Anomalies in Exact Renomalization Group

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A short summary

- Our earlier studies on the long standing problem in ERG: gauge symmetry vs cutoff
  - Using Batalin-Vilkovisky formalism,
    we can write the quantum master equation (QME) $\bar{\Sigma}_\Lambda = 0$ for finite cutoff Λ.
    Therefore, gauge symmetry is present even in the presence of a cutoff.
    The 1PI expression of QME is the modified ST identity.

- For anomalous theory, $\bar{\Sigma}_\Lambda$, the QM operator does not vanish.

  $\bar{\Sigma}_\Lambda \equiv \mathcal{A} \sim$ ghost $\times$ anomaly.

  - $\mathcal{A}$, a functional of fields. We discuss its properties for any Λ.
  - Some thoughts on the Wess-Zumino condition.
  - Concrete evaluation of the functional for a simple example.

Earlier works on anomalies in ERG

- Bonini, D'Attanasio and Marchesini, PLB329 (1994) 249
- Bonini and Vian, NPB511 (1998) 469
- Pernici, Raciti and Riva, NPB520 (1998) 469

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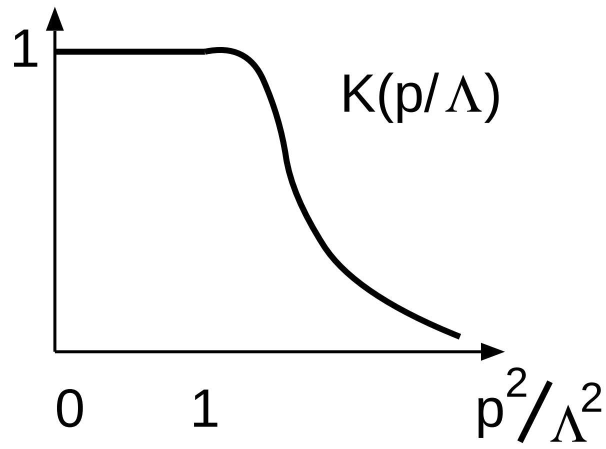
1.1 Path integral formulation

The cutoff function



The UV action with the cutoff Λ_0

$$
\begin{aligned}
S_{\Lambda_0}[\phi] &= \frac{1}{2}\phi \cdot K_0^{-1} D \cdot \phi + S_{I,\Lambda_0}[\phi] \,. \\
\mathcal{Z}_\phi[J] &= \int \mathcal{D}\phi \exp\left(-S_{\Lambda_0}[\phi] - K_0^{-1} J \cdot \phi\right) \,.
\end{aligned}
$$

$$
K_0(p) \equiv K(p/\Lambda_0)
$$

$$\phi \cdot D \cdot \phi \;=\; \int \frac{d^4p}{(2\pi)^4} \phi^A(-p) D_{AB}(p) \phi^B(p), \qquad J \cdot \phi = \int \frac{d^4p}{(2\pi)^4} J_A(-p) \phi^A(p)$$

Introduce a cutoff $\Lambda(< \Lambda_0)$ with $K(p/\Lambda)$, and decompose ϕ^A into IR fields Φ^A and UV fields χ^A:

$$K_0 D^{-1} = K D^{-1} + (K - K_0) D^{-1}$$

Integration over the UV fields gives the interaction aciton $S_{I,\Lambda}[\Phi]$

$$\exp\left(-S_{I,\Lambda}[\Phi]\right) \equiv \int \mathcal{D}\chi \exp\left[-\frac{1}{2}\chi \cdot (K_0 - K)^{-1} D \cdot \chi - S_{I,\Lambda_0}[\Phi + \chi]\right] \,.$$

The Wilson action with the cutoff Λ is $S_\Lambda[\Phi] \equiv \frac{1}{2}\Phi \cdot K^{-1} D \cdot \Phi + S_{I,\Lambda}[\Phi]$
and the partiction function is

$$Z_\Phi[J] \equiv \int \mathcal{D}\Phi \exp\left(-S_\Lambda[\Phi] - K^{-1} J \cdot \Phi\right)$$

The two partition functions are related as

$$\mathcal{Z}_\phi[J] = N_J Z_\Phi[J],$$

The normalization factor N_J is given by

$$\ln N_J \;=\; -\frac{(-)^{\epsilon_A}}{2} J_A K_0^{-1} K^{-1} (K_0 - K) \left(D^{-1}\right)^{AB} J_B \; .$$

where ϵ_A is the Grassmann parity of Φ^A.

