Universality of the ERG formalism, revisited

Hidenori SONODA

Physics Department, Kobe University, Japan

1 September 2011 at Yukawa Institute
for the EMMI program on the RG approach
From Ultra Cold Atoms to the Hot QGP

Abstract

I collect what has been known for the convenience of those who wish to understand the formal aspects of ERG. With care I introduce an anomalous dimension of the field to the ERG differential equation, and show that the anomalous dimension at a fixed point of ERG is independent of the choice of a cutoff function.

Plan of the talk

1. Introduction

2. Properties of the ERG differential equation

3. Change of variables

4. Introduction of γ

5. ERG differential equation for a fixed point

6. Dependence on the choice of a cutoff function

7. Concluding remarks

Introduction

1. Universality of the ERG formalism has been shown by Latorre and Morris.[JHEP 2000]

2. We would like to improve their discussion on the universality of the anomalous dimension.

3. We do two things:
   (a) a self-consistent introduction of the anomalous dimension into the ERG diff eq (based upon Appendix C of Igarashi, Itoh, Sonoda [PTPS 2009]
   (b) show that the anomalous dimension at a fixed point is independent of the choice of a cutoff function for ERG [HS 2011?]

Properties of the ERG differential equation

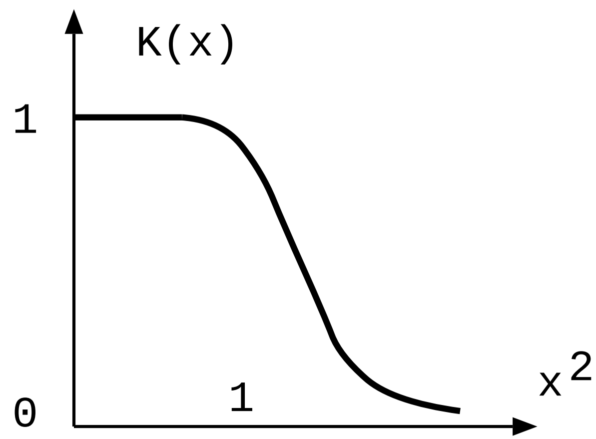
1. Consider a real scalar theory with a Wilson action [Wilson 1973; Polchinski 1983]

$$S_\Lambda = -\int_p \frac{p^2}{K(p/\Lambda)} \frac{1}{2}\phi(p)\phi(-p) + S_{I,\Lambda}[\phi]$$

2. The cutoff function $K(x)$ has the properties:

$$\begin{aligned} K(x) &= 1 \quad (x^2 \simeq 0) \\ &\longrightarrow 0 \quad (x^2 \gg 1) \end{aligned}$$

The modes with $p^2 > \Lambda^2$ are suppressed.



3. The Λ dependence of $S_{I,\Lambda}$ is [Polchinski 1983]

$$-\Lambda \frac{\partial}{\partial \Lambda} S_{I,\Lambda} = \int_p \frac{\Delta(p/\Lambda)}{p^2} \frac{1}{2} \left\{ \frac{\delta S_{I,\Lambda}}{\delta \phi(p)} \frac{\delta S_{I,\Lambda}}{\delta \phi(-p)} + \frac{\delta^2 S_{I,\Lambda}}{\delta \phi(p) \delta \phi(-p)} \right\}$$

Alternatively, the Λ dependence of S_Λ is

$$\begin{aligned} -\Lambda \frac{\partial}{\partial \Lambda} S_\Lambda &= \int_p \left[\frac{\Delta(p/\Lambda)}{K(p/\Lambda)} \phi(p) \frac{\delta S_\Lambda}{\delta \phi(p)} \right. \\ &\left. + \frac{\Delta(p/\Lambda)}{p^2} \frac{1}{2} \left\{ \frac{\delta S_\Lambda}{\delta \phi(p)} \frac{\delta S_\Lambda}{\delta \phi(-p)} + \frac{\delta^2 S_\Lambda}{\delta \phi(p) \delta \phi(-p)} \right\} \right] \end{aligned}$$

where $\Delta(p/\Lambda) \equiv \Lambda \partial_\Lambda \ln K(p/\Lambda)$.

