Nonperturbative functional renormalization group for disordered systems:

The case of the random field Ising model

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Physics of "disordered systems"

- Systems in the presence of quenched disorder (due to impurities, dislocations, random environment, etc, frozen on the relevant time scale) pose new challenges to statistical physics:
  - new phases and phase transitions (spin glass, glassy phases, Griffiths phases,...)
  - new phenomena (localization, pinning,...)
  - slow relaxation, aging and hysteresis.

- One often needs new theoretical tools

  => Nonperturbative functional RG (NP-FRG)

- Here: focus on the equilibrium behavior of classical systems.

Random field model

- Prototypical model in theory of "disordered systems"

In the field-theoretical description (RFIM) :

$$\boxed{S_h[\phi] = S_B[\phi] - \int_x h(x)\phi(x);\ S_B = \int d^dx \left\{ \frac{1}{2}(\partial_\mu \phi(x))^2 + \frac{\tau}{2}\phi(x)^2 + \frac{u}{4!}\phi(x)^4 \right\}}$$

with a <u>quenched</u> random field drawn from a given probability distribution (e.g., Gaussian)

$$\left| \begin{array}{l} \overline{h(x)} = 0, \\ \overline{h(x)h(y)} = \Delta_B\, \delta^{(d)}(x-y) \end{array} \right.$$

- Physical realizations in soft and hard condensed matter:

  - ✳ Near critical fluids in disordered porous materials
  - ✳ Dilute antiferromagnets in a uniform magnetic fluid
  - ✳ Hysteresis in dirty magnets
  - ✳ Vortex phases in disordered type-II superconductors

Generic difficulties of disordered systems

- Due to quenched disorder (h), one loses translational invariance.

  Way out: average over disorder, but what ?, how ?

- Presence of many low-energy (low-action) "metastable states".

- Possible influence of rare events, rare spatial regions or rare samples.

Average over the disorder

["self-averaging", "replica trick", etc.]

- RFIM equilibrium partition function in a given random-field sample h :

$$\boxed{Z_h[J] = e^{W_h[J]} = \int \mathcal{D}\phi \, e^{-S_h[\phi] + \int_x J(x)\phi(x)}}$$

- $W_h[J]$ is a random functional of the source =>

  * in principle, one needs its whole probability distribution

  * or equivalently, the infinite set of its disorder-averaged cumulants:

$$W_1[J] = \overline{W_h[J]}, \, W_2[J_1, J_2] = \overline{W_h[J_1]W_h[J_2]}|_c, \, \cdots$$

Known results about the RFIM

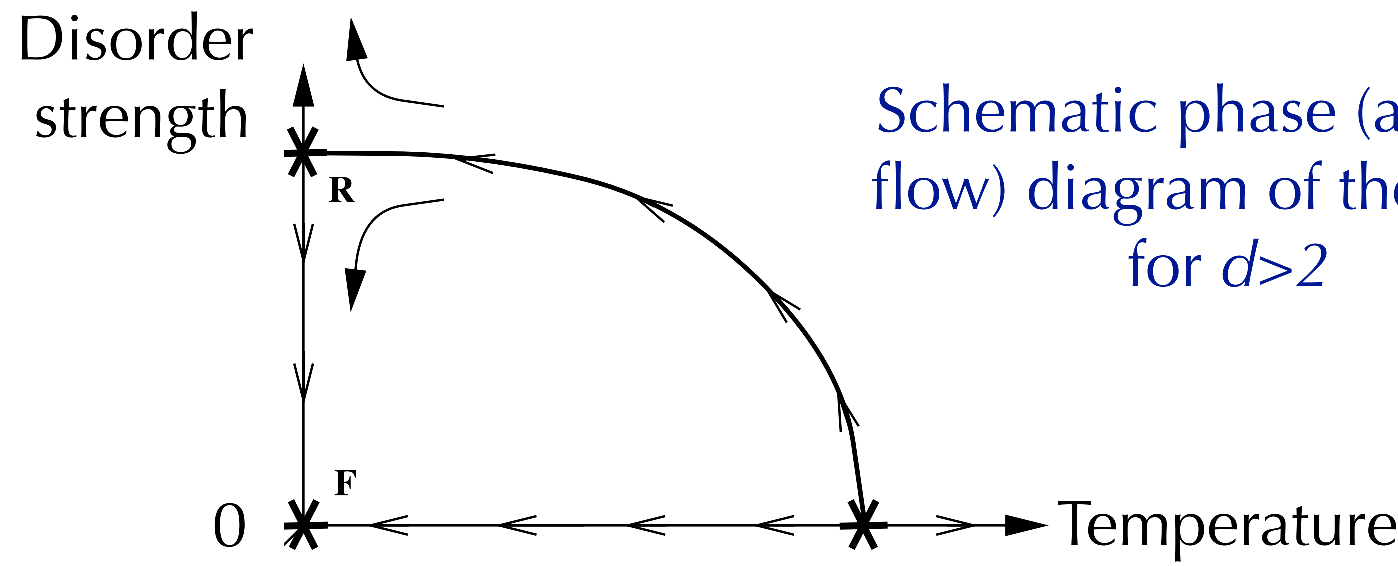
- Existence of a Z_2 symmetry breaking transition for $d>2$ for the Ising version [transition for $d<4$ for the $O(N>2)$ version]. The upper critical dimension is $d_{UC}=6$.

- The critical behavior is associated with a zero-temperature fixed point (thermal fluctuations are formally irrelevant) and one can directly work at $T=0$.

- For a given realization of the disorder $h(x)$, the ground state is unique [except for rare values/configurations of the external source $J(x)$].

Known results about the RFIM (contd.)

Zero-temperature fixed point and its consequences



Schematic phase (and RG flow) diagram of the RFIM for *d*>2

- Additional exponent for the temperature flow: $\theta > 0$

- Two distinct pair correlation functions:

$$\overline{<\phi(x)><\phi(x')>} \sim \frac{1}{|x-x'|^{d-4+\bar{\eta}}} \text{ , with } \theta = 2+\eta-\bar{\eta},$$

$$\overline{<\phi(x)\phi(x')> - <\phi(x)><\phi(x')>} \sim \frac{T}{|x-x'|^{d-2+\eta}}$$

- For *T*>*0*: very slow "activated" critical dynamics, $\tau \sim \exp(c\,\xi^{\theta})$,

with ξ the correlation length (that diverges at the critical point).