1.2 Flow and composite operator

The gradual integration gives a RG flow, or the Polchinski equation

$$\Lambda \frac{\partial}{\partial \Lambda} S_\Lambda = -\int_p \left(K^{-1}\dot{K}\right)(p)\left[\Phi^A(p)\frac{\partial^l S_\Lambda}{\partial \Phi^A(p)}\right]$$
$$+\frac{1}{2}\int_p (-)^{\epsilon_A}\left(\dot{K}D^{-1}(p)\right)^{AB}\left[\frac{\partial^l S_\Lambda}{\partial \Phi^B(-p)}\frac{\partial^r S_\Lambda}{\partial \Phi^A(p)} - \frac{\partial^l \partial^r S_\Lambda}{\partial \Phi^B(-p)\partial \Phi^A(p)}\right]$$

with the initial condition

$$S_{\Lambda=\Lambda_0} = S_{\Lambda_0}$$

The functional integration is equivallent to solving the Polchinski equation.

The composite operator is a useful notion.

Equivallent definitions for the composite operator $\mathcal{O}_\Lambda[\Phi]$

1. Via the linearized Polchinski equation, with an initial condition at Λ_0

$$\Lambda \frac{\partial}{\partial \Lambda} \mathcal{O}_\Lambda[\Phi] = -\mathcal{D}\ \mathcal{O}_\Lambda[\Phi]$$

$$\mathcal{D} \equiv \int_p \left[\left(K^{-1}\dot{K}\right)\Phi^A \frac{\partial^l}{\partial \Phi^A} + (-)^{\epsilon_A}\left(\dot{K}D^{-1}\right)^{AB}\left(\frac{\partial^l S_\Lambda}{\partial \Phi^B}\frac{\partial^r}{\partial \Phi^A} - \frac{1}{2}\frac{\partial^l \partial^r}{\partial \Phi^B \partial \Phi^A}\right)\right]$$

2. Given an operator $\mathcal{O}_{\Lambda_0}[\phi]$ at the UV scale Λ_0, the corresponding IR composite operator $\mathcal{O}_\Lambda[\Phi]$ may be constructed as

$$\mathcal{O}_\Lambda[\Phi]\ e^{-S_I[\Phi;\Lambda]} \equiv \int \mathcal{D}\chi\, \mathcal{O}_{\Lambda_0}[\Phi + \chi]\ e^{-\frac{1}{2}\chi\cdot(K_0 - K)^{-1}D\cdot\chi - \mathcal{S}_I[\Phi+\chi;\Lambda_0]}$$

3. The expectation values in the presence of arbitrary sources satisfy

$$\left\langle \mathcal{O}_\Lambda[\Phi]\right\rangle_{\Phi, K^{-1}J} = N_J^{-1}\left\langle \mathcal{O}_{\Lambda_0}[\phi]\right\rangle_{\phi, K_0^{-1}J}$$

Two important composite operators for later discussion:

$$\begin{aligned}\varphi^A_\Lambda &\equiv \frac{K_0}{K}\Phi^A - (K_0 - K)(D^{-1})^{AB}\frac{\partial^l S_\Lambda}{\partial \Phi^B},\\ &= \Phi^A - (K_0 - K)(D^{-1})^{AB}\frac{\partial^l S_{I,\Lambda}}{\partial \Phi^B}\end{aligned}$$

and

$$K\Big(\frac{\partial^r S_\Lambda}{\partial \Phi^A}\mathcal{O}'_\Lambda - \frac{\partial^r \mathcal{O}'_\Lambda}{\partial \Phi^A}\Big)$$

for a composite operator $\mathcal{O}'_\Lambda$.