4. Connected correlation functions

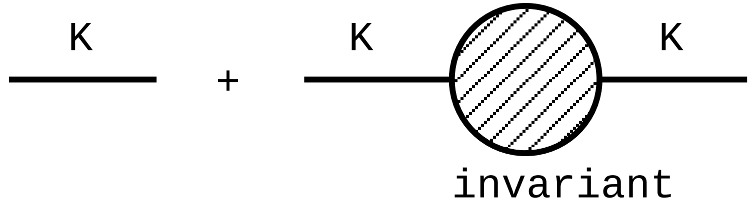
$$\langle\, \phi(p_1) \cdots \phi(p_n)\, \rangle_{S_\Lambda} \delta(p_1 + \cdots + p_n) \equiv \int [d\phi]\, \phi(p_1) \cdots \phi(p_n)\, \mathrm{e}^{S_\Lambda}$$

5. Λ independent correlation functions [HS 2007] (called **ERG invariants** [Rosten 2008]):

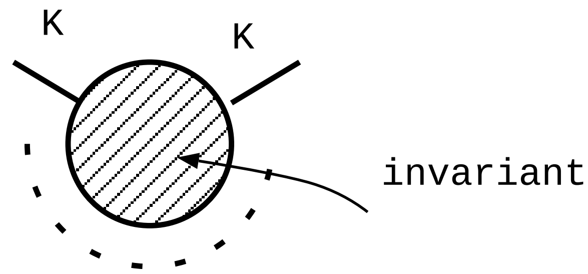
$$\langle\, \phi(p)\phi(-p)\, \rangle^{\infty}_{S_\Lambda} \equiv \frac{1}{p^2} + \left(\langle\, \phi(p)\phi(-p)\, \rangle_{S_\Lambda} - \frac{K(p/\Lambda)}{p^2} \right) \Big/ K(p/\Lambda)^2$$

$$\langle\, \phi(p_1) \cdots \phi(p_n)\, \rangle^{\infty}_{S_\Lambda} \equiv \langle\, \phi(p_1) \cdots \phi(p_n)\, \rangle_{S_\Lambda} \Big/ \prod_{i=1}^{n} K(p_i/\Lambda)$$

$$\langle\, \phi(p)\phi(-p)\, \rangle_{S_\Lambda} =$$



$$\langle\, \phi(p_1)\cdots\phi(p_n)\, \rangle_{S_\Lambda} =$$



Change of variables

1. Linear change of variables:

$$\phi(p) \longrightarrow \phi(p) + \epsilon F(p)\phi(p)$$

The action changes by

$$\delta S_\Lambda = \epsilon \int_p F(p)\phi(p) \frac{\delta S_\Lambda}{\delta \phi(p)} \quad \text{“}F\text{-term”}$$

The correlation functions change by

$$\begin{cases} \langle\, \phi(p)\phi(-p) \,\rangle_{S_\Lambda + \delta S_\Lambda} &= (1 - 2\epsilon F(p)) \langle\, \phi(p)\phi(-p) \,\rangle_{S_\Lambda} \\ \langle\, \phi(p_1) \cdots \phi(-p) \,\rangle_{S_\Lambda + \delta S_\Lambda} &= (1 - \epsilon \sum_{i=1}^n F(p_i)) \langle\, \phi(p_1) \cdots \phi(p_n) \,\rangle_{S_\Lambda} \end{cases}$$

2. Change of variables by an equation of motion:

$$\phi(p) \longrightarrow \phi(p) + \epsilon G(p) \frac{1}{2} \frac{\delta S_\Lambda}{\delta \phi(p)}$$

The action changes by

$$\delta S_\Lambda = \epsilon \int_p G(p) \frac{1}{2} \left\{ \frac{\delta S_\Lambda}{\delta \phi(p)} \frac{\delta S_\Lambda}{\delta \phi(-p)} + \frac{\delta^2 S_\Lambda}{\delta \phi(p) \delta \phi(-p)} \right\} \quad \textbf{“}G\textbf{ term”}$$

where the second term is the jacobian.

Only the two-point function changes:

$$\left\{ \begin{array}{rcl} \langle\, \phi(p)\phi(-p)\,\rangle_{S_\Lambda + \delta S_\Lambda} & = & \langle\, \phi(p)\phi(-p)\,\rangle_{S_\Lambda} + \epsilon G(p) \\ \langle\, \phi(p_1)\cdots\phi(p_n)\,\rangle_{S_\Lambda + \delta S_\Lambda} & = & \langle\, \phi(p_1)\cdots\phi(p_n)\,\rangle_{S_\Lambda} \end{array} \right.$$

3. The choice

$$F(p) = \frac{\Delta(p/\Lambda)}{K\,(p/\Lambda)}, \quad G(p) = \frac{\Delta(p/\Lambda)}{p^2}$$

gives rise to the ERG differential equation:

$$\begin{aligned} -\Lambda \frac{\partial}{\partial \Lambda} S_\Lambda &= \int_p \underbrace{\frac{\Delta(p/\Lambda)}{K\,(p/\Lambda)}}_{F(p)} \phi(p) \frac{\delta S_\Lambda}{\delta \phi(p)} \\ &\quad + \int_p \underbrace{\frac{\Delta(p/\Lambda)}{p^2}}_{G(p)} \frac{1}{2} \left\{ \frac{\delta S_\Lambda}{\delta \phi(p)} \frac{\delta S_\Lambda}{\delta \phi(-p)} + \frac{\delta^2 S_\Lambda}{\delta \phi(p) \delta \phi(-p)} \right\} \end{aligned}$$