Metastable states

- At zero temperature, the equilibrium behavior of the RFIM is determined by the ground state configuration [absolute minimum of $S_h = S_B - (h+J)\phi$], which is solution of the stochastic field equation:

$$\boxed{\frac{\delta S_B[\phi]}{\delta \phi(x)} = h(x) + J(x)}$$

- However, for low disorder strength and in the region of interest (near the critical point), the equation has many solutions => many minima of the bare action ("metastable states").

- What is their effect on the long-distance properties ?
  [Also known to go with slow relaxation, hysteresis and "glassiness"]

Parisi-Sourlas supersymmetric approach of the RFIM

- At *T=0*, generating functional of the correlation functions:

$$\boxed{\mathcal{Z}_h[J,\hat{J}] = \int \mathcal{D}\phi\, \delta\Big[\frac{\delta S_B[\phi]}{\delta\phi} - h - J\Big] \left|\frac{\delta^2 S_B[\phi]}{\delta\phi\delta\phi}\right| e^{\int_x \hat{J}(x)\phi(x)}}$$

* If there is a unique solution of the stochastic field equation, usual manipulations:

Introduce auxiliary fields $\hat{\phi}(x)$, $\psi(x)$, $\overline{\psi}(x)$,
average over disorder *h* (Gaussian probability distribution),
introduce a superspace with 2 Grassmann coordinates $\underline{x} = (x, \overline{\theta}, \theta)$
and supermetric $d\underline{x}^2 = dx^2 + \dfrac{4}{\Delta_B} d\bar{\theta} d\theta$,
a superLaplacian $\Delta_{SS} = \partial_\mu^2 + \Delta_B \partial_\theta \partial_{\overline{\theta}}$,
a superfield $\Phi(\underline{x}) = \phi(x) + \overline{\theta}\psi(x) + \overline{\psi}(x)\theta + \overline{\theta}\theta\hat{\phi}(x)$, super-etc...

Parisi-Sourlas supersymmetric approach of the RFIM (contd.)

- The generating functional $\overline{\mathcal{Z}_h}$ can then be obtained from a superfield theory with action:

$$\boxed{S_{SUSY}[\Phi] = \int_{\underline{x}} \left\{ -\frac{1}{2}\Phi(\underline{x})\Delta_{SS}\Phi(\underline{x}) + \frac{\tau}{2}\Phi(\underline{x})^2 + \frac{u}{4!}\Phi(\underline{x})^4 \right\}}$$

- Invariant under SUSY (super-rotations in superspace)

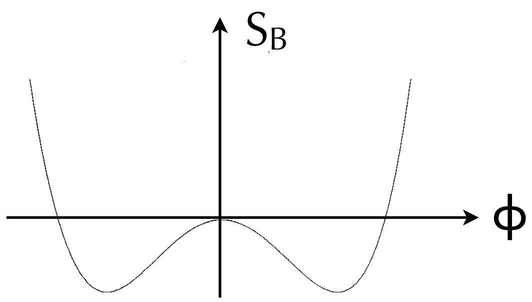
=> leads to <u>"dimensional reduction"</u>: RFIM in *d* dimensions is equivalent to the pure theory in *d-2*.

$$\left[\int d^d x d\theta d\bar{\theta}\, f(x^2 + \frac{4}{\Delta_B}\theta\bar{\theta}) = \left(\frac{4\pi}{\Delta_B}\right) \int d^{d-2}x\, f(x^2) \right]$$

Beautiful, but wrong!!
Problem with multiple solutions!!

Rare events: toy model (d=0 RFIM)

Stochastic equation: $\dfrac{\delta S_B(\phi)}{\delta \phi} = J + h$ with

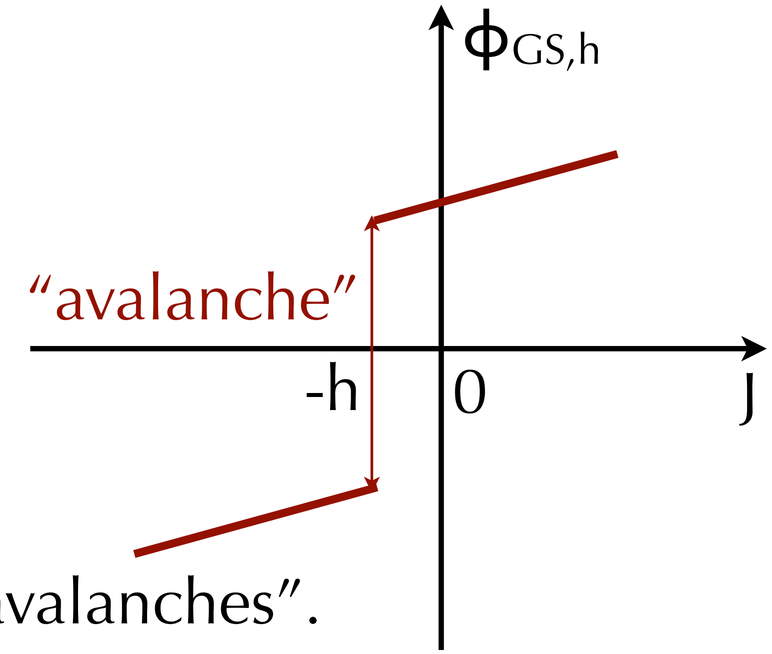


- <u>For zero temperature *T=0*</u>, select the ground state:

  The pair correlation function for slightly different sources,

$$\boxed{\begin{aligned}&\overline{\phi_{GS,h}(J+\delta J)\phi_{GS,h}(J-\delta J)} = \\ &\overline{\phi_{GS,h}(J)^2} + A(J)|\delta J| + O(\delta J^2)\end{aligned}}$$

  has a nonanalytic behavior (a “cusp”) when J->0 due to the “avalanches”.



- <u>For a small *T>0*</u>, Boltzmann weighting of the minima ($e^{-\frac{S_B(\phi)-(J+h)\phi}{T}}$):

  The cusp is rounded in a “thermal boundary layer”

$$\boxed{\overline{<\phi(J+\delta J+h)><\phi(J-\delta J+h)>} = \overline{<\phi(J+h)>^2} + T\, f(J, \frac{|\delta J|}{T}) + \cdots}$$

- ISSUE: Does this persist at long distance when *d>0* ?

Long-standing puzzles concerning random-field systems

- Critical behavior: what is the way out of dimensional reduction?

- What is the phase diagram of the *d*-dimensional random-field *O(N)* model in the whole *(N,d)* plane?

Why does one need a nonperturbative functional RG ?