These obeys the flow equation:

$$\Lambda\frac{\partial}{\partial\Lambda}\mathcal{O}_\Lambda = -\mathcal{D}\mathcal{O}_\Lambda$$

2. Realization of symmetry: consider some transformation

$$\phi^A \to \phi'^A = \phi^A + \delta_\lambda \phi^A \ , \qquad \delta_\lambda \phi^A = \delta\phi^A \lambda = K_0 \mathcal{R}^A[\phi;\Lambda_0]\ \lambda\ .$$

$$\int \mathcal{D}\phi \ \left(K_0^{-1} J \cdot \delta\phi + \Sigma_{\Lambda_0}[\phi]\right) \ \exp\left(-S_{\Lambda_0}[\phi] - K_0^{-1} J \cdot \phi\right) = 0$$

where the quantity $\Sigma_{\Lambda_0}[\phi]$ is given as

$$\Sigma_{\Lambda_0}[\phi] \equiv \frac{\partial^r S_{\Lambda_0}}{\partial \phi^A} \delta\phi^A - \frac{\partial^r}{\partial \phi^A} \delta\phi^A\ .$$

The second term is the contribution from the functional measure $\mathcal{D}\phi$

$$\delta_\lambda \ln \mathcal{D}\phi = (-)^{\epsilon_A} \frac{\partial^r}{\partial \phi^A} \delta_\lambda \phi^A = \frac{\partial^r}{\partial \phi^A} \delta\phi^A \lambda\ .$$

- $\Sigma_{\Lambda_0}[\phi] = 0$ implies that the UV theory is invariant under $\delta\phi$

- Appropriate to call $\Sigma_{\Lambda_0}[\phi]$ as the WT operator

Let us see how the transformation and the WT operator changes as the scale changes. To find $\delta\Phi$ and Σ_Λ at the scale Λ, use the definition of composite operator

$$\begin{aligned}
\langle K^{-1}\delta\Phi^A\rangle_{\Phi,K^{-1}J} &= N_J^{-1}\langle K_0^{-1}\delta\phi^A\rangle_{\phi,K_0^{-1}J} \\
\langle\Sigma_\Lambda[\Phi]\rangle_{\Phi,K^{-1}J} &= N_J^{-1}\langle\Sigma_{\Lambda_0}[\phi]\rangle_{\phi,K_0^{-1}J}
\end{aligned}$$

Starting from the transformation $K_0^{-1}\delta\phi = \mathcal{R}[\phi;\Lambda_0]$

$$N_J^{-1}\langle K_0^{-1}\delta\phi^A\rangle_{\phi,K_0^{-1}J} = N_J^{-1}\mathcal{R}^A[K_0\partial_J^l;\Lambda_0]\mathcal{Z}_\phi[J] = \left(N_J^{-1}\mathcal{R}^A[K_0\partial_J^l;\Lambda_0]N_J\right)Z_\Phi[J]$$

Writing the transformation of IR fields as $\delta\Phi^A = KR^A[\Phi]$, we may equate the above expression with the following

$$\langle K^{-1}\delta\Phi^A\rangle_{\Phi,K^{-1}J} = R^A[K\partial_J^l]Z_\Phi[J]$$

We find the relation that gives transformation of the IR fields

$$R^A[K\partial^l_J] = N_J^{-1}\mathcal{R}^A[K_0\partial^l_J;\Lambda_0]N_J$$

- Note here ∂^l_J acts on N_J, that produces the scale change of the transformation.

Using the transformation $\delta\Phi$, we find the WT operator as

$$\Sigma_\Lambda[\Phi] = \frac{\partial^r S_\Lambda[\Phi]}{\partial\Phi^A}\delta\Phi^A - \frac{\partial^r}{\partial\Phi^A}\delta\Phi^A$$

The relation

$$\langle\Sigma_\Lambda[\Phi]\rangle_{\Phi,K^{-1}J} = N_J^{-1}\langle\Sigma_{\Lambda_0}[\phi]\rangle_{\phi,K_0^{-1}J}$$

implies that if the WT operator vanishes at the scale Λ_0, it does at any lower scale.