4. The corresponding changes in the correlation functions are

$$-\Lambda\frac{\partial}{\partial\Lambda}\langle\,\phi(p)\phi(-p)\,\rangle_{S_\Lambda} = -2\frac{\Delta(p/\Lambda)}{K(p/\Lambda)}\langle\,\phi(p)\phi(-p)\,\rangle_{S_\Lambda} + \frac{\Delta(p/\Lambda)}{p^2}$$

$$-\Lambda\frac{\partial}{\partial\Lambda}\langle\,\phi(p_1)\cdots\phi(p_n)\,\rangle_{S_\Lambda} = -\sum_{i=1}^{n}\frac{\Delta(p_i/\Lambda)}{K(p_i/\Lambda)}\times\langle\,\phi(p_1)\cdots\phi(p_n)\,\rangle_{S_\Lambda}$$

This explains the Λ independence of

$$\left(\langle\,\phi(p)\phi(-p)\,\rangle_{S_\Lambda} - \frac{K(p/\Lambda)}{p^2}\right)\Big/ K\left(p/\Lambda\right)^2$$

$$\langle\,\phi(p_1)\cdots\phi(p_n)\,\rangle_{S_\Lambda}\Big/\prod_{i=1}^{n}\frac{1}{K\left(p_i/\Lambda\right)}$$

5. **Equation of motion operator**: With the choice

$$F(p) = -1, \quad G(p) = -2\frac{K(p/\Lambda)(1-K(p/\Lambda))}{p^2}$$

we obtain

$$\begin{aligned} \delta S_\Lambda &= -\epsilon \int_p \phi(p) \frac{\delta S_\Lambda}{\delta \phi(p)} \\ & \quad -\epsilon \int_p \frac{2K(p/\Lambda)(1-K(p/\Lambda))}{p^2} \frac{1}{2} \left\{ \frac{\delta S_\Lambda}{\delta \phi(p)} \frac{\delta S_\Lambda}{\delta \phi(-p)} + \frac{\delta^2 S_\Lambda}{\delta \phi(p) \delta \phi(-p)} \right\} \\ &\equiv \epsilon \mathcal{N} \end{aligned}$$

This rescales the Λ independent correlation functions: $(n = 2, \cdots)$

$$\delta \langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_\Lambda}^\infty = \epsilon \cdot n \langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_\Lambda}^\infty$$

6. Since

$$\langle\, \phi(p_1) \cdots \phi(p_n)\,\rangle^{\infty}_{S_\Lambda + \epsilon\,\mathcal{N}} = (1 + n\epsilon)\,\langle\, \phi(p_1) \cdots \phi(p_n)\,\rangle^{\infty}_{S_\Lambda}$$

are independent of Λ, $S_\Lambda + \epsilon\mathcal{N}$ satisfies the same ERG differential equation as S_Λ.

Introduction of the anomalous dimension γ

1. Introduction by hand:

$$-\Lambda \frac{\partial}{\partial \Lambda} S_\Lambda = \int_p \left[\left(\frac{\Delta(p/\Lambda)}{K(p/\Lambda)} - \gamma \right) \phi(p) \frac{\delta S_\Lambda}{\delta \phi(p)} + \frac{\Delta(p/\Lambda)}{p^2} \frac{1}{2} \left\{ \frac{\delta S_\Lambda}{\delta \phi(p)} \frac{\delta S_\Lambda}{\delta \phi(-p)} + \frac{\delta^2 S_\Lambda}{\delta \phi(p) \delta \phi(-p)} \right\} \right]$$

where γ is an arbitrary constant.

Disadvantage: the Λ independence of $\langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_\Lambda}^{\infty}$ is lost.

An additional term is necessary to restore this independence.

2. **Alternative**: introduce a scale for the wave function renormalization

(a) Expansion of $S_{I,\Lambda}$ in powers of ϕ:

$$S_{I,\Lambda} = \int_p \mathcal{V}_2(\Lambda; p, -p) \frac{1}{2} \phi(p) \phi(-p) + \int_{p_1, \cdots, p_4} \delta(p_1 + \cdots + p_4) \mathcal{V}_4(\Lambda; p_1, \cdots, p_4) \frac{1}{4!} \phi(p_1) \cdots \phi(p_4) + \cdots$$

(b) Wave function renormalization at $\Lambda = \mu$:

$$\frac{\partial}{\partial p^2} \mathcal{V}_2(\Lambda = \mu; p, -p) \Big|_{p^2=0} = 0$$

(c) Let $S_{\Lambda,\mu}$ be a solution of the ERG diff eq and the above normalization condition.

