Eikonal method for halo nuclei

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Motivation

To study elastic scattering and breakup cross sections of $^{11}\text{Li}$ in a four-body eikonal model.

- Bound states
- Continuum states
- Dipole strengths

- Four-body elastic scattering cross sections
Motivation

To study elastic scattering and breakup cross sections of $^{11}$Li in a four-body eikonal model.

- Bound states
- Continuum states
- Dipole strengths
- Breakup cross sections
- Angular distributions
**Introduction**

- High-energy reactions are widely used to investigate Halo nuclei.

- High incident energies permits to handle the Schrödinger equation in a simplified way: Eikonal approximation.

- Non-microscopic 2-Body and 3-Body descriptions of the projectile has been introduced in the eikonal method.

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**Elastic scattering, breakup**

Ex: $^{11}\text{Be} + ^{208}\text{Pb} = (^{10}\text{Be} + n) + ^{208}\text{Pb}$


Ex: $^{6}\text{He} + ^{208}\text{Pb} = (\alpha + n + n) + ^{208}\text{Pb}$

We have to solve the Schrödinger equation

\[ \left[ -\frac{\hbar^2}{2\mu_{PT}} \Delta + V_{PT}(r) \right] \Phi(r) = E \Phi(r). \]

At high-energies the wave function: Smooth deviation from a plane wave

\[ \Phi(r) = \frac{1}{(2\pi)^{3/2}} e^{iKZ} \tilde{\Phi}(r), \]

we have

\[ -\frac{\hbar^2}{2\mu_{PT}} \left[ \Delta + 2iK \frac{\partial}{\partial Z} + V_{PT}(r) \right] \tilde{\Phi}(r) = 0. \]

At high-energies \( |\Delta \tilde{\Phi}| \ll K \left| \frac{\partial \tilde{\Phi}}{\partial Z} \right| \), then

\[ \Phi^{\text{eik}} = \frac{1}{(2\pi)^{3/2}} \exp[iKZ - i\hbar \int_{-\infty}^{Z} V_{PT}(b, Z') dZ']. \]
Eikonal approximation for one body projectile

Ex: Elastic scattering of an incident uncharged particle

The elastic amplitude

\[ f(\theta) = iK \int_0^\infty J_0(qb)[1 - e^{i\chi(b)}]b db; \quad q = 2K \sin \frac{\theta}{2} \]

The eikonal phase

\[ \chi(b) = -\frac{1}{\hbar \nu} \int_{-\infty}^\infty V_{PT}(b, Z) dZ; \quad \nu = \frac{\hbar K}{\mu_{PT}} \]

Extension to charge particles

\[ \chi(b) = \chi_N(b) + \chi_C(b) \]

Nuclear Coulomb

Corrected to overcome divergences due to the Coulomb potential.
Elastic cross sections for $n^+^{208}\text{Pb}$ at different incident energies

Fig 1. The energies are shown in MeV. The $n^+^{208}\text{Pb}$ potential is taken from A. J. Kooning and J. P. Delaroche, Nucl. Phys. A 713, 231 (2003).

- The agreement improves when the energy increases and $\theta$ decreases.
Four-body eikonal

\[ H_{4B} \Phi = E_T \Phi, \quad E_T = E_0 + \frac{\hbar^2 K^2}{2\mu_{PT}} \]

\[ E_0 \rightarrow \text{G. S. energy of the projectile} \]

\[ \frac{\hbar^2 K^2}{2\mu_{PT}} \rightarrow \text{Initial relative P.T. energy} \]

\[ H_{4B} = -\frac{\hbar^2}{2\mu_{PT}} \nabla_R^2 + V_{PT} + H_{3B}, \]

Nuclear optical potentials+Coulomb

\[ V_{PT} = V_{CT} + V_{Tn} + V_{Tn} \]

Factorizing:
\[ \Phi(\vec{R}, \vec{x}, \vec{y}) = e^{iKZ} \phi(\vec{R}, \vec{x}, \vec{y}) \]
\[ \rightarrow \left(-\frac{\hbar^2}{2\mu_{PT}} \nabla_R^2 + i\hbar \partial_Z + V_{PT} \right) \phi = 0 \]

The eikonal approx. (High-energies)
\[ |\nabla^2 \phi| \ll K |\partial_Z \phi| \]
Four-body eikonal

Eikonal w. f.
\[ \Phi^{\text{eik}}(R, x, y) \approx \Psi_0(x, y) \exp \left[ -\frac{i}{\hbar v} \int_{-\infty}^{Z} V_{PT}(b, Z', x, y) dZ' \right] \]

Eikonal elastic amplitude
\[ S(b) = \left\langle \psi_{J_0 M_0' \pi_0} \right| e^{i\chi(b)} \left| \psi_{J_0 M_0 \pi_0} \right\rangle \]
3B bound state
3B bound state
Elastic Cross sections

Eikonal breakup amplitude
\[ S(b) \propto \left\langle \psi_{K_y K_y}^{J_0 M_0} \xi(E) \right| e^{i\chi(b)} \left| \psi_{J_0 M_0 \pi_0} \right\rangle \]
3B scattering
State R-matrix
3B bound state
Bup obs.

