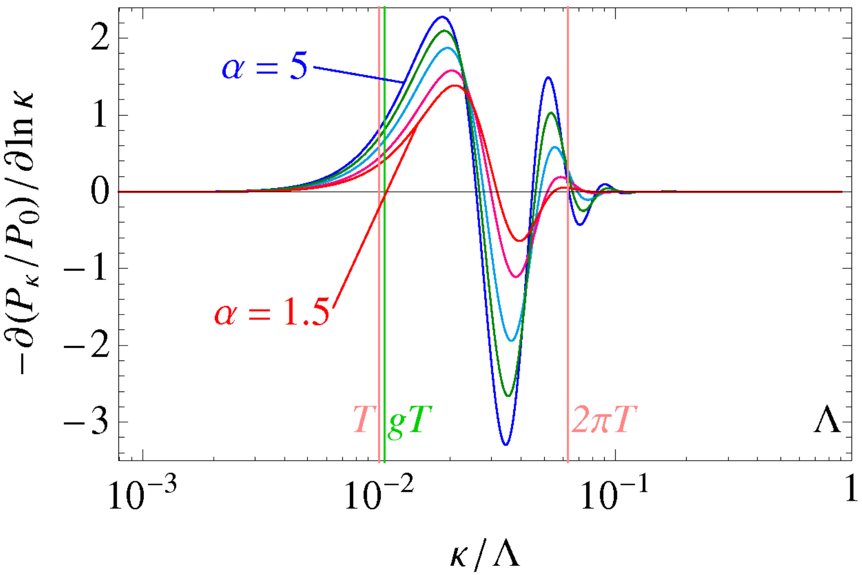


Even though regulators seem similar, the pressure
is sensitive to the choice of the regulator

