# $N_f = 2$ linear sigma model in presence of axial anomaly from Functional Renormalization Group

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# Outline of the talk

1. The $O(4)$-conjecture
   - The universality hypothesis...
   - ...applied to two-flavor QCD
2. Functional Renormalization Group (FRG) Method
   - Local Potential Approximation (LPA)
3. Two-flavor linear sigma model in absence of anomaly, LPA
4. Infrared fixed points in presence of the anomaly
5. Conclusion

# The universality hypothesis

Experimental facts for (simple) systems:
power law behavior near critical point

- order parameter $\propto |T - T_c|^\beta$
- specific heat: $C \propto |T - T_c|^\alpha$
- susceptibility: $\chi \propto |T - T_c|^{-\gamma}$
- correlation length: $\xi \propto |T - T_c|^{-\nu}$
- etc.

scaling relations

- $\alpha = 2 - \nu D$
- $\beta = \frac{\nu}{2}(D - 2 + \eta)$
- $\gamma = \nu(2 - \eta)$
- etc.

# The universality hypothesis

**Universality hypothesis (R.B.Griffiths, 1970)** The critical exponents (primarily) depend on three properties:

1. the lattice dimensionality (i.e. the spatial dimensionality $D$ of the system)
2. the spin dimensionality (i.e. the number of components of the order parameter)
3. the range of interaction between the spins on the lattice

# The universality hypothesis

**Universality hypothesis (Bruce, 1980)** The critical exponents depend on at least four properties:

1. the spatial dimensionality $D$ of the system
2. the number of components of the order parameter
3. the symmetry properties of the order parameter
4. the range and angular dependence of the interaction

# The universality hypothesis

Wilson: stable IR fixed point $\Rightarrow \exists$ 2nd order phase transition

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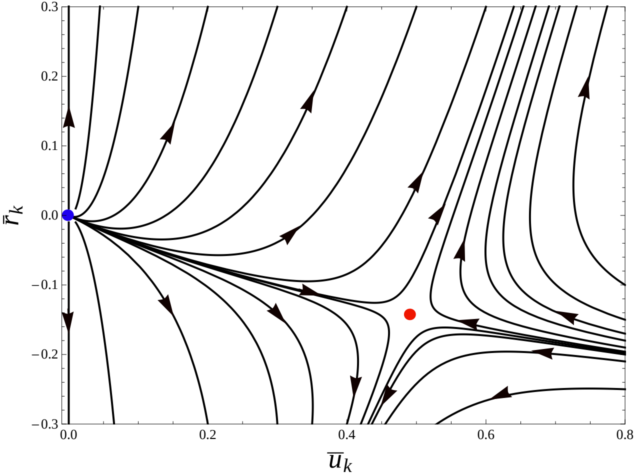

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$$U(N_f = 2)_L \times U(N_f = 2)_R \simeq U(1)_A \times SU(2)_A \times SU(2)_V \times U(1)_V$$

**Most general renormalizable Lagrangian for a complex 2x2 matrix field $\Phi$ (i.e. 8 d.o.f.) invariant under $U(2)_A \times U(2)_V$**

$$\mathscr{L} = \frac{1}{2} Tr(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + \frac{1}{2} m_\Phi^2 \, Tr \Phi^\dagger \Phi$$
$$+ \frac{\pi^2}{3} g_1 (Tr \Phi^\dagger \Phi)^2 + \frac{\pi^2}{3} g_2 \, Tr(\Phi^\dagger \Phi)^2 \; ,$$

where $\Phi = (\sigma + i\eta)\, t_0 + \vec{t} \cdot (\vec{a} + i\vec{\pi})$.