- RG, because one is interested in the long-distance properties near to the critical point; in particular, the “metastable states” of potential relevance are not those of the bare action but those of a scale-dependent renormalized functional;

- Functional, because the influence of the rare events (avalanches and droplets) can only be described through a singular dependence of the cumulants of the renormalized disorder on their arguments;

- Nonperturbative, because standard perturbation theory completely fails (dimensional reduction), relevant dimensions are far from d=6, disorder grows strong under coarse graining.

Program for RG study of RFIM

[Search for the proper T=0 IR (critical) fixed point]

- Select with high probability the ground state at the running IR scale *k* among the solutions if several of them and ensure that only the ground state is considered when *k -> 0*.
- Describe full functional dependence of cumulants of renormalized disorder and allow for nonanalytical dependence on their arguments.
- Start the RG flow with a "regularized" stochastic field equation having a unique solution.
- Use a nonperturbative truncation and be able to recover dimensional reduction if it has a range of validity.

=> NP-FRG in a superfield setting

Superfield formalism for the RFIM

- Several copies + a weighting factor => Generating functional:

$$\boxed{\begin{aligned}\mathcal{Z}_h^{(\beta)}[\{J_a,\hat{J}_a\}] = \prod_a \int \mathcal{D}\phi_a \delta\Big[\frac{\delta S_B[\phi_a]}{\delta \phi_a} - h - J_a\Big] \det\left(\frac{\delta^2 S_B[\phi_a]}{\delta\phi_a\delta\phi_a}\right) \\ \times\, e^{\int_x \hat{J}_a(x)\phi_a(x)}\; e^{-\beta\left(S_B[\phi_a] - \int_x [h(x)+J_a(x)]\phi_a(x)\right)}\end{aligned}}$$

Average over disorder generates cumulants with full functional dependence:

$$\overline{\mathcal{Z}_h[\{J_a,\hat{J}_a\}]} = \overline{\prod_a e^{\mathcal{W}_h[J_a,\hat{J}_a]}} = e^{\sum_a \overline{\mathcal{W}_h[J_a,\hat{J}_a]} + \frac{1}{2}\sum_{ab}\overline{\mathcal{W}_h[J_a,\hat{J}_a]\mathcal{W}_h[J_b,\hat{J}_b]}|_c + \cdots}$$

- Introduce superfields and a "curved" Grassmannian space

$$\Phi(\underline{\theta}) = \phi + \bar{\theta}\psi + \bar{\psi}\theta + \bar{\theta}\theta\hat{\phi}\,;\; \int_{\underline{\theta}} = \int\int d\theta d\bar{\theta}(1+\beta\bar{\theta}\theta)$$

$$\boxed{\begin{aligned}=>\; S^{(\beta)}[\{\Phi_a\}] = \sum_a \int_{\underline{\theta}} S_1[\Phi_a(\underline{\theta})] + \frac{1}{2}\sum_{ab}\int\int_{\underline{\theta}_1\underline{\theta}_2} S_2[\Phi_a(\underline{\theta}_1),\Phi_a(\underline{\theta}_2)] \\ S_1 = \int_x \left[\frac{1}{2}\big(\partial_\mu \Phi_a(\underline{\theta},x)\big)^2 + U_B(\Phi_a(\underline{\theta},x)\right]\,;\; S_2 = \int_x \Delta_B\,\Phi_a(\underline{\theta}_1,x)\Phi_b(\underline{\theta}_2,x)\end{aligned}}$$

Superfield formalism (contd.)

- Add coupling to supersources $\sum_a \int_{\underline{\theta},x} \mathcal{J}_a(\underline{\theta},x)\Phi_a(\underline{\theta},x)$ -> $\mathcal{W}^{(\beta)}[\{\mathcal{J}_a\}]$
  + Legendre transform -> Effective action $\Gamma^{(\beta)}[\{\Phi_a\}]$

- The action is invariant under a large group of symmetries and supersymmetries (S_n between copies, global Z_2 and Euclidean translations + rotations, isometries of the curved Grassmann subspace copy by copy).

- The expansion in increasing number of sums over copies generates the "cumulant expansion" of the 1PI generating functional (effective action):

$$\boxed{\Gamma^{(\beta)}[\{\Phi_a\}] = \sum_{a_1} \Gamma_1^{(\beta)}[\Phi_{a_1}] - \frac{1}{2}\sum_{a_1,a_2} \Gamma_2^{(\beta)}[\Phi_{a_1},\Phi_{a_2}] + \cdots}$$

NP-FRG in superfield formalism

- Add an IR regulator to the action:

$$\Delta S_k^{(\beta)}[\{\Phi_a\}] = \frac{1}{2}\sum_{ab}\int_{\underline{x}_1}\int_{\underline{x}_2}\Phi_a(\underline{x}_1)\mathcal{R}_{k,ab}(\underline{x}_1,\underline{x}_2)\Phi_b(\underline{x}_2)$$

$\mathcal{R}_{k,ab}(\underline{x}_1,\underline{x}_2) = \delta_{\underline{\theta}_1,\underline{\theta}_2}\widehat{R}_k(q^2) + \widetilde{R}_k(q^2)$: suppresses fluctuations of ϕ field <u>and</u> random field

- ERGE for the effective average action at scale *k*:

$$\boxed{\partial_k\Gamma_k^{(\beta)}[\{\Phi_a\}] = \frac{1}{2}\sum_{ab}\int_{\underline{x}_1}\int_{\underline{x}_2}\partial_k\mathcal{R}_{k,ab}(\underline{x}_1,\underline{x}_2)\left(\Gamma_k^{(2)}[\{\Phi_a\}] + \mathcal{R}_k\right)^{-1}_{(b,\underline{x}_2)(a,\underline{x}_1)}}$$

- Through the expansion of $\Gamma_k^{(\beta)}[\{\Phi_a\}]$ in increasing number of copies:

  Hierarchy of coupled ERGE's for the cumulants (functionals of the superfields):

$$\partial_k\Gamma_{k,1}^{(\beta)}[\Phi_1] = \cdots\,,\ \partial_k\Gamma_{k,2}^{(\beta)}[\Phi_1,\Phi_2] = \cdots\,,\ \text{etc}$$

"Grassmannian ultralocality" and superrotational invariance

- Property of the generating functionals when a unique solution of the stochastic equation is included:
  "Grassmannian ultralocality": $\mathcal{W}_h^{(\beta)}[\mathcal{J}] = \int_{\underline{\theta}} W[\mathcal{J}(\underline{\theta})]$
- When $\beta \to \infty$, "ultralocality" (UL) becomes exact, with the *p*th cumulant of the effective average action given by (more later!)

$$\boxed{\begin{aligned}&\Gamma_{k,p}^{(\beta)}[\Phi_{a_1}, ..., \Phi_{a_p}] = \\ &\int_{\underline{\theta}_{a_1}} ... \int_{\underline{\theta}_{a_p}} \left(\Gamma_{k,p}^{(UL)}[\Phi_{a_1}(\theta_{a_1}), ..., \Phi_{a_p}(\theta_{a_p})] + NUL \,\text{corrections}\right)\end{aligned}}$$

- When "Grassm. UL", β drops out of the FRG equations.
  Then, for supersources that reduce the theory to a 1-copy problem, the theory is invariant under superrotations (SUSY)
  => Ward-Takahashi (WT) identities.