2.1 The anti-field formalism a la Batalin-Vilkovisky

For a classical gauge fixed action $S_{cl}[\phi]$ for a generic gauge theory, define an extended action as

$$\bar{S}_{cl}[\phi, \phi^*] \equiv S_{cl}[\phi] + \phi^*_A \delta \phi^A$$

the canonical structure via the antibracket for any field variables X and Y, we define

$$(X, Y) \equiv \frac{\partial^r X}{\partial \phi^A} \frac{\partial^l Y}{\partial \phi^*_A} - \frac{\partial^r X}{\partial \phi^*_A} \frac{\partial^l Y}{\partial \phi^A}$$

$$(\bar{S}_{cl}, \bar{S}_{cl}) = 2(\delta S_{cl} + \phi^*_A \delta^2 \phi^A)$$

Classical master equation (CME): $(\bar{S}_{cl}, \bar{S}_{cl}) = 0 \Leftrightarrow$ action invariance *and* the nilpotency.

Generalize the consideration for $\bar{S}[\phi,\phi^*]$ that defines a quantum system via the functional integration over ϕ. Under the BRST transformation of fields

$$\delta\phi^A \equiv (\phi^A, \bar{S}) = \frac{\partial^l \bar{S}}{\partial\phi^*_A},$$

the changes of the action and the functional measure are summed up to the quantum master operator:

$$\bar{\Sigma}[\phi,\phi^*] \equiv \frac{\partial^r \bar{S}}{\partial\phi^A}\frac{\partial^l \bar{S}}{\partial\phi^*_A} - \frac{\partial^r}{\partial\phi^A}\delta\phi^A = \frac{1}{2}(\bar{S},\ \bar{S}) - \Delta\bar{S}\,,$$

$$\Delta \equiv (-)^{\epsilon_A+1}\frac{\partial^r}{\partial\phi^A}\frac{\partial^r}{\partial\phi^*_A}\,.$$

The system is BRST invariant quantum mechanically if the two contributions cancel:

$$\bar{\Sigma}[\phi, \phi^*] = 0 \ . \quad \text{(QME)}$$

The quantum BRST transformation as

$$\delta_Q X \equiv (X, \bar{S}) - \Delta X$$

We have two important algebraic identities without assuming QME:

$$\delta_Q \bar{\Sigma}[\phi, \phi^*] = 0 \, ,$$

$$\delta_Q^2 X = (X, \bar{\Sigma}[\phi, \phi^*]) \, .$$

The quantum BRST transformation is nilpotent if and only if QME holds.

Also useful to remember that QME = WT identity + nilpotency

2.2 The effective average action $\bar{\Gamma}_{B,\Lambda}$

$$\exp\Big(-\bar{W}_{B,\Lambda}[J,\phi^*]\Big) \equiv \int \mathcal{D}\phi \exp\Big(-\frac{1}{2}\phi\cdot(K_0-K)D\cdot\phi - \bar{S}_{I,B}[\phi,\phi^*] - K_0^{-1}J\cdot\phi\Big)$$

- The modes with $\Lambda^2 < p^2 < \Lambda_0^2$ contribute to the path integral since the factor $K_0 - K \sim 1$ for $\Lambda^2 < p^2 < \Lambda_0^2$.
- $\bar{S}_{B,\Lambda} \equiv \frac{1}{2}\phi\cdot(K_0-K)D\cdot\phi + \bar{S}_{I,B}$ differs from $\bar{S}_B$ only in the kinetic term. In $\Lambda \to 0$, two actions are the same.

Define the effective average action as

$$\bar{\Gamma}_{B,\Lambda}[\varphi_\Lambda,\phi^*] \equiv \bar{W}_{B,\Lambda}[J,\phi^*] - K_0^{-1}J\cdot\varphi_\Lambda, \qquad \varphi_\Lambda(p) \equiv K_0(p)\frac{\partial^l \bar{W}_{B,\Lambda}[J,\phi^*]}{\partial J(-p)}$$

The limit of $\Lambda \to 0$ leads to the ordinary generating functional and effective action

$$\lim_{\Lambda\to 0}\bar{W}_{B,\Lambda}[J,\phi^*] = \bar{W}_B[J,\phi^*]\ , \qquad \lim_{\Lambda\to 0}\bar{\Gamma}_{B,\Lambda}[\varphi_\Lambda,\phi^*] = \bar{\Gamma}_B[\varphi,\phi^*]$$