(d) $S_{\Lambda,\mu(1-\Delta t)} \equiv S_{\Lambda,\mu} + \Delta t \cdot \gamma \mathcal{N}$ satisfies the normalization condition at $\Lambda = \mu(1-\Delta t)$ if we choose

$$\gamma = \frac{-\frac{\partial}{\partial p^2} \int_q \frac{\Delta(q/\mu)}{q^2} \frac{1}{2} \mathcal{V}_4(\mu; q, -q, p, -p)\Big|_{p^2=0}}{2 - \frac{\partial}{\partial p^2} \int_q \frac{2K(q/\mu)(1-K(q/\mu))}{q^2} \frac{1}{2} \mathcal{V}_4(\mu; q, -q, p, -p)\Big|_{p^2=0}}$$

which is μ dependent.

(e) Thus, we have two solutions $S_{\Lambda,\mu}$, $S_{\Lambda,\mu(1-\Delta t)}$ of the **same** ERG diff eq: one satisfying the normalization condition at $\Lambda = \mu$, and the other at $\Lambda = \mu(1-\Delta t)$.

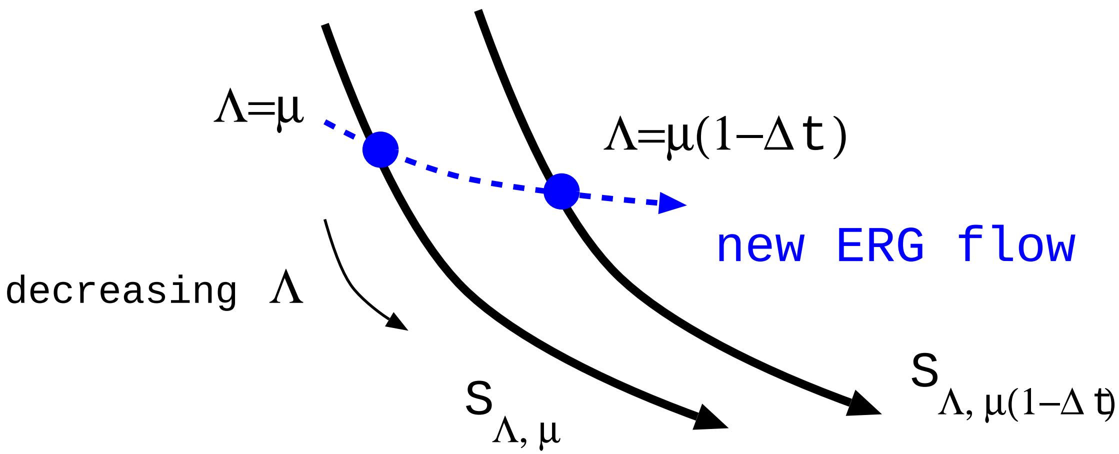
$$\langle\, \phi(p_1) \cdots \phi(p_n) \,\rangle^{\infty}_{S_{\Lambda,\mu(1-\Delta t)}} = (1 + \Delta t \cdot n\gamma) \langle\, \phi(p_1) \cdots \phi(p_n) \,\rangle^{\infty}_{S_{\Lambda,\mu}}$$

are Λ independent for $n = 2, \cdots$.

(f) Choosing $\Lambda = \mu$, we obtain a new ERG flow

$$-\mu \frac{\partial}{\partial \mu} S_{\mu,\mu} = \int_p \left[\left(\frac{\Delta(p/\mu)}{K(p/\mu)} - \gamma \right) \phi(p) \frac{\delta S_{\mu,\mu}}{\delta \phi(p)} \right.$$

$$\left. + \frac{1}{p^2} \left\{ \Delta(p/\mu) - 2\gamma K(p/\mu)(1 - K(p/\mu)) \right\} \frac{1}{2} \left\{ \frac{\delta S_{\mu,\mu}}{\delta \phi(p)} \frac{\delta S_{\mu,\mu}}{\delta \phi(-p)} + \frac{\delta^2 S_{\mu,\mu}}{\delta \phi(p) \delta \phi(-p)} \right\} \right]$$



(g) Under the new ERG flow,

$$-\mu\frac{\partial}{\partial\mu}\langle\,\phi(p_1)\cdots\phi(p_n)\,\rangle^{\infty}_{S_{\mu,\mu}} = n\gamma(\mu)\,\langle\,\phi(p_1)\cdots\phi(p_n)\,\rangle^{\infty}_{S_{\mu,\mu}}$$

thanks to the extra G-term.