NP-FRG and SUSY breaking

- Grassm. ultralocality => hierarchy of ERGE's for the cumulants with physical field arguments ($\Phi \equiv \phi$):

$$\begin{aligned} &\partial_t \Gamma_{k1}[\phi] = \frac{1}{2}\widetilde{\partial}_t\, Tr\Big\{\big[\Gamma^{(2)}_{k1}[\phi] + \hat{R}_k\big]^{-1}\big[\Gamma^{(11)}_{k2}[\phi,\phi] - \widetilde{R}_k\big]\Big\} \\ &\partial_t \Gamma_{k2}[\phi_1,\phi_2] = \cdots \end{aligned} \qquad [t = ln(k/\Lambda)]$$

!!! Recall: The auxiliary parameter β then drops out of the ERGE's !!!

- As a result, superrotational invariance for 1 copy is *a priori* preserved along the RG flow: From the WT identities, one can show that it leads (nonperturbatively) to dimensional reduction.

- What can go wrong ?
  - ✳ Spontaneous breaking of superrotation invariance: some 1PI vertex blows up when copy fields become equal.
  - ✳ Dimension reduction is broken when a cusp $\Gamma^{(11)}_{k,2}(\varphi_1,\varphi_2) - \Gamma^{(11)}_{k,2}(\varphi_1,\varphi_1) \sim |\varphi_2 - \varphi_1|$ as $\varphi_2 \to \varphi_1$ appears at a finite scale k_L.

SUSY-compatible approximation and RG flow

- Ansatz for effective average action (under "Grassm. ultralocality"):

$$\boxed{\begin{aligned} &\Gamma_{k1}[\phi] = \int_x \left[U_k(\phi(x)) + \frac{1}{2} Z_k(\phi(x))(\partial_\mu \phi(x))^2\right] \\ &\Gamma_{k2}[\phi_1, \phi_2] = \int_x V_k(\phi_1(x), \phi_2(x)), \qquad \Gamma_{k,p>2} = 0 \end{aligned}}$$

+ Regulators: $\hat{R}_k = Z_k k^2\, r(q^2/k^2)$, $\tilde{R}_k = -(\Delta_k/Z_k)\partial_{q^2}\hat{R}(q^2)$

[SUSY WT identity: $\Delta_k = \Delta_B\, Z_k$]

- Introduce scaling dimensions for T=0 fixed point (critical). Then,

$$\begin{aligned} &\partial_t u_k'(\varphi) = \cdots \\ &\partial_t z_k(\varphi) = \cdots \\ &\partial_t \delta_k(\varphi_1, \varphi_2) = \partial_t v_k^{(11)}(\varphi_1, \varphi_2) = \cdots \end{aligned}$$

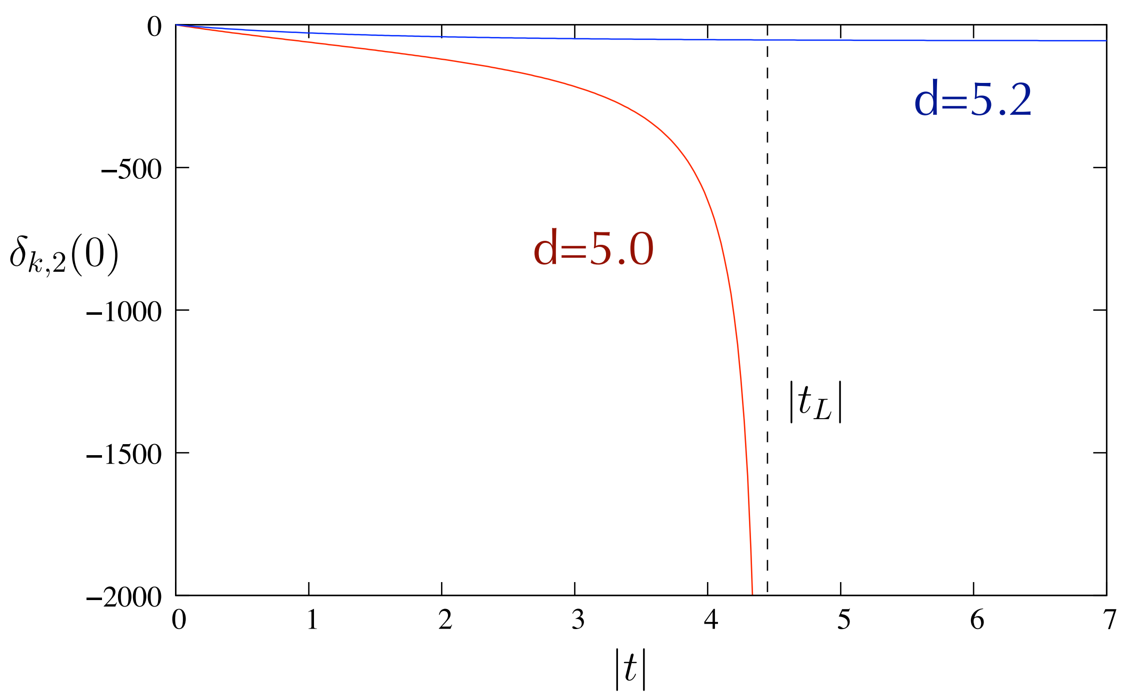
$$\left|\begin{aligned} \eta_k &= -\partial_t Z_k \\ \bar{\eta}_k &= 2\eta_k + \partial_t \Delta_k \end{aligned}\right.$$

- If no linear cusp in $\delta_k(\varphi_1, \varphi_2)$, then $\partial_t \delta_k(\varphi, \varphi) = \partial_t z_k(\varphi)$ (WT id.) and exact dim. reduction follows: found for $\underline{d > d_{DR} \simeq 5.1}$
- <u>Numerical resolution on a grid.</u>