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# The $O(4)$-conjecture

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$$\Phi = (\sigma + i\eta)\, t_0 + \vec{t} \cdot (\vec{a} + i\vec{\pi})$$

$$SU(2) \times SU(2)/Z(2) \simeq SO(4) \sim O(4)$$
$$\Phi_1 = \sigma t_0 + i\vec{t} \cdot \vec{\pi} \ , \quad \Phi_2 = i\eta t_0 + \vec{t} \cdot \vec{a}$$

# FRG in Local Potential Approximation (LPA)

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$$U_k = \frac{1}{2} m_{\Phi,k}^2 \, Tr\Phi^\dagger\Phi + \frac{\pi^2}{3} g_{1,k} (Tr\Phi^\dagger\Phi)^2 + \frac{\pi^2}{3} g_{2,k} \, Tr(\Phi^\dagger\Phi)^2$$

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**Wetterich equation, LPA with Litim regulator**

$$\partial_k U_k[\phi_i] = K_d k^{d+1} \sum_i \frac{1}{E_i^2}$$

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$$E_i^2 \equiv k^2 + M_i^2 \ , \quad M_{ij} \equiv \frac{\partial^2 U_k}{\partial\phi_i \partial\phi_j} \ , \quad i,j = 1, ..., 8$$

# $U(2)_V \times U(2)_A$ linear sigma model

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New:
$c\left(\det\Phi^\dagger + \det\Phi\right) + y\left(\det\Phi^\dagger + \det\Phi\right) Tr\Phi^\dagger\Phi + z\left[\left(\det\Phi^\dagger\right)^2 + \left(\det\Phi\right)^2\right]$
8 d.o.f.

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$$+ c \left( \det \Phi^\dagger + \det \Phi \right) + y \left( \det \Phi^\dagger + \det \Phi \right) Tr \Phi^\dagger \Phi$$

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$$
\begin{aligned}
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\end{aligned}
$$

$$
\lambda_1 \equiv 4! \frac{\pi^2}{3} \left( g_1 + \frac{1}{2} g_2 \right) \ , \quad \lambda_2 \equiv 2 \frac{\pi^2}{3} g_2 \ , \quad \mu^2 \equiv m_\Phi^2
$$

$$
\Phi = (\sigma + i\eta)\, t_0 + \vec{t} \cdot (\vec{a} + i\vec{\pi})
$$

# $U(2)_V \times SU(2)_A$ linear sigma model

$$U(\varphi, \xi, \alpha) = \frac{1}{2}\mu^2\varphi + \frac{1}{4!}\lambda_1\varphi^2 + \lambda_2\xi + c\alpha + y\alpha\varphi + z\beta$$

$$\varphi \equiv \sigma^2 + \vec{\pi}^2 + \eta^2 + \vec{a}^2 \ , \quad \xi = (\sigma^2 + \vec{\pi}^2)(\eta^2 + \vec{a}^2) - (\sigma\eta - \vec{\pi}\cdot\vec{a})^2 \ ,$$

$$\alpha \equiv \sigma^2 - \eta^2 + \vec{\pi}^2 - \vec{a}^2 \ ,$$

$$\beta \equiv \frac{1}{2}\left(\eta^2 + \vec{a}^2 - \sigma^2 - \vec{\pi}^2 - 2\vec{a}\cdot\vec{\pi} + 2\eta\sigma\right) \ \times$$

$$\times \ \left(\eta^2 + \vec{a}^2 - \sigma^2 - \vec{\pi}^2 + 2\vec{a}\cdot\vec{\pi} - 2\eta\sigma\right) \ .$$

# Results $c = y = z = 0$

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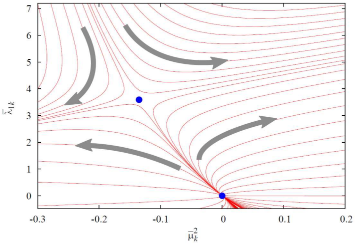
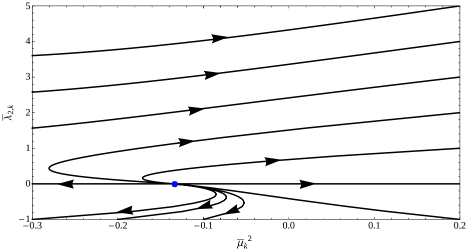



[Fukushima et al.]




unstable direction

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$$(S_{ij}) \equiv \left( \frac{\partial (k \partial_k \bar{p}_i)}{\partial \bar{p}_j} \right) \Big|_{\bar{p}=\bar{p}*}$$

$$\xi \propto |T - T_c|^{-\nu}$$

$$FP_0 \equiv (\bar{\mu}_*^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}) = (-0.135, 3.591, 0)$$