where $\varphi \equiv \lim_{\Lambda\to 0}\varphi_\Lambda$

QME and the modified ST identity

The path integral average of the QM operator $\bar{\Sigma}_B[\phi, \phi^*]$

$$
\begin{aligned}
\bar{\Sigma}^{1PI}_{B,\Lambda}[\varphi_\Lambda, \phi^*] &\equiv \exp[\bar{W}_{B,\Lambda}[J, \phi^*]] \int \mathcal{D}\phi \bar{\Sigma}_{\Lambda_0}[\phi, \phi^*] \exp\Big(-\bar{S}_{B,\Lambda}[\phi, \phi^*] - K_0^{-1} J \cdot \phi\Big) \\
&= \frac{\partial^r \bar{\Gamma}_{B,\Lambda}}{\partial \varphi^A_\Lambda} \frac{\partial^l \bar{\Gamma}_{B,\Lambda}}{\partial \phi^*_A} + [R_\Lambda]_{BA}\Big(-(\bar{\Gamma}^{(2)})^{-1}_{B,\Lambda} \frac{\partial^l}{\partial \varphi^C_\Lambda} \frac{\partial^l \bar{\Gamma}_{B,\Lambda}}{\partial \phi^*_A} + \varphi^B_\Lambda \frac{\partial^l \bar{\Gamma}_{B,\Lambda}}{\partial \phi^*_A}\Big)
\end{aligned}
$$

$$
[R_\Lambda(p)]_{BA} \equiv D_{BA}(p)\left(\frac{1}{K_0 - K} - \frac{1}{K_0}\right) \qquad \to 0 \ \text{ as } \Lambda \to 0
$$

- $\bar{\Sigma}_{\Lambda_0}[\phi, \phi^*] = 0$ implies the presence of a symmetry.
- $\bar{\Sigma}^{1PI}_{B,\Lambda} = 0$ is the modified Slavnov-Taylor identity. (Ellwanger 1994)

Since $R_\Lambda \to 0$ in the limit of $\Lambda \to 0$, we find

$$\bar{\Sigma}_B^{1PI} \equiv \lim_{\Lambda\to 0} \bar{\Sigma}_{B,\Lambda}^{1PI} = \frac{\partial^r \bar{\Gamma}_B}{\partial \varphi^A} \frac{\partial^l \bar{\Gamma}_B}{\partial \phi_A^*} \ .$$

Vanishing of the last expression is the Zinn-Justin equation for the effective action $\bar{\Gamma}_B$.

2. Anomaly

Where to find an anomaly?

- The vanishing of the WT operator implies symmetry: $\Sigma \neq 0$ for an anomalous theory.

- The WT operator Σ evolves as a composite operator.

- ghost number of $\Sigma = 1$

- We will see in an example: $\Sigma_\Lambda[\Phi] \to \mathrm{ghost} \times \mathrm{anomaly} \quad \mathrm{as}\ \Lambda \to \infty$.

- We also know the Zinn-Justin equation may be broken by an anomaly in a similar manner.

3.1 QM operator as anomaly composite operator

The QM operator is a composite operator

$$-\Lambda\frac{\partial}{\partial\Lambda}\bar{\Sigma}_\Lambda = \mathcal{D}\bar{\Sigma}_\Lambda$$

In the UV limit, it becomes a ghost times an anomlay

$$\lim_{\Lambda\to\infty}\lim_{\Lambda_0\to\infty}\bar{\Sigma}_\Lambda = \mathcal{A}[\phi]$$

where ϕ is the bare field. This will be calculated explicitly later for a simple example.

Also known that

$$\bar{\Sigma}_B^{1PI} \equiv \lim_{\Lambda\to 0}\bar{\Sigma}_{B,\Lambda}^{1PI} = \frac{\partial^r\bar{\Gamma}_B}{\partial\varphi^A}\frac{\partial^l\bar{\Gamma}_B}{\partial\phi^*_A} = \mathcal{A}'[\varphi]$$

$\mathcal{A}'$ satisfies the Wess-Zumino condition: $(\mathcal{A}', \bar{\Gamma}_B)_{\varphi,\phi^*} = 0$

- Here, the antibracket is defined w.r.t. φ, ϕ^*.

In the following we explain:

- The form of QM operator with finite Λ and its relation to $\mathcal{A}$ and $\mathcal{A}'$.

- Algebraic relations satisfied by the QM operator and the effective average action.