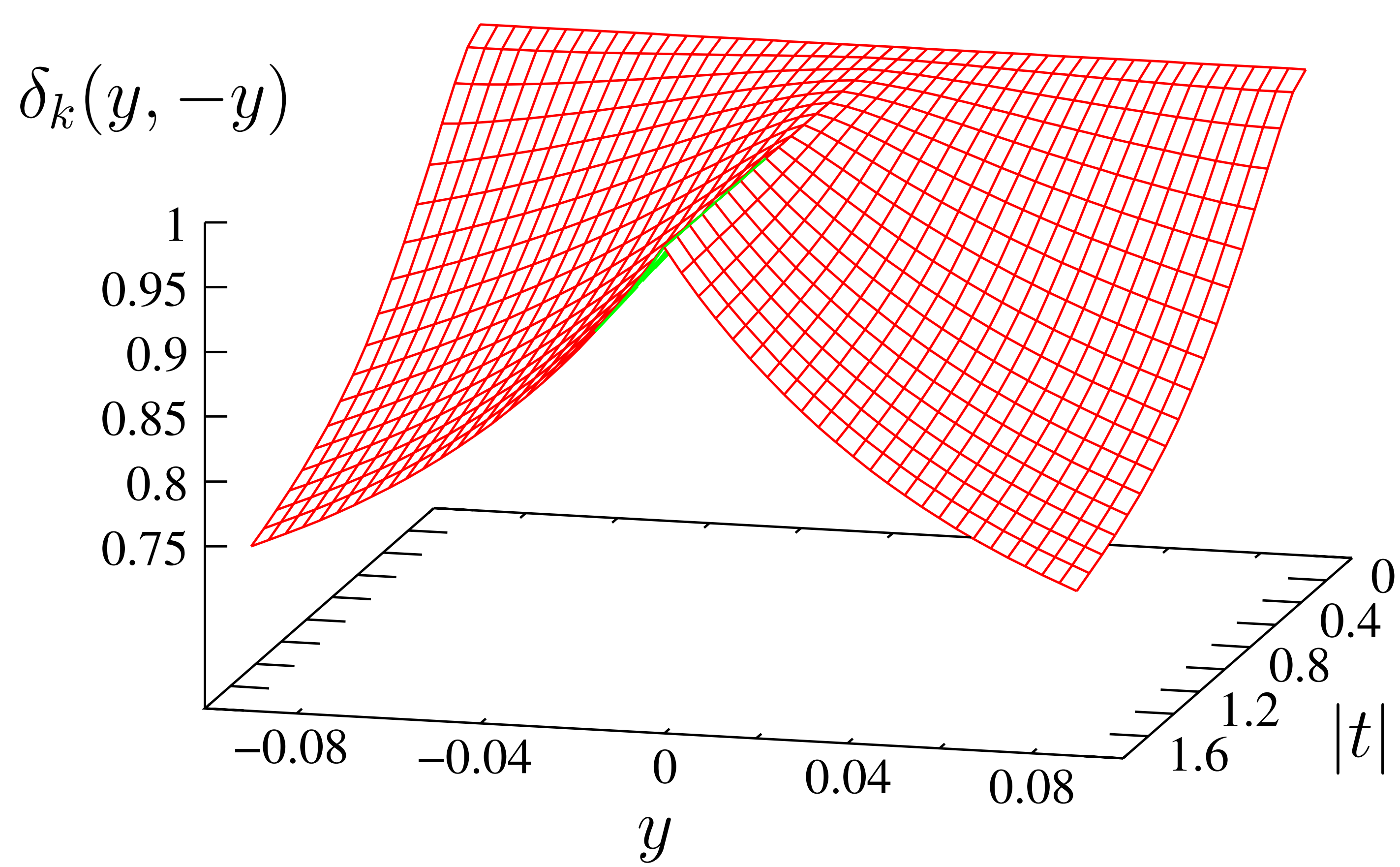
Results

Above $d_{DR} \simeq 5.1$: no cusp in $\delta_k(\varphi_1, \varphi_2)$.
Below d_{DR}: cusp in $\delta_k(\varphi_1, \varphi_2)$ and SUSY breaking in a finite RG time.

A second order derivative of δ_k blows up in a finite RG time for $d<d_{DR}$ (red curve), not for $d>d_{DR}$ (blue curve) => SUSY breaking



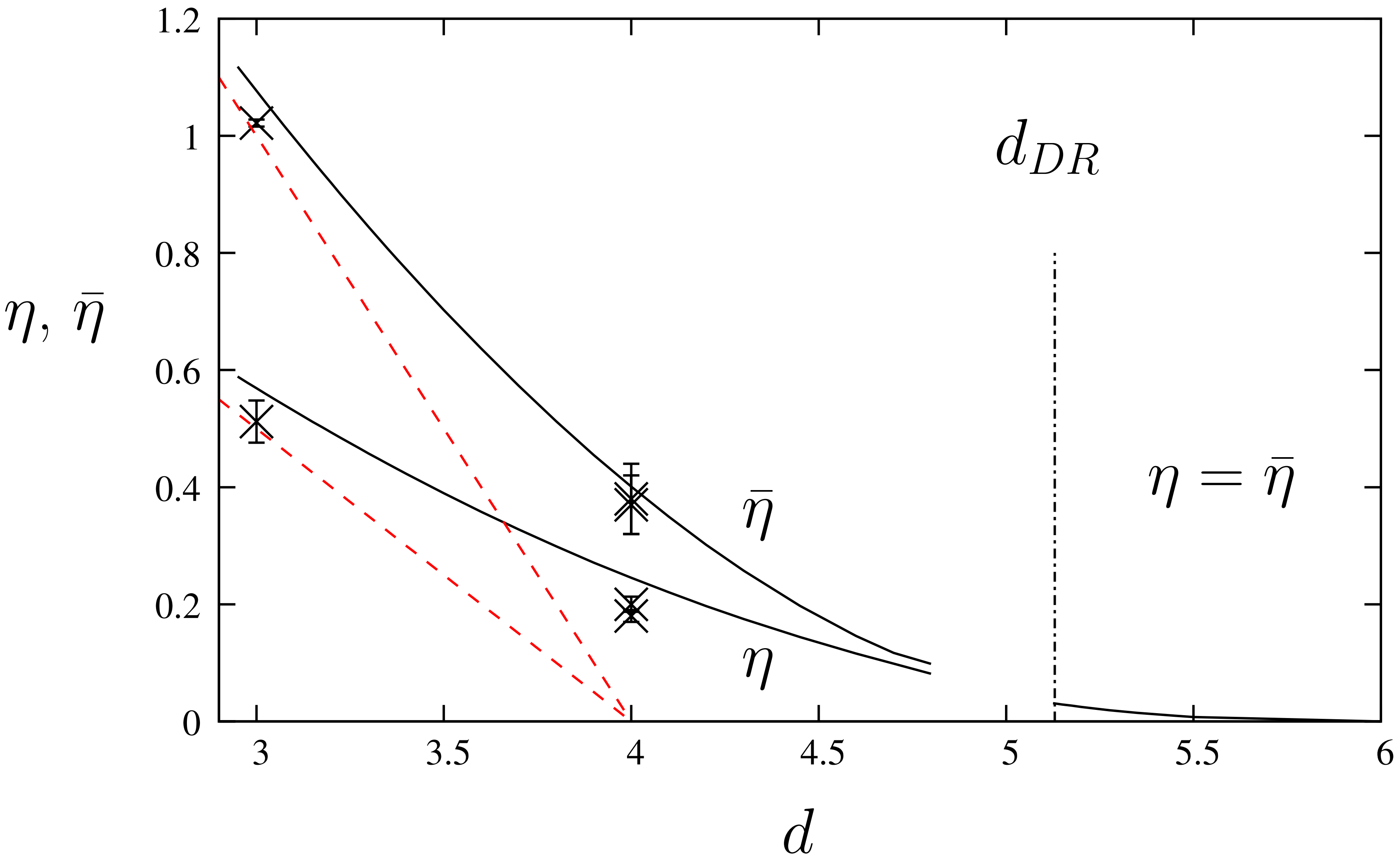
Flow of the dimensionless second cumulant δ_k in d=4