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$$U = \frac{1}{2}\mu^2 \sum_{n=1}^{N} \phi_i^2 + \frac{\lambda_1}{24} \left( \sum_{n=1}^{N} \phi_i^2 \right)^2$$

| $N$ | $\frac{1}{2}\bar{\mu}^2_*$ | $\bar{\lambda}_{1*}$ | $\nu = -1/y_1$ | $y_2$ |
|---|---|---|---|---|
| 1 | -0.03846 | 7.76271 | 0.54272=-1/-1.84256 | 1.1759 |
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$$m_1^2 \equiv \frac{1}{2}\mu^2 + c\ ,\ m_2^2 = \frac{1}{2}\mu^2 - c\ ,$$

$$\delta \equiv \frac{1}{12}\lambda_1 + \lambda_2\ ,\ \lambda_\sigma \equiv \frac{\lambda_1}{24} + y\ ,\ \lambda_a \equiv \frac{\lambda_1}{24} - y\ .$$

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Ambiguous flow equations for $c \neq 0$, $y = 0$, $z = 0$

Our results for $c \neq 0$, $y = 0$, $z = 0$ assume $\sigma \gg a^1$

# Results $c \neq 0$, $y \neq 0$, $z = 0$

$$\bar{m}^2_{i,k} \equiv k^{-2} m^2_{i,k} \quad \text{etc.}$$

$$A \equiv (\bar{m}^2_{1*}, \bar{m}^2_{2*}, \bar{\delta}_*, \bar{\lambda}_{\sigma*}, \bar{\lambda}_{a*}) = (-0.0599, -0.0083, 0.0406, 0.2067, 0.0050)$$

stability matrix eigenvalues: $\{-1.99923, -1.78902, 1.28814, -0.942028, -0.57213\}$

$$A' \equiv (\bar{m}^2_{1*}, \bar{m}^2_{2*}, \bar{\delta}_*, \bar{\lambda}_{\sigma*}, \bar{\lambda}_{a*}) = (-0.0083, -0.0599, 0.0406, 0.0050, 0.2067)$$

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$$FP_0 \equiv (\bar{m}^2_{1*}, \bar{m}^2_{2*}, \bar{\delta}_*, \bar{\lambda}_{\sigma*}, \bar{\lambda}_{a*}) = (-0.0676, -0.0676, 0.2993, 0.1496, 0.1496)$$

stability matrix eigenvalues: $\{-1.98804, -1.71971, 1.34471, 0.613041, -0.25\}$

$$B \equiv (\bar{m}^2_{1*}, \bar{m}^2_{2*}, \bar{\delta}_*, \bar{\lambda}_{\sigma*}, \bar{\lambda}_{a*}) = (0.4343, -0.8398, 22.4235, -31.1684, 1.4998)$$

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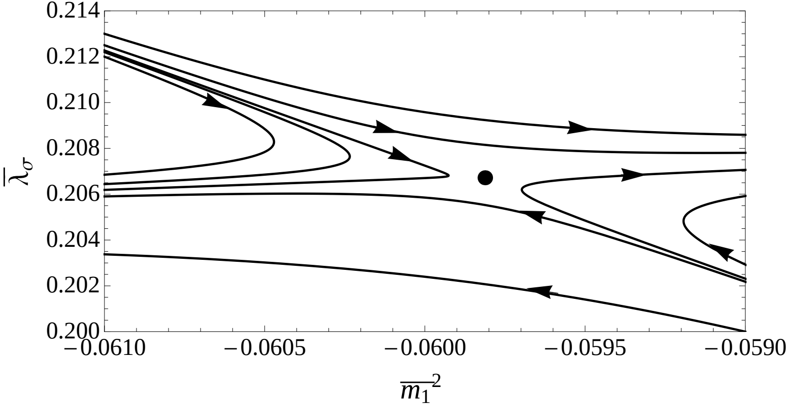
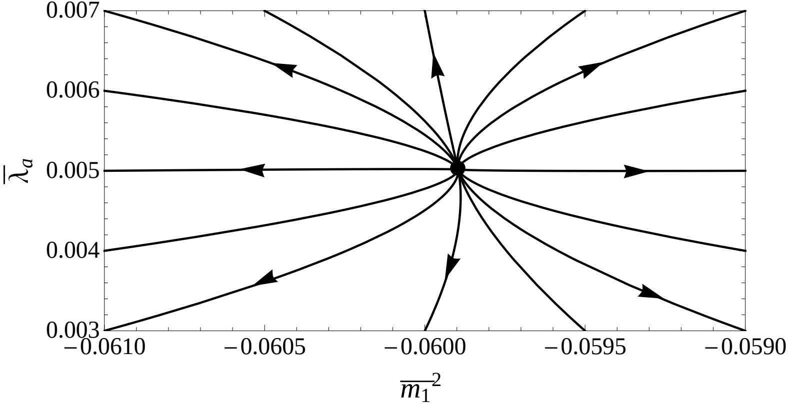
# Results $c \neq 0$, $y \neq 0$, $z = 0$, Fixed Point $A$