- An explicit calculation of $\mathcal{A}$

The form of QM operator with finite Λ and its relation to $\mathcal{A}$ and $\mathcal{A}'$.

The relation between $\bar{\Sigma}_\Lambda$ and $\bar{\Sigma}^{1PI}_{B,\Lambda}$,

$$\bar{\Sigma}_\Lambda[\Phi, \Phi^*] = \bar{\Sigma}^{1PI}_{B,\Lambda}[\varphi_\Lambda, \phi^*] \;, \quad K_0 \phi^*_A = K \Phi^*_A$$

$$\varphi^A_\Lambda = \frac{K_0}{K}\Phi^A + (K_0 - K)(D^{-1})^{AB} \frac{\partial^l \bar{S}_\Lambda}{\partial \Phi^B} \;.$$

- $\bar{\Sigma}_\Lambda$ is a functional of φ_Λ and ϕ^*, where φ_Λ is a composite operator by itself.

$$\bar{\Sigma}_\Lambda = \bar{\mathcal{A}}[\varphi_\Lambda, \phi^*; \Lambda] \;.$$

The QM operator depends on Λ via φ_Λ and coefficients.

Consider the flow equation for $\bar{\mathcal{A}}[\varphi_\Lambda, \phi^*; \Lambda]$,

$$-\Lambda \frac{\partial}{\partial \Lambda} \bar{\mathcal{A}}[\varphi_\Lambda, \phi^*; \Lambda] = \mathcal{D} \bar{\mathcal{A}}[\varphi_\Lambda, \phi^*; \Lambda] \ ,$$

where

$$\mathcal{D} \equiv (D^{-1}\Delta)^{AB} \Big(\frac{\partial^l \bar{S}_{I,\Lambda}}{\partial \Phi^B} \frac{\partial^l}{\partial \Phi^A} + \frac{1}{2} \frac{\partial^l}{\partial \Phi^B} \frac{\partial^l}{\partial \Phi^A} \Big) \ .$$

Since φ_Λ is a composite operator by itself, the other scale dependence of $\bar{\mathcal{A}}[\varphi_\Lambda, \phi^*; \Lambda]$ follows the equation

$$\Big(-\Lambda \frac{\partial}{\partial \Lambda} \Big)' \bar{\mathcal{A}}[\varphi_\Lambda, \phi^*; \Lambda] = \mathcal{D}' \bar{\mathcal{A}}[\varphi_\Lambda, \phi^*; \Lambda],$$

where

$$\mathcal{D}' \equiv \frac{1}{2} (-)^{\epsilon_A + \epsilon_B(\epsilon_A + \epsilon_C)} (D^{-1}\Delta)^{AB} \Big(\frac{\partial^l \varphi_\Lambda^C}{\partial \Phi^A} \frac{\partial^l \varphi_\Lambda^D}{\partial \Phi^B} \Big) \frac{\partial^l}{\partial \varphi_\Lambda^D} \frac{\partial^l}{\partial \varphi_\Lambda^C} \ .$$

Make the loop expansion of $\bar{\mathcal{A}}$. Since there is no tree-level contribution, we find

$$\left(-\Lambda\frac{\partial}{\partial\Lambda}\right)'\bar{\mathcal{A}}^{(1)}[\varphi_\Lambda,\phi^*;\Lambda]=0$$

for the one-loop contribution. At the one-loop level, the scale dependence originates solely from φ_Λ .

Let us assume that the one-loop calculation is exact. $\bar{\mathcal{A}}[\varphi_\Lambda,\phi^*]$ is the functional such that $\lim_{\Lambda\to 0}\bar{\mathcal{A}}[\varphi_\Lambda,\phi^*]=\bar{\mathcal{A}}'[\varphi]$ and $\lim_{\Lambda\to\infty}\lim_{\Lambda_0\to\infty}\bar{\mathcal{A}}[\varphi_\Lambda,\phi^*]=\bar{\mathcal{A}}[\phi]$.

All the known facts are consistent with the following expression for the QM operator,

$$\bar{\Sigma}_\Lambda[\Phi,\Phi^*]=\mathcal{A}[\varphi_\Lambda]\ .$$

The scale dependence comes solely through the composite operator φ_Λ.