Results: Critical exponents η and $\bar{\eta}$

Breakdown of dimensional reduction appears continuously in dimension *d*

- Dimensional reduction: $\bar{\eta} = \eta \ [= \eta^{(pure,d-2)}]$
- Below d_{DR}: $\bar{\eta} > \eta$



Very good agreement with "best estimates":

In d=3,

η=0.57 ± 0.05

[0.51±0.04]

$\bar{\eta}$=1.08 ± 0.05

[1.02-1.10]

Role of the (auxiliary) temperature

- If one adds the first "non-ultralocal" corrections, one finds that they go to finite fixed-point values and that the flow of the ultralocal quantities is generically of the form

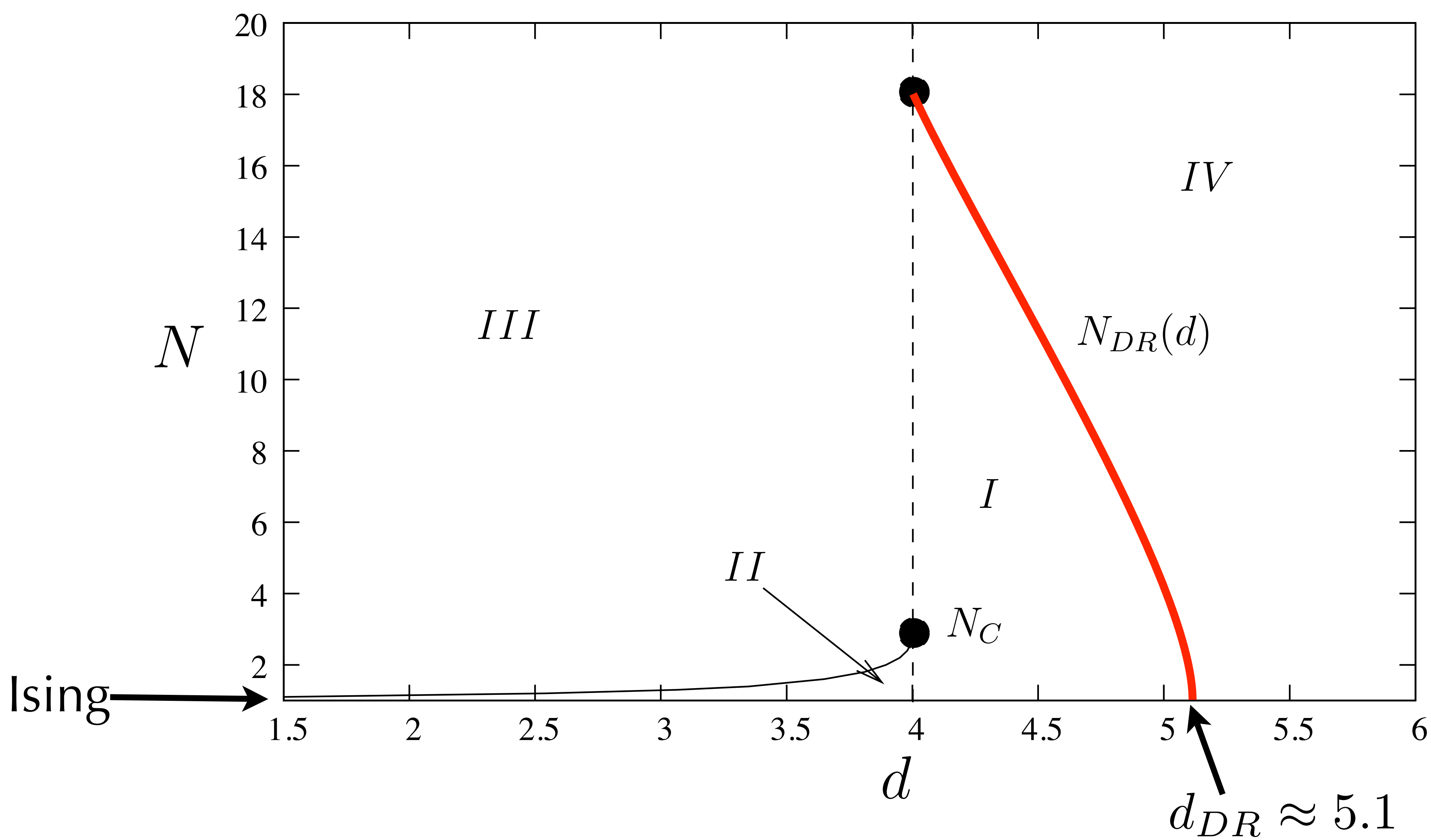
$$\partial_t \delta_k(\varphi_1, \varphi_2) = \beta^{(UL)}_{\delta_k}(\varphi_1, \varphi_2) + \left(\frac{k^\theta}{\beta}\right) \beta^{(NUL)}_{\delta_k}(\varphi_1, \varphi_2)$$

- The second term drops out when (1/β)=0 (zero auxiliary temperature) and one is back to the purely "ultralocal" contributions.