# Results $c \neq 0$, $y \neq 0$, $z = 0$, Fixed Point $A$



# Results $c \neq 0,\ y = 0,\ z = 0$

$$FP_0 \equiv (\bar{\mu}_*^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{c}_*) = (-0.135, 3.591, 0, 0)$$

stability matrix eigenvalues: $\{-1.71971, 1.34471, -0.25, -1.875\}$

$$FP_1 \equiv (\bar{\mu}_*^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{c}_*) = (-0.8512, 6.0564, -0.5797, 0.21699)$$

stability matrix eigenvalues: $\{-23.4681, -15.048, 6.752, -1.66877\}$

$$FP_2 \equiv (\bar{\mu}_*^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{c}_*) = (-1.1683, 1.8341, 0.8484, 0.21675)$$

stability matrix eigenvalues:

$$\{29.4962, -2.7835 + 9.9434i, -2.7835 - 9.9434i, -1.06612\}$$

# Results $c \neq 0$, $y = 0$, $z = 0$

$$FP_0 \equiv (\bar{\mu}_*^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{c}_*) = (-0.135, 3.591, 0, 0)$$

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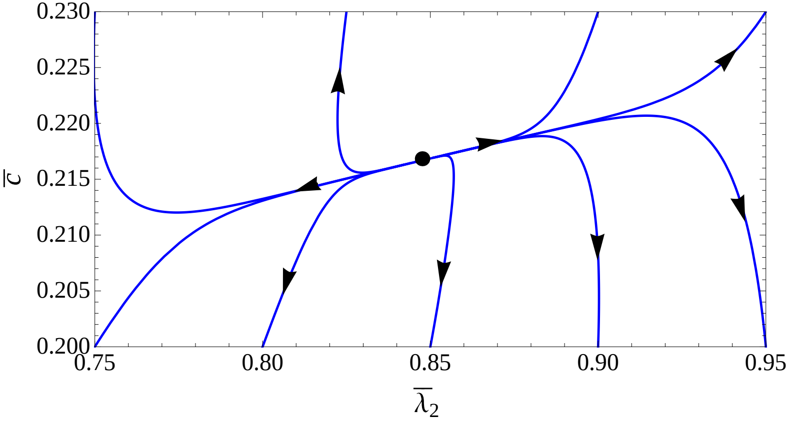
| $O(N)$ | $\frac{1}{2}\bar{\mu}_*^2$ | $\bar{\lambda}_{1*}$ | $\nu = -1/y_1$ | $y_2$ |
|---|---|---|---|---|
| 1 | -0.03846 | 7.76271 | 0.54272=-1/-1.84256 | 1.1759 |
| 2 | -0.04545 | 6.67366 | 0.55149=-1/-1.81327 | 1.21327 |
| 4 | -0.05556 | 5.1988 | 0.564751=-1/ -1.77069 | 1.27069 |
| 8 | -0.06757 | 3.59143 | 0.581495=-1/-1.71971 | 1.34471 |

# Results $c \neq 0$, $y = 0$, $z = 0$, Fixed Point $FP_2$

The only unstable plane:



# Results $c \neq 0$, $y \neq 0$, $z \neq 0$

$$U(\varphi, \xi, \alpha) = \frac{1}{2}\mu^2\varphi + \frac{1}{4!}\lambda_1\varphi^2 + \lambda_2\xi + c\alpha + y\alpha\varphi + z\beta$$

$$\partial_k U_k[\phi_i] \Big|_{\sigma, a_1, \eta, \pi_1 \neq 0} = K_d k^{d+1} \sum_i \frac{1}{E_i^2} \Big|_{\sigma, a_1, \eta, \pi_1 \neq 0}$$

## Results $c \neq 0$, $y \neq 0$, $z \neq 0$

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$$U_k = a_1^2 m_{2,k}^2 + \eta^2 m_{2,k}^2 + \sigma^2 m_{1,k}^2 + \pi_1^2 m_{1,k}^2 + \left(a_1^4 + \eta^4\right)\lambda_{a\eta} + \left(\sigma^4 + \pi_1^4\right)\lambda_{\sigma\pi}$$
$$+\delta_1\left(\pi_1^2 a_1^2 + \eta^2\sigma^2\right) + \delta_2 a_1^2\eta^2 + \delta_0\left(a_1^2\sigma^2 + \pi_1^2\eta^2\right) + \kappa\pi_1 a_1\eta\sigma + \delta_3\pi_1^2\sigma^2 \ ,$$

$$\lambda_{a\eta} \equiv \left(\frac{\lambda_1}{24} - y + \frac{z}{2}\right) \ , \quad \lambda_{\sigma\pi} \equiv \left(\frac{\lambda_1}{24} + y + \frac{z}{2}\right) \ , \quad \delta_0 \equiv \left(\frac{\lambda_1}{12} + \lambda_2 - z\right) \ ,$$
$$\delta_1 \equiv \left(\frac{\lambda_1}{12} - 3z\right) \ , \quad \delta_2 \equiv \left(\frac{\lambda_1}{12} + z - 2y\right) \ , \quad \delta_3 \equiv \left(\frac{\lambda_1}{12} + z + 2y\right) \ , \quad \kappa \equiv 4z + 2\lambda_2 \ .$$

*Note that*

$$\delta_3 = 2\lambda_{\sigma\pi} \ , \quad \delta_2 = 2\lambda_{a\eta} \ , \quad \delta_0 = \delta_1 + \frac{\kappa}{2} \ , \quad y = \frac{\lambda_{\sigma\pi}}{2} - \frac{\lambda_{a\eta}}{2} \ ,$$
$$z = -\frac{\delta_1}{4} + \frac{\lambda_{\sigma\pi}}{4} + \frac{\lambda_{a\eta}}{4} \ , \quad \lambda_1 = 3\delta_1 + 9\lambda_{a\eta} + 9\lambda_{\sigma\pi} \ , \quad \lambda_2 = \frac{\delta_1}{2} + \frac{\kappa}{2} - \frac{\lambda_{\sigma\pi}}{2} - \frac{\lambda_{a\eta}}{2} \ .$$

# Results $c \neq 0$, $y \neq 0$, $z \neq 0$

$$F_1 \equiv (\bar{m}^2_{1*}, \bar{m}^2_{2*}, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.05556, -0.05556, 3.89910, -0.21662, 0, -0.10831)$$

$$\text{stability matrix eigenvalues: } \{-2, -1.77069, 1.27069, -0.66667, 0, 0\}$$

$$F_3 \equiv (\bar{m}^2_{1*}, \bar{m}^2_{2*}, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.05556, 0, 1.94955, -0.10831, 0.10831, 0.05415)$$

$$\text{stability matrix eigenvalues: } \{-2, -1.77069, 1.27069, -1, -0.83333, -0.5\}$$

$$F_5 \equiv (\bar{m}^2_{1*}, \bar{m}^2_{2*}, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.05556, -0.05556, 3.89910, -0.21662, 0, 0.10831)$$

$$\text{stability matrix eigenvalues: } \{-1.77069, -1.77069, 1.27069, 1.27069, -0.66667, 0\}$$

$$FP_0 \equiv (\bar{m}^2_{1*}, \bar{m}^2_{2*}, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.06757, -0.06757, 3.59143, 0, 0, 0)$$

$$\text{stability matrix eigenvalues: } \{-1.98804, -1.71971, 1.34471, 0.61304, -0.25000, -0.25000\}$$

$$F_{IRS} \equiv (\bar{m}^2_{1*}, \bar{m}^2_{2*}, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.334, -1.347, -152.996, 14.242, 8.581, -4.155)$$

$$\text{stability matrix eigenvalues:}$$

$$\{29.6235, 14.1 + 4.2i, 14.1 - 4.2i, 0.9 + 9.6i, 0.9 - 9.6i, -1.17232\}$$

(other fixed points not relevant for discussion)