3.2 An algebraic relation

We will show the following relation for finite cutoffs Λ and Λ_0:

$$\left(\mathcal{A}_{B,\Lambda},\ \bar{\Gamma}_{B,\Lambda}\right)_{\varphi_\Lambda,\phi^*} = e^{\bar{W}_{B,\Lambda}} \int \mathcal{D}\phi \left(\delta'_Q \bar{\Sigma}_{B,\Lambda}\right) e^{-\bar{S}_{B,\Lambda} - K_0^{-1} J\cdot\phi} .$$

$\mathcal{A}_{B,\Lambda}$ stands for the quantity

$$\mathcal{A}_{B,\Lambda} \equiv \frac{\partial^r \bar{\Gamma}_{B,\Lambda}}{\partial \varphi_\Lambda^A} \frac{\partial^l \bar{\Gamma}_{B,\Lambda}}{\partial \phi_A^*} = e^{\bar{W}_{B,\Lambda}} \int \mathcal{D}\phi\ \bar{\Sigma}_{B,\Lambda} e^{-\bar{S}_{B,\Lambda} - K_0^{-1} J\cdot\phi} ,$$

δ'_Q and $\bar{\Sigma}_{B,\Lambda}$ are the BRST transformation and QM operator defined with the action $\bar{S}_{B,\Lambda}$ respectively:

$$\begin{aligned} \delta'_Q X &\equiv (X, \bar{S}_{B,\Lambda}) - \Delta X , \\ \bar{\Sigma}_{B,\Lambda} &\equiv \frac{1}{2}(\bar{S}_{B,\Lambda}, \bar{S}_{B,\Lambda}) - \Delta \bar{S}_{B,\Lambda} . \end{aligned}$$

The difference between $\bar{S}_B$ and $\bar{S}_{B,\Lambda}$ vanishes in $\Lambda \to 0$. Therefore, in this limit,

$$\delta'_Q \to \delta_Q, \quad \bar{\Sigma}_{B,\Lambda} \to \bar{\Sigma}_B \ .$$

We also know that in the same limit,

$$\bar{W}_{B,\Lambda} \to \bar{W}_B, \quad \bar{\Gamma}_{B,\Lambda} \to \bar{\Gamma}_B$$

Sending $\Lambda \to 0$, we find

$$\Big(\frac{\partial^r \bar{\Gamma}_B}{\partial \varphi^A}\frac{\partial^l \bar{\Gamma}_B}{\partial \phi^*_A},\ \bar{\Gamma}_B\Big)_{\varphi,\phi^*} = e^{\bar{W}_B} \int \mathcal{D}\phi \Big(\delta_Q \bar{\Sigma}_B\Big) e^{-\bar{S}_B - K_0^{-1} J\cdot\phi} \ .$$

This relates the Wess-Zumino conditon and the algebraic relation on the QM operator.

3.3 Evaluation of WT operator for $U(1)_V \times U(1)_A$ gauge theory:

WT identities and BRST transformations

Two sets of gauge sector: $(A_\mu, h_V, c_V, \bar{c}_V)$ and $(B_\mu, h_A, c_A, \bar{c}_A)$

$$S_{\Lambda_0}[\phi] = \frac{1}{2}\phi K_0^{-1} \cdot D \cdot \phi + S_{I,\Lambda_0}[\phi]$$

$$\begin{aligned}
\frac{1}{2}\phi K_0^{-1} \cdot D \cdot \phi &= \int_p K_0^{-1}\Big[\bar{\psi}(-p)\not{p}\psi(p) \\
&+ \frac{1}{2}A_\mu(p^2\delta_{\mu\nu} - p_\mu p_\nu)A_\nu - h_V\Big(ip\cdot A + \frac{\xi_V}{2}h_V\Big) + \bar{c}_V i p^2 c_V \\
&+ \frac{1}{2}B_\mu(p^2\delta_{\mu\nu} - p_\mu p_\nu)B_\nu - h_A\Big(ip\cdot B + \frac{\xi_A}{2}h_A\Big) + \bar{c}_A i p^2 c_A\Big]
\end{aligned}$$

We will explain our calculation for the axial transformation.