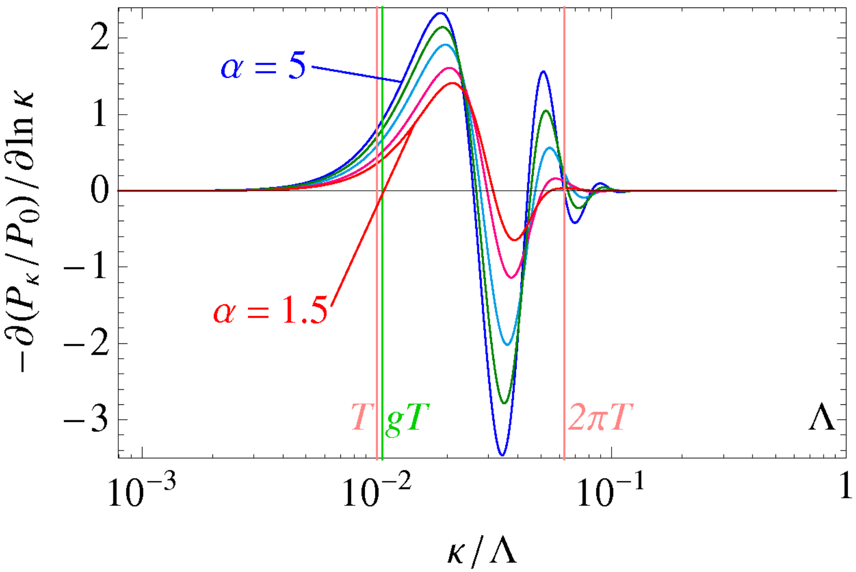
Flow of the potential

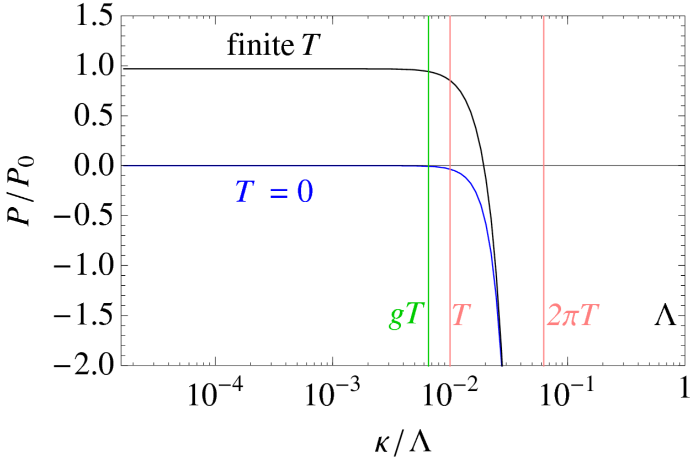
LPA

$$r(q) = \alpha q^2 / (e^{q^2} - 1)$$



$$r(q) = \alpha e^{-\beta q^2 - \gamma q^4}$$





Difference of the flow of the potential at finite and zero temperature

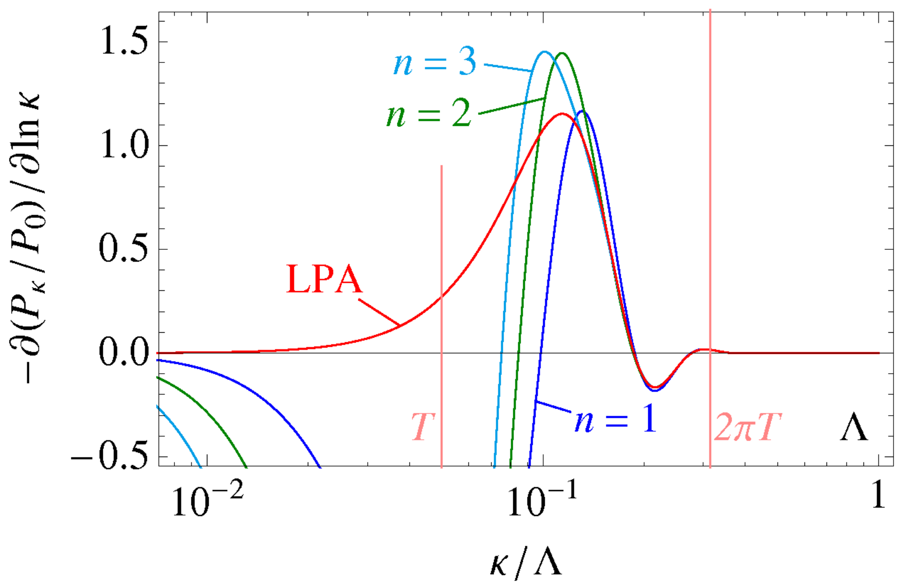
$$R_\kappa(Q) = Z_\kappa \kappa^2 r(Q/\kappa)$$

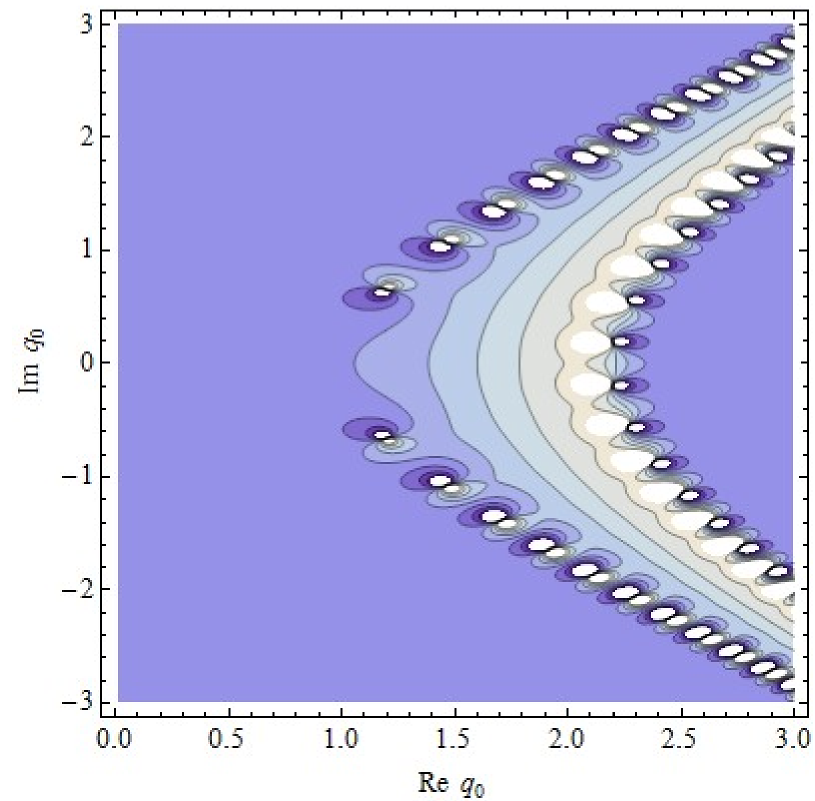
Oscillations due to regulator

Oscillations can be understood in complex q_0 plane

$$\partial_t V_\kappa(\rho) = \frac{1}{2} T \sum_n \int \frac{d^3\mathbf{q}}{(2\pi)^3} \partial_t R_\kappa(Q)\left(\Gamma^{(2)}_\kappa(\mathbf{Q},-\mathbf{Q};\rho)+R_\kappa(Q)\right)^{-1}$$

Pole structure





$$r(q) = \frac{\alpha q^2}{e^{q^2}-1}$$

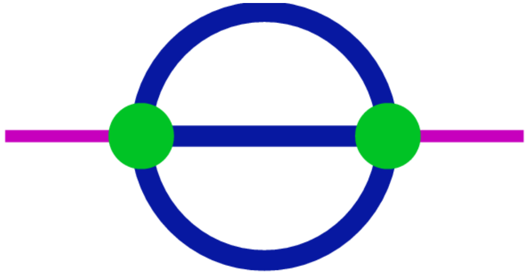
Difficulty of finding suitable regulator for realtime quantities?
Analytic continuation from imaginary to real axis?

Outlook

- Calculation of real-time quantities at finite temperature like particle width are not readily accessible within LPA.

- Need approximation scheme beyond LPA.

- But: analytic continuation may turn out to be tricky.



Particle width: imaginary part of 2-point function

Perturbative calculation of damping rate

Parvani (1992), Wang and Heinz (1996)

Summary

- **Non-perturbative renormalization group**
  - Approximation necessary
- **BMW at finite temperature**
  - 4D regulator more appropriate
  - Results for pressure and mass similar to local potential approximation
- **Outlook**
  - Calculate real time quantities