# Results $c \neq 0$, $y \neq 0$, $z \neq 0$

$$F_1 \equiv (\bar{m}^2_{1*}, \bar{m}^2_{2*}, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.05556, -0.05556, 3.89910, -0.21662, 0, -0.10831)$$

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stability matrix eigenvalues: $\{-1.77069, -1.77069, 1.27069, 1.27069, -0.66667, 0\}$

$$FP_0 \equiv (\bar{m}^2_{1*}, \bar{m}^2_{2*}, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.06757, -0.06757, 3.59143, 0, 0, 0)$$

stability matrix eigenvalues: $\{-1.98804, -1.71971, 1.34471, 0.61304, -0.25000, -0.25000\}$

$$F_{IRS} \equiv (\bar{m}^2_{1*}, \bar{m}^2_{2*}, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.334, -1.347, -152.996, 14.242, 8.581, -4.155)$$

stability matrix eigenvalues:

$$\{29.6235, 14.1 + 4.2i, 14.1 - 4.2i, 0.9 + 9.6i, 0.9 - 9.6i, -1.17232\}$$

IR stable, BUT: one (rescaled) mass matrix eigenvalue negative $\Rightarrow$ reject $F_{IRS}$

(other fixed points not relevant for discussion)

# Results $c \neq 0$, $y \neq 0$, $z \neq 0$

$$F_1 \equiv (\bar{m}^2_{1*}, \bar{m}^2_{2*}, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.05556, -0.05556, 3.89910, -0.21662, 0, -0.10831)$$

stability matrix eigenvalues: $\{-2, -1.77069, 1.27069, -0.66667, 0, 0\}$

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| $O(N)$ | $\frac{1}{2}\bar{\mu}^2_*$ | $\bar{\lambda}_{1*}$ | $\nu = -1/y_1$ | $y_2$ |
|---|---|---|---|---|
| 1 | -0.03846 | 7.76271 | 0.54272=-1/-1.84256 | 1.1759 |
| 2 | -0.04545 | 6.67366 | 0.55149=-1/-1.81327 | 1.21327 |
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# Results $c \neq 0$, $y \neq 0$, $z \neq 0$

$$F_1 \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.05556, -0.05556, 3.89910, -0.21662, 0, -0.10831)$$

stability matrix eigenvalues: $\{-2, -1.77069, 1.27069, -0.66667, 0, 0\}$

Comparison with $c \neq 0,\ y \neq 0,\ z = 0$ :

$$A \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\delta}_*, \bar{\lambda}_{\sigma *}, \bar{\lambda}_{a*}) = (-0.0599, -0.0083, 0.0406, 0.2067, 0.0050)$$

stability matrix eigenvalues: $\{-1.99923, -1.78902, 1.28814, -0.942028, -0.57213\}$

$$A' \equiv (\bar{m}_{1*}^2, \bar{m}_{2*}^2, \bar{\delta}_*, \bar{\lambda}_{\sigma *}, \bar{\lambda}_{a*}) = (-0.0083, -0.0599, 0.0406, 0.0050, 0.2067)$$

stability matrix eigenvalues: $\{-1.99923, -1.78902, 1.28814, -0.942028, -0.57213\}$

| $O(N)$ | $\frac{1}{2}\bar{\mu}_*^2$ | $\bar{\lambda}_{1*}$ | $\nu = -1/y_1$ | $y_2$ |
|---|---|---|---|---|
| 1 | -0.03846 | 7.76271 | 0.54272=-1/-1.84256 | 1.1759 |
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# Strong anomaly limit $m^2_{2,k} \to \infty$

Jungnickel and Wetterich **Phys.Rev.,D53:5142** (1996): limit $c \to -\infty$ should be closer to reality.
Quark-meson model: $k \sim 600\ MeV$ ,
$|c| \sim (958\ MeV)^2$ (for $m_\eta = 958\ MeV$, $N_f = 2$).