- When β finite and *k -> 0,* nonuniform convergence to the "ultralocal" cuspy fixed point: cusp in $\delta_k(\varphi_1, \varphi_2)$ is rounded in a thermal boundary layer in $|\varphi_1 - \varphi_2|/k^\theta$.

- The boundary layer is related to the presence of rare "droplet" excitations.

Results: N-d phase diagram of the RFO(N)M



Region IV: Weak non-analyticity (at fixed pt.); dim. red. predictions O.K.

Regions I and II: Spontaneous SUSY breaking at finite RG scale; cusp in renormalized second cumulant; breakdown of dim. red. (II: QLRO)

Region III: No phase transition

Conclusion

- The description of the long-distance physics of systems with quenched disorder requires special theoretical tools to account for loss of translational invariance/average over disorder, rare events, metastable states, etc...

- NP-FRG in a superfield setting (with introduction of many copies and of a weighting factor to select the proper solution) solves the 30-year-old pending problems concerning the critical behavior in random field systems.

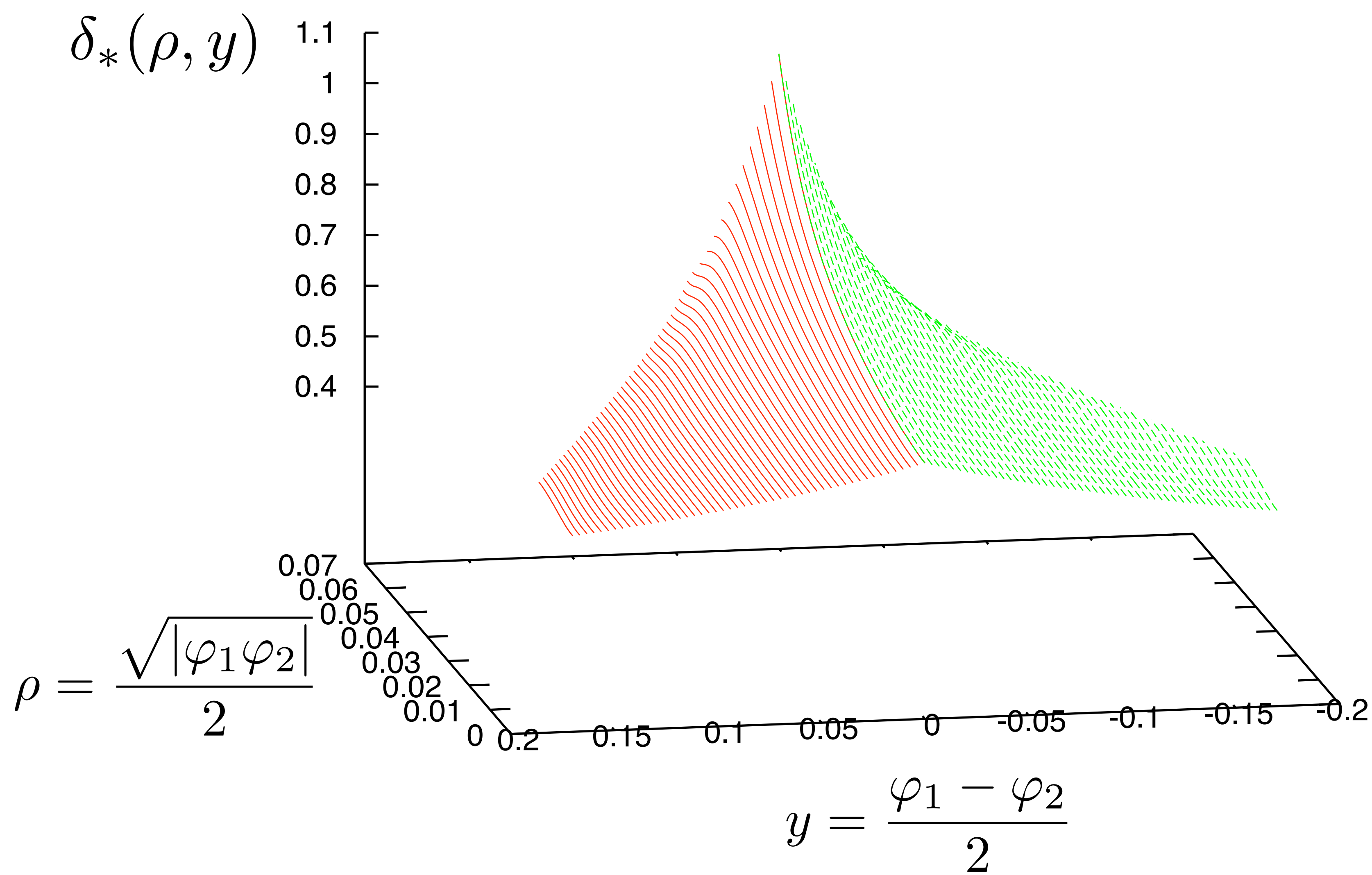
- It could be a useful formalism for other problems described by a stochastic field equation with multiple solutions (metastable states in glasses, shocks in fluid turbulence, Gribov copies in non-Abelian gauge theories,...).

Results (contd)