ERG for a fixed point

1. For ERG to have a fixed point, we must rescale the momentum and field:

$$\begin{cases} p &= \mu\, \bar{p} \\ \phi(p) &= \mu^{-\frac{D+2}{2}}\, \bar{\phi}(\bar{p}) \end{cases}$$

so that

$$\int_p \frac{p^2}{K(p/\mu)} \phi(p)\phi(-p) = \mu^D \int_{\bar{p}} \frac{\mu^2 \bar{p}^2}{K(\bar{p})} \mu^{-(D+2)} \bar{\phi}(\bar{p})\bar{\phi}(-\bar{p}) = \int_{\bar{p}} \frac{\bar{p}^2}{K(\bar{p})} \bar{\phi}(\bar{p})\bar{\phi}(-\bar{p})$$

We define

$$\bar{S}_t[\bar{\phi}] \equiv S_{\mu,\mu}[\phi]$$

where $\mu = \mu_0 \mathrm{e}^{-t}$. (Trade μ for t.)

2. ERG diff eq for $\bar{S}_t$:

$$\begin{aligned}\partial_t S_t \quad = \quad & \int_p \left\{ \vec{p} \cdot \frac{\partial \phi(p)}{\partial \vec{p}} + \left(\frac{D+2}{2} - \gamma(t) + \frac{\Delta(p)}{K(p)} \right) \phi(p) \right\} \frac{\delta S_t}{\delta \phi(p)} \\ & + \int_p \frac{1}{p^2} \left(\Delta(p) - 2\gamma(t) K(p)(1 - K(p)) \right) \frac{1}{2} \left\{ \frac{\delta S_t}{\delta \phi(p)} \frac{\delta S_t}{\delta \phi(-p)} + \frac{\delta^2 S_t}{\delta \phi(p) \delta \phi(-p)} \right\}\end{aligned}$$

where the bars above S_t, ϕ, and p are all omitted for simplicity.

3. The cutoff independent correlation functions satisfy

$$\partial_t \left\langle \phi(p_1 \mathrm{e}^t) \cdots \phi(p_n \mathrm{e}^t) \right\rangle_{S_t}^{\infty} = \left(D + n \left(-\frac{D+2}{2} + \gamma(t) \right) \right) \left\langle \phi(p_1 \mathrm{e}^t) \cdots \phi(p_n \mathrm{e}^t) \right\rangle_{S_t}^{\infty}$$

for $n = 2, \cdots$.

4. **Fixed point** S^* with an anomalous dimension $\eta = 2\gamma^*$

(a)

$$0 = \int_p \left\{ \vec{p} \cdot \frac{\partial \phi(p)}{\partial \vec{p}} + \left(\frac{D+2}{2} - \gamma^* + \frac{\Delta(p)}{K(p)} \right) \phi(p) \right\} \frac{\delta S^*}{\delta \phi(p)} + \int_p \frac{1}{p^2} \left(\Delta(p) - 2\gamma^* K(p)(1 - K(p)) \right) \frac{1}{2} \left\{ \frac{\delta S^*}{\delta \phi(p)} \frac{\delta S^*}{\delta \phi(-p)} + \frac{\delta^2 S^*}{\delta \phi(p) \delta \phi(-p)} \right\}$$

(b) **Scaling relations** for the cutoff independent correlation functions:

$$\langle\, \phi(p_1 \mathrm{e}^t) \cdots \phi(p_n \mathrm{e}^t) \,\rangle_{S^*}^{\infty} = \exp\left[t \left(D + n \left(-\frac{D+2}{2} + \gamma^* \right) \right) \right] \langle\, \phi(p_1) \cdots \phi(p_n) \,\rangle_{S^*}^{\infty}$$

Dependence on K

1. Under an infinitesimal change of the cutoff function K by δK, Latorre and Morris found

$$\begin{aligned}\delta S_\Lambda &= \int_p \left[\frac{\delta K(p/\Lambda)}{K(p/\Lambda)} \phi(p) \frac{\delta S_\Lambda}{\delta \phi(p)} \right. \\ &\quad \left. + \frac{\delta K(p/\Lambda)}{p^2} \frac{1}{2} \left\{ \frac{\delta S_\Lambda}{\delta \phi(p)} \frac{\delta S_\Lambda}{\delta \phi(-p)} + \frac{\delta^2 S_\Lambda}{\delta \phi(p) \delta \phi(-p)} \right\} \right]\end{aligned}$$

This gives

$$\begin{aligned}\langle \phi(p)\phi(-p) \rangle_{S_\Lambda + \delta S_\Lambda} &= \left(1 + 2 \frac{\delta K(p/\Lambda)}{K(p/\Lambda)} \right) \langle \phi(p)\phi(-p) \rangle_{S_\Lambda} - \frac{\delta K(p/\Lambda)}{p^2} \\ \langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_\Lambda + \delta S_\Lambda} &= \left(1 + \sum_{i=1}^n \frac{\delta K(p_i/\Lambda)}{K(p_i/\Lambda)} \right) \times \langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_\Lambda}\end{aligned}$$

Hence, for $n = 2, \cdots$,

$$\langle\, \phi(p_1) \cdots \phi(p_n)\,\rangle^{\infty}_{S_\Lambda + \delta S_\Lambda; K+\delta K} = \langle\, \phi(p_1) \cdots \phi(p_n)\,\rangle^{\infty}_{S_\Lambda; K}$$

2. $S_\Lambda + \delta S_\Lambda$ solves the ERG diff eq for $K + \delta K$.