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Quark-meson model: $k \sim 600\ MeV$ ,
$|c| \sim (958\ MeV)^2$ (for $m_\eta = 958\ MeV,\ \ N_f = 2$).

$$m^2_{2,k} = \frac{1}{2}\mu^2_k - c_k \to \infty \ , \ \text{note that}\ m^2_{1,k} \equiv \frac{1}{2}\mu^2_k + c_k \ .$$

# Strong anomaly limit $m^2_{2,k} \to \infty$, $c \neq 0$, $y = 0$, $z = 0$

$$(\bar{m}^2_{1*}, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}) = (-0.05556, 5.1988, -0.3713)$$

stability matrix eigenvalues: $\{-1.77069, 1.27069, -0.58333\}$ .

$$M^2_\sigma = 2/9\ , \quad M^2_{\pi_i} = 0\ , \quad M^2_\eta \to \infty\ , \quad M^2_{a_i} \to \infty\ .$$

| $O(N)$ | $\frac{1}{2}\bar{\mu}^2_*$ | $\bar{\lambda}_{1*}$ | $\nu = -1/y_1$ | $y_2$ |
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| 1 | -0.03846 | 7.76271 | 0.54272=-1/-1.84256 | 1.1759 |
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## Strong anomaly limit $m_{2,k}^2 \to \infty$, $c \neq 0$, $y = 0$, $z = 0$

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stability matrix eigenvalues: $\{-1.77069, 1.27069, -0.58333\}$ .

$$M_\sigma^2 = 2/9 \ , \quad M_{\pi_i}^2 = 0 \ , \quad M_\eta^2 \to \infty \ , \quad M_{a_i}^2 \to \infty \ .$$

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| 1 | -0.03846 | 7.76271 | 0.54272=-1/-1.84256 | 1.1759 |
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# Strong anomaly limit $m^2_{2,k} \to \infty$, $c \neq 0$, $y \neq 0$, $z \neq 0$

$$(\bar{m}^2_{1*}, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.05556, 1.94955, -0.10831, 0.10831, 0.05415)$$

stability matrix eigenvalues: $\{-1.77069, 1.27069, -1, -0.83334, -0.5\}$ .

$$M^2_\sigma = 2/9 \ , \quad M^2_{\pi_i} = 0 \ , \quad M^2_\eta \to \infty \ , \quad M^2_{a_i} \to \infty \ .$$

| $O(N)$ | $\frac{1}{2}\bar{\mu}^2_*$ | $\bar{\lambda}_{1*}$ | $\nu = -1/y_1$ | $y_2$ |
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# Strong anomaly limit $m^2_{2,k} \to \infty$, $c \neq 0$, $y \neq 0$, $z \neq 0$

$$(\bar{m}^2_{1*}, \bar{\lambda}_{1*}, \bar{\lambda}_{2*}, \bar{y}_*, \bar{z}_*) = (-0.05556, 1.94955, -0.10831, 0.10831, 0.05415)$$

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$$M^2_\sigma = 2/9 \ , \quad M^2_{\pi_i} = 0 \ , \quad M^2_\eta \to \infty \ , \quad M^2_{a_i} \to \infty \ .$$

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# Strong anomaly limit $m^2_{2,k} \to \infty$, $c \neq 0$, $y \neq 0$, $z = 0$