Below $d_{DR} \simeq 5.1$: cuspy fixed point for

$\delta_k(\varphi_1, \varphi_2)$

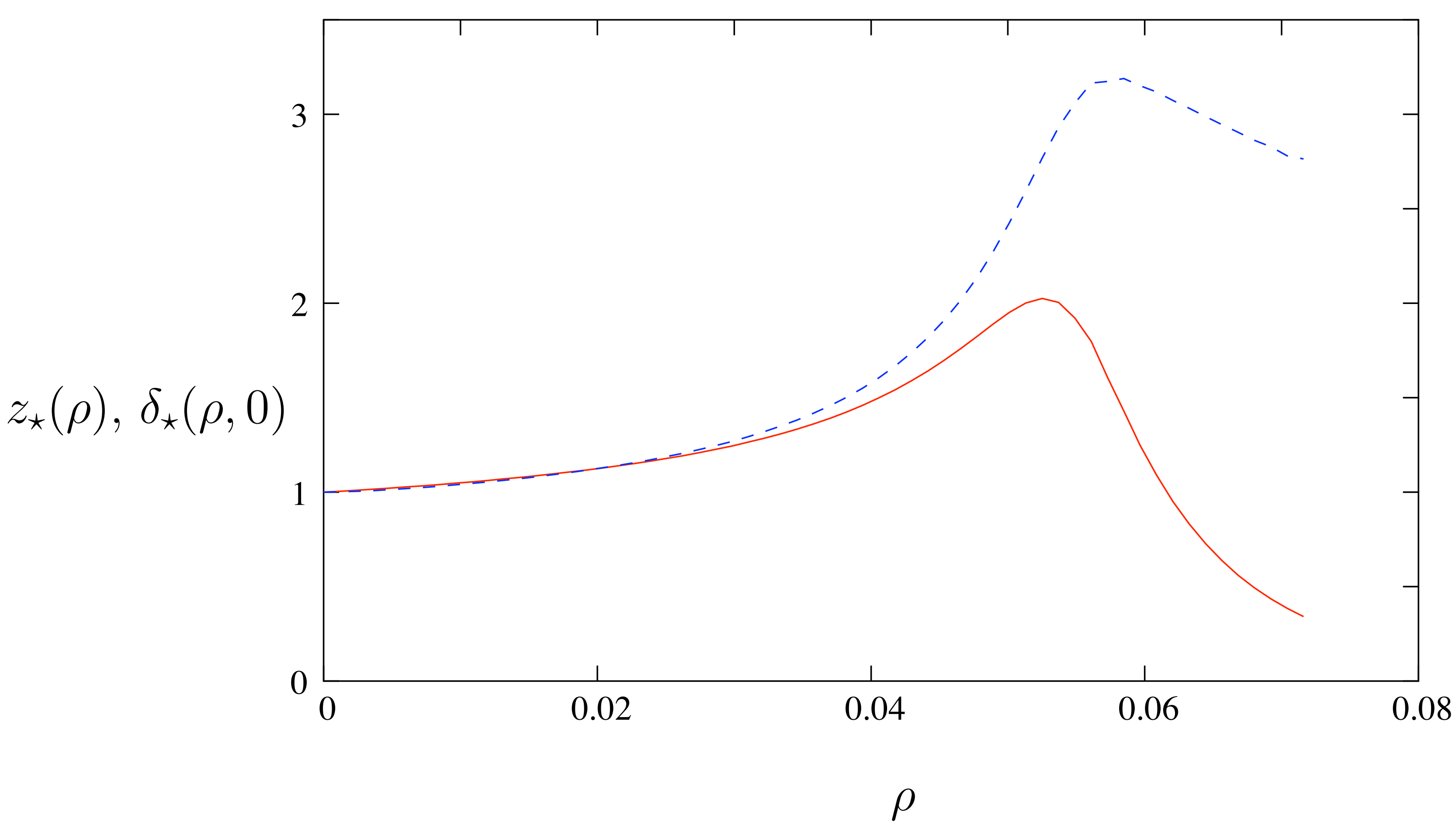
Dimensionless second cumulant at the fixed point in d=3



Results (contd)

Below d_{DR}: cusp in $\delta_k(\varphi_1, \varphi_2)$ & spontaneous SUSY breaking

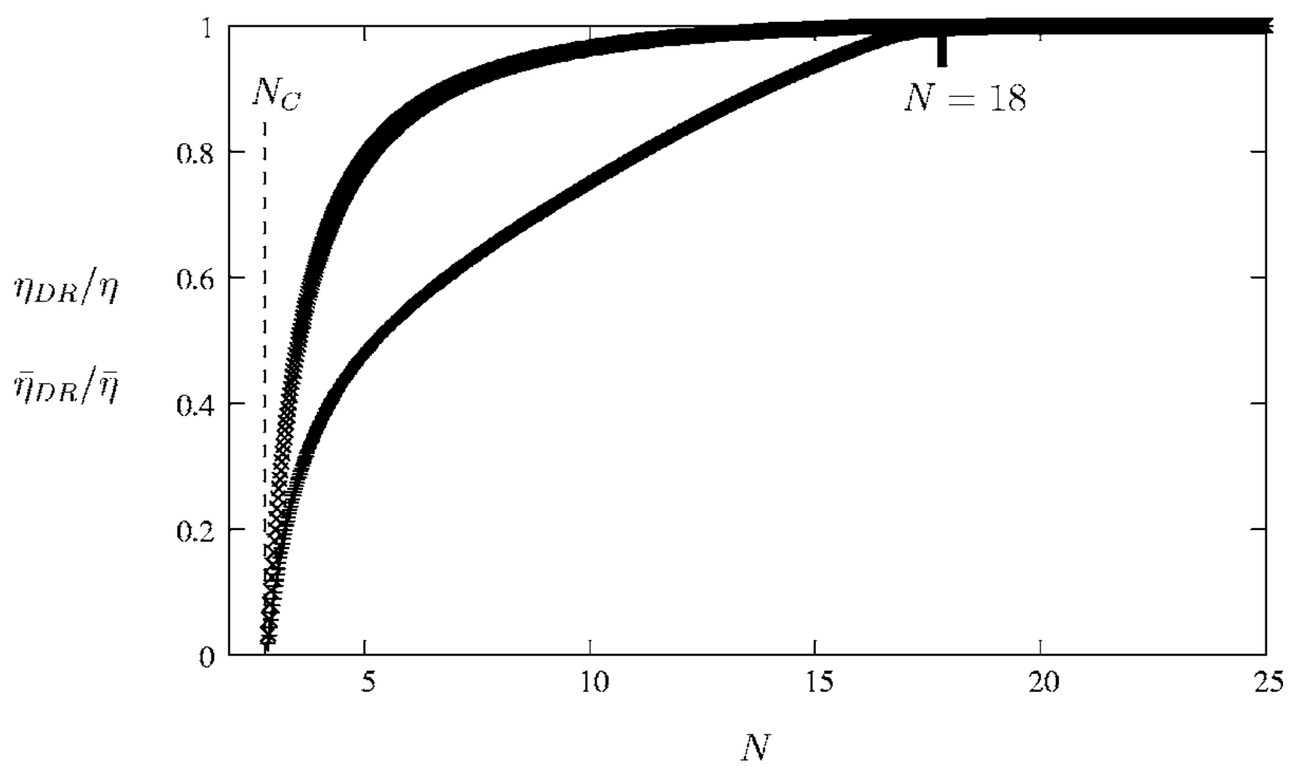
Breakdown of the (SUSY) WT identity at the fixed point in d=3



Results (contd.)

Above N_{DR} =18: no cusp and dimensional reduction
Below N_{DR}=18: cusp and breakdown of dimensional reduction

Anomalous dimensions in d=4+ε (1-loop exact FRG)



Results: Critical exponents η and η̃ (contd.)

Optimization of the cut-off (to ensure a better stability of the results)