3. Even if S_Λ satisfies the normalization condition at $\Lambda = \mu$, $S_\Lambda + \delta S_\Lambda$ does not.

4. $S_\Lambda + \delta S_\Lambda + \delta z \mathcal{N}$ satisfies the ERG diff eq for $K + \delta K$ and the normalization condition at $\Lambda = \mu$, if

$$\delta z = \frac{- \frac{\partial}{\partial p^2} \int_q \frac{\delta K(q/\mu)}{q^2} \frac{1}{2} \mathcal{V}_4(\mu; q, -q, p, -p) \Big|_{p^2=0}}{2 - \frac{\partial}{\partial p^2} \int_q \frac{2K(q/\mu)(1-K(q/\mu))}{q^2} \frac{1}{2} \mathcal{V}_4(\mu; q, -q, p, -p) \Big|_{p^2=0}}$$

5. $S^K_{\Lambda,\mu}$ for K and $S^{K+\delta K}_{\Lambda,\mu}$ for $K+\delta K$, satisfying the respective ERG diff eq and the normalization condition at $\Lambda = \mu$.

$$\langle\, \phi(p_1) \cdots \phi(p_n) \,\rangle^\infty_{S^K_{\Lambda,\mu}} = (1 - n\delta z) \,\langle\, \phi(p_1) \cdots \phi(p_n) \,\rangle^\infty_{S^{K+\delta K}_{\Lambda,\mu}}$$

This is independent of Λ, and we can set $\Lambda = \mu$:

$$\langle\, \phi(p_1) \cdots \phi(p_n) \,\rangle^\infty_{S^K_{\mu,\mu}} = (1 - n\delta z) \,\langle\, \phi(p_1) \cdots \phi(p_n) \,\rangle^\infty_{S^{K+\delta K}_{\mu,\mu}}$$

This is compared with the defining equation for the anomalous dimension:

$$\begin{aligned} -\mu\frac{\partial}{\partial\mu} \langle\, \phi(p_1) \cdots \phi(p_n) \,\rangle^\infty_{S^K_{\mu,\mu}} &= n\gamma \,\langle\, \phi(p_1) \cdots \phi(p_n) \,\rangle^\infty_{S^K_{\mu,\mu}} \\ -\mu\frac{\partial}{\partial\mu} \langle\, \phi(p_1) \cdots \phi(p_n) \,\rangle^\infty_{S^{K+\delta K}_{\mu,\mu}} &= n(\gamma + \delta\gamma) \,\langle\, \phi(p_1) \cdots \phi(p_n) \,\rangle^\infty_{S^{K+\delta K}_{\mu,\mu}} \end{aligned}$$

Hence,

$$\delta\gamma = -\mu \frac{d}{d\mu} \delta z$$

6. We wish to show $\delta\gamma = 0$ at the fixed point. Compare the expansions:

$$\begin{aligned} S_{\mu,\mu}[\phi] &= \int_p \left(-\frac{p^2}{K(p/\mu)} + \mathcal{V}_2(\mu; p, -p) \right) \frac{1}{2} \phi(p) \phi(-p) \\ &+ \int_{p_1, \cdots, p_4} \delta(p_1 + \cdots + p_4) \mathcal{V}_4(\mu; p_1, \cdots, p_4) \frac{1}{4!} \phi(p_1) \cdots \phi(p_4) + \cdots \\ \bar{S}_t[\bar{\phi}] &= \int_{\bar{p}} \left(-\frac{\bar{p}^2}{K(p)} + \bar{\mathcal{V}}_2(t; \bar{p}, -\bar{p}) \right) \frac{1}{2} \bar{\phi}(\bar{p}) \bar{\phi}(-\bar{p}) \\ &+ \int_{\bar{p}_1, \cdots, \bar{p}_4} \delta(\bar{p}_1 + \cdots + \bar{p}_4) \bar{\mathcal{V}}_4(t; \bar{p}_1, \cdots, \bar{p}_4) \frac{1}{4!} \bar{\phi}(\bar{p}_1) \cdots \bar{\phi}(\bar{p}_4) + \cdots \end{aligned}$$

where

$$\begin{cases} \mathcal{V}_2(\mu; p, -p) &= \mu^2 \bar{\mathcal{V}}_2(t; \bar{p}, -\bar{p}) \\ \mathcal{V}_4(\mu; p_1, \cdots, p_4) &= \mu^{4-D} \bar{\mathcal{V}}_4(t; \bar{p}_1, \cdots, \bar{p}_4) \end{cases}$$