The BRST transformation for axial gauge symmetry

$$\begin{aligned}
\delta B_\mu(p) &= -iK_0(p)p_\mu c_A(p), \quad \delta\bar{c}_A = iK_0(p)h_A(p), \quad \delta c_A(p) = \delta h_A(p) = 0 \\
\delta\psi(p) &= -ie_A K_0(p)\int_k \gamma_5\psi(p-k)c_A(k), \\
\delta\bar{\psi}(-p) &= -ie_A K_0(p)\int_k \bar{\psi}(-p-k)c_A(k)\gamma_5\,,
\end{aligned}$$

$$\delta A_\mu(p) = \delta h_V(p) = \delta c_V(p) = \delta\bar{c}_V(p) = 0$$

WT operator for axial transformation:

$$\Sigma_\Lambda^A = \bar{\Sigma}_\Lambda^A|_{\Phi^*=0} = \frac{\partial^r S_\Lambda}{\partial\Phi^A}\delta\Phi^A + \frac{\partial^l}{\partial\Phi^A}\delta\Phi^A$$

One-loop contributions

$$\Sigma_\Lambda^{A(1)} = \frac{\partial^r S_\Lambda^{(1)}}{\partial\Phi^A}\delta\Phi^A + \frac{\partial^r S_\Lambda}{\partial\Phi^A}\Big[\delta\Phi^A\Big]^{(1)} + \Big[\frac{\partial^l}{\partial\Phi^A}\delta\Phi^A\Big]^{(1)}$$

- The high mometum modes ($\Lambda^2 < p^2 < \Lambda_0^2$) produce the one-loop action $S_\Lambda^{(1)}$.

- $K^{-1}\delta\Phi^A$ evolves as a composite operator. For example,

$$\begin{aligned} \delta\psi(p) &= -ie_A K(p)\int_k \gamma_5[\psi(p-k)]_\Lambda c_A(k) \\ \big[\psi(q)\big]_\Lambda &= \psi(q) + \frac{K_0(q)-K(q)}{\not p}\frac{\partial^l S_{I,\Lambda}}{\partial\bar\psi(-q)} \end{aligned}$$

  - $K(p)$ in $\delta\phi(p)$ restricts the momentum, $p^2 < \Lambda^2$
  - Propagators in $S_{I,\Lambda}$ have $K_0 - K$ to allow only high momentum modes to propagate.

The momentum integration in

$$\int_p \frac{\partial^l}{\partial\psi(p)}\Big[\delta\psi(p)\Big]^{(0)} + \int_p \frac{\partial^l}{\partial\bar{\psi}(p)}\Big[\delta\bar{\psi}(p)\Big]^{(0)}$$

is restricted to $p^2 \sim \Lambda^2$.

- It was found that the above terms produce non-zero contributions to $\Sigma_\Lambda^{A(1)}$ in $\Lambda, \Lambda_0 \to \infty$ limit.

$$\begin{aligned}
\Sigma_\Lambda^{A(1)} &\to \frac{e_A}{48\pi^2}\int_x c_A(x)\ \epsilon_{\mu\nu\rho\sigma}\left(e_V^2 F_{\mu\nu}^V(x) F_{\rho\sigma}^V(x) + e_A^2 F_{\mu\nu}^A(x) F_{\rho\sigma}^A(x)\right) \\
\Sigma_\Lambda^{V(1)} &\to 2 \times \frac{e_A e_V^2}{48\pi^2}\int_x \ c_V(x)\epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}^A(x) F_{\rho\sigma}^V(x)
\end{aligned}$$

Adding a counter term to the Wilson action, we may keep the vector symmetry.

Summary

- $\bar{\Sigma}_\Lambda$ changes along the flow as a composite operator.

- When a gauge symmetry exits, we find an expression, $\bar{\Sigma}_\Lambda = 0$.
  - $\bar{\Sigma}_\Lambda|_{\phi^*=0} = 0$ is the Ward-Takahashi identity (cutoff dependent).

- For anomalous symmetry, $\bar{\Sigma}_\Lambda = \mathcal{A}$ is the anomaly composite operator.
  - If the one loop result is exact, the cutoff dependence of $\mathcal{A}$ comes solely from φ_Λ
  - Discussed the Wess-Zumino condition
  - An explicit calculation is explained.