$$P_1 \equiv (\bar{m}^2_{1*}, \bar{\delta}_*, \bar{\lambda}_{\sigma *}, \bar{\lambda}_{a*}) = (0, 0, 0, 14.8044)$$
$$\text{stability matrix eigenvalues: } \{-2, -1, -1, 1\}\ ,$$
$$P_2 \equiv (\bar{m}^2_{1*}, \bar{\delta}_*, \bar{\lambda}_{\sigma *}, \bar{\lambda}_{a*}) = (-0.05556, 1.3680, 0.2166, 9.3595)$$
$$\text{stability matrix eigenvalues: } \{-1.77069, 1.27069, 0.886734, -0.628083\}\ ,$$
$$P_3 \equiv (\bar{m}^2_{1*}, \bar{\delta}_*, \bar{\lambda}_{\sigma *}, \bar{\lambda}_{a*}) = (-0.05556, 0.03166, 0.2166, 0.0050)$$
$$\text{stability matrix eigenvalues: } \{-1.77069, 1.27069, -0.961396, -0.579306\}\ .$$
$$M^2_\sigma = 2/9\ ,\quad M^2_{\pi_i} = 0\ ,\quad M^2_\eta \to \infty\ ,\quad M^2_{a_i} \to \infty\ .$$

| $O(N)$ | $\frac{1}{2}\bar{\mu}^2_*$ | $\bar{\lambda}_{1*}$ | $\nu = -1/y_1$ | $y_2$ |
|---|---|---|---|---|
| 1 | -0.03846 | 7.76271 | 0.54272=-1/-1.84256 | 1.1759 |
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# Strong anomaly limit $m^2_{2,k} \to \infty$, $c \neq 0$, $y \neq 0$, $z = 0$

$$P_1 \equiv (\bar{m}^2_{1*}, \bar{\delta}_*, \bar{\lambda}_{\sigma *}, \bar{\lambda}_{a*}) = (0, 0, 0, 14.8044)$$

$$\text{stability matrix eigenvalues: } \{-2, -1, -1, 1\} \ ,$$

$$P_2 \equiv (\bar{m}^2_{1*}, \bar{\delta}_*, \bar{\lambda}_{\sigma *}, \bar{\lambda}_{a*}) = (-0.05556, 1.3680, 0.2166, 9.3595)$$

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$$P_3 \equiv (\bar{m}^2_{1*}, \bar{\delta}_*, \bar{\lambda}_{\sigma *}, \bar{\lambda}_{a*}) = (-0.05556, 0.03166, 0.2166, 0.0050)$$

$$\text{stability matrix eigenvalues: } \{-1.77069, 1.27069, -0.961396, -0.579306\} \ .$$

$$M^2_\sigma = 2/9 \ , \quad M^2_{\pi_i} = 0 \ , \quad M^2_\eta \to \infty \ , \quad M^2_{a_i} \to \infty \ .$$

# Strong anomaly limit $m^2_{2,k} \to \infty$, $c \neq 0$, $y \neq 0$, $z = 0$

$$P_1 \equiv (\bar{m}^2_{1*}, \bar{\delta}_*, \bar{\lambda}_{\sigma*}, \bar{\lambda}_{a*}) = (0, 0, 0, 14.8044)$$

$$\text{stability matrix eigenvalues: } \{-2, -1, -1, 1\} \ ,$$

$$P_2 \equiv (\bar{m}^2_{1*}, \bar{\delta}_*, \bar{\lambda}_{\sigma*}, \bar{\lambda}_{a*}) = (-0.05556, 1.3680, 0.2166, 9.3595)$$

$$\text{stability matrix eigenvalues: } \{-1.77069, 1.27069, 0.886734, -0.628083\} \ ,$$

$$P_3 \equiv (\bar{m}^2_{1*}, \bar{\delta}_*, \bar{\lambda}_{\sigma*}, \bar{\lambda}_{a*}) = (-0.05556, 0.03166, 0.2166, 0.0050)$$

$$\text{stability matrix eigenvalues: } \{-1.77069, 1.27069, -0.961396, -0.579306\} \ .$$

$$M^2_\sigma = 2/9 \ , \quad M^2_{\pi_i} = 0 \ , \quad M^2_\eta \to \infty \ , \quad M^2_{a_i} \to \infty \ .$$

$$SU(2) \times SU(2)/Z(2) \simeq SO(4) \sim O(4)$$

$$\Phi_1 = \sigma t_0 + i\vec{t} \cdot \vec{\pi} \ , \quad \Phi_2 = i\eta t_0 + \vec{t} \cdot \vec{a}$$

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ご清聴ありがとうございました。

**Thank you for your attention!**

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