Hence, we can rewrite

$$\delta z(t) = \frac{- \frac{\partial}{\partial \bar{p}^2} \int_{\bar{q}} \frac{\delta K(\bar{q})}{\bar{q}^2} \frac{1}{2} \bar{\mathcal{V}}_4(t; \bar{q}, -\bar{q}, \bar{p}, -\bar{p}) \Big|_{\bar{p}^2=0}}{2 - \frac{\partial}{\partial \bar{p}^2} \int_{\bar{q}} \frac{2K(\bar{q})(1-K(\bar{q}))}{\bar{q}^2} \frac{1}{2} \bar{\mathcal{V}}_4(t; \bar{q}, -\bar{q}, \bar{p}, -\bar{p}) \Big|_{\bar{p}^2=0}}$$

Under δK,

$$\delta\gamma(t) = \frac{d}{dt} \delta z(t)$$

Since $\bar{\mathcal{V}}_4(t; \bar{p}_1, \cdots, \bar{p}_4)$ is independent of t at $\bar{S}^*$, we obtain

$$\boxed{\delta\gamma^* = 0}$$

Concluding remarks

1. Though we have only discussed a real scalar theory, the flow of discussion applies to any theory (as long as it has a fixed point).

2. Because of a one-to-one correspondence

$$\langle \phi(p_1) \cdots \phi(p_n) \rangle^{\infty}_{S_\Lambda, K} = \langle \phi(p_1) \cdots \phi(p_n) \rangle^{\infty}_{S_\Lambda + \delta S_\Lambda, K + \delta K}$$

   universality of critical exponents (other than the anomalous dimension of the field) is almost obvious.

3. Is the formal understanding any useful for practical (and precise) calculations of the critical exponents?

Appendix A. Gaussian theory

1. Quadratic action

$$S_\Lambda = \int_p \left(-\frac{p^2}{K\left(p/\Lambda\right)} + \mathcal{V}_2(\Lambda; p, -p) \right) \frac{1}{2} \phi(p) \phi(-p)$$

where $\mathcal{V}_2$ satisfies

$$-\Lambda \frac{\partial}{\partial \Lambda} \mathcal{V}_2(\Lambda; p, -p) = \mathcal{V}_2(\Lambda; p, -p) \frac{\Delta(p/\Lambda)}{p^2} \mathcal{V}_2(\Lambda; p, -p)$$

The solution satisfying $\frac{\partial}{\partial p^2} \mathcal{V}_2(\Lambda; p, -p)\Big|_{p^2=0} = 0$ is given by

$$\mathcal{V}_2(\Lambda; p, -p) = \frac{-(m^2 + p^4 f(p))}{1 + (m^2 + p^4 f(p)) \frac{1 - K(p/\Lambda)}{p^2}} \stackrel{p^2 \to 0}{\longrightarrow} -(m^2 + p^4 f(0))$$

2. The Λ independent two-point function is

$$\langle\, \phi(p)\phi(-p)\,\rangle_{S_\Lambda}^{\infty} = \frac{1}{m^2 + p^2 + p^4 f(p)}$$

3. Rescaling gives

$$\begin{aligned}\bar{\mathcal{V}}_2(t;\bar{p},-\bar{p}) &\equiv \mu^{-2}\mathrm{e}^{2t}\mathcal{V}_2(\Lambda=\mu\mathrm{e}^{-t};\bar{p}\mu\mathrm{e}^{-t},-\bar{p}\mu\mathrm{e}^{-t})\\ &= \frac{-1}{\frac{1}{\mathrm{e}^{2t}\frac{m^2}{\mu^2}+\mathrm{e}^{-2t}\bar{p}^4\mu^2 f(\bar{p}\mu\mathrm{e}^{-t})} + \frac{1-K(\bar{p})}{\bar{p}^2}}\end{aligned}$$

4. At the critical point $m^2 = 0$, $\bar{\mathcal{V}}_2(t;\bar{p},-\bar{p}) \stackrel{t\to\infty}{\longrightarrow} 0$, and

$$\bar{S}_t \stackrel{t\to\infty}{\longrightarrow} \bar{S}^* \equiv -\int_{\bar{p}} \frac{\bar{p}^2}{K(\bar{p})}\frac{1}{2}\bar{\phi}(\bar{p})\bar{\phi}(-\bar{p})$$

Appendix B. UV renormalizable ϕ^4 in $D = 3$

1. The ϕ^4 theory in $D = 3$ is super-renormalizable.

2. The renormalized theory has two parameters: a squared mass m^2 and a self-coupling λ.

3. In ERG formalism, we can introduce the parameters by

$$\begin{aligned} \mathcal{V}_2(\Lambda = \mu; p, -p)\Big|_{p^2=0} &= -m^2 \\ \frac{\partial}{\partial p^2}\mathcal{V}_2(\Lambda = \mu; p, -p)\Big|_{p^2=0} &= 0 \\ \mathcal{V}_4(\Lambda = \mu; p_i = 0) &= -\lambda \end{aligned}$$

where the second is a normalization condition.

4. **UV renormalizability** amounts to the following asymptotic behavior:

$$S_{I,\Lambda} \quad \stackrel{\Lambda\to\infty}{\longrightarrow} \quad \int_p \left\{ I\Lambda + J \ln \Lambda/\mu + z\, p^2 + w \right\} \frac{1}{2}\phi(p)\phi(-p)$$
$$-\lambda' \int_{p_1,\cdots,p_3} \frac{1}{4!}\phi(p_1)\phi(p_2)\phi(p_3)\phi(-p_1-p_2-p_3)$$

where I, J, z, w, λ' are constants independent of Λ.

5. The μ dependence is given by

$$-\mu\frac{\partial}{\partial\mu}S_\Lambda = -\beta_m\frac{\partial}{\partial m^2}S_\Lambda - \beta\frac{\partial}{\partial\lambda}S_\Lambda + \gamma\mathcal{N}$$

   where β_m, β, γ are functions of m^2, λ, and μ.

6. More precisely, $\frac{\beta_m}{\mu^2}, \frac{\beta}{\mu}, \gamma$ are functions of $\frac{m^2}{\mu^2}$ and $\frac{\lambda}{\mu}$.

7. The Wilson-Fisher fixed point is determined by the invariance of $\frac{m^2}{\mu^2}$ and $\frac{\lambda}{\mu}$:

$$2m^2 + \beta_m = 0, \quad \lambda + \beta = 0$$

8. The Λ independent correlation functions satisfy

$$\left(-\mu\frac{\partial}{\partial\mu} + \beta_m \frac{\partial}{\partial m^2} + \beta\frac{\partial}{\partial\lambda}\right) \langle\, \phi(p_1)\cdots\phi(p_n)\,\rangle^{\infty}_{m^2,\lambda;\mu} = n\gamma\, \langle\, \phi(p_1)\cdots\phi(p_n)\,\rangle^{\infty}_{m^2,\lambda;\mu}$$

9. Under δK, the Wilson action changes by

$$\begin{aligned}\delta S_\Lambda \quad = \quad & \int_p \frac{\delta K\left(p/\Lambda\right)}{K\left(p/\Lambda\right)} \phi(p) \frac{\delta S_\Lambda}{\delta\phi(p)} \\ & + \int_p \frac{\delta K\left(p/\Lambda\right)}{p^2} \frac{1}{2} \left\{ \frac{\delta S_\Lambda}{\delta\phi(p)} \frac{\delta S_\Lambda}{\delta\phi(-p)} + \frac{\delta^2 S_\Lambda}{\delta\phi(p)\delta\phi(-p)} \right\} \\ & + \delta z\, \mathcal{N}\end{aligned}$$

where the last term is added to preserve the normalization condition. (δz, proportional to δK, is now a function of λ/μ & m^2/μ.)

10. Evaluating $\delta\mathcal{V}_2$ and $\delta\mathcal{V}_4$ at zero momenta, we obtain the following change of parameters:

$$\begin{aligned}
\delta m^2 &= -\delta\mathcal{V}_2(\Lambda=\mu;0,0)\\
&= -2\delta z\, m^2 - \int_q \frac{1}{q^2}\left\{\delta K(q/\mu) - 2\delta z K(q/\mu)(1-K(q/\mu))\right\}\frac{1}{2}\mathcal{V}_4(\mu;q,-q,0,0)\\
\delta\lambda &= -\delta\mathcal{V}_4(\Lambda=\mu;0,0,0,0)\\
&= -4\delta z\,\lambda - \int_q \frac{1}{q^2}\left\{\delta K(q/\mu) - 2\delta z K(q/\mu)(1-K(q/\mu))\right\}\frac{1}{2}\mathcal{V}_6(\mu;q,-q,0,0,0,0)
\end{aligned}$$

11. Thus, a different choice of the cutoff function K amounts to **reparametrization**.

12. Critical exponents are invariant under a regular change of variables, hence they do not depend on the choice of K.