Eikonal phase (Dynamics information)
\[ \chi(b) = -\frac{i}{\hbar v} \int_{-\infty}^{\infty} [V_{CT}(b) + V_{nT}(b) + V_{nT}(b)] dZ \]
**Three-body model of the projectile**

\[ H_{3B} \psi^{J\pi} = E \psi^{J\pi} \]

- \( E < 0 \rightarrow \) Bound state
- \( E > 0 \rightarrow \) Scattering states

\[ \rho^2 = x^2 + y^2: \text{ Hyperradius} \]

\[ \alpha = \arctan \left( \frac{y}{x} \right): \text{ Hyperangle} \]

\[ \Omega_5 = (\alpha, \Omega_x, \Omega_y) \]

\[ E = -S_{2n} \]

- \( E = 0 \text{ MeV, core} + n + n \)
Three-body model of the projectile

\[ H_{3B} \psi_{J\pi} = E \psi_{J\pi} \]

\[ H_{3B} = -\frac{\hbar^2}{2m} \nabla_x^2 - \frac{\hbar^2}{2m} \nabla_y^2 + T_{c.m.} + \sum_{i<j} V_{ij} \]

2B potentials, Vcn Gaussian, W. Saxon

\[ \psi_{J\pi} = \rho^{-5/2} \sum_{K=0}^{\infty} \sum_{\gamma} \chi_{\gamma K}^{J\pi}(\rho) \chi_{\gamma K}^{J M}(\Omega_5) \]

Kmax

Hyperradial Function (Unknown)

Eigenfunction of angular momentum \( K \) (Known)

\( \pi = (-1)^K \rightarrow \text{Parity of the relative motion of the 3B} \)

\( \gamma = (l_x, l_y, L, S) \)

\( \hat{L} = \hat{l}_x + \hat{l}_y \)

\( \hat{S} = \hat{S}_1 + \hat{S}_2 \)

\( \hat{f} = \hat{L} + \hat{S} \)
Three-body bound states

\[ H_{3B} \Psi^{J\pi} = E \Psi^{J\pi} \]

\[ \Psi^{J\pi} = \rho^{-5/2} \sum_{K=0}^{\infty} \sum_{\gamma} \chi^{J\pi}_{\gamma K}(\rho) y^{JM}_{\gamma K}(\Omega_5) \]

\[ \chi^{J\pi}_{\gamma K}(\rho) = \sum_{i=1}^{N} C^{J\pi}_{\gamma K i} u_i(\rho), \]

Eigenvalue problem

Lagrange basis

It facilitates the calculations
Three-body continuum states: R-matrix

Internal region

\[ \chi_{\gamma K}^{J\pi}(\rho) = \sum_{i=1}^{N} C_{\gamma Ki}^{J\pi} u_i(\rho) \]

External region

\[ \chi_{\gamma K}^{J\pi}(\rho \to \infty) \]

Nuclear + Coulomb + Centrifugal potentials

\[ \chi_{\gamma K}^{J\pi}(\rho \to \infty) = A_{\gamma K}^{J\pi} \left[ H_{\gamma K}^{-}(k\rho) \delta_{\gamma \gamma'} \delta_{K K'} - U_{\gamma K, \gamma' K'}^{J\pi} H_{\gamma K}^{+}(k\rho) \right] \]

Hankel functions

\[ U_{\gamma K, \gamma' K'}^{J\pi} \to \text{Collision matrix} \to e^{2i\delta} \to \text{Eigenphases} \]

Large matrix for typical \( \gamma K \) values
### Dimension of the R-matrix calculations

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\[ \gamma = (l_x, l_y, L, S) \]

\[ N \rightarrow \text{Number of Lagrange basis, typical } N = 40 \]

\[ \gamma K \rightarrow \text{Channels number} \]

Matrices of: \[ \gamma KN \times \gamma KN \]

**Example:** \( J = 2^+ \) and \( K_{\text{max}} = 20 \)

Matrices of: \[ \gamma KN \times \gamma KN = 265 \cdot 40 \times 265 \cdot 40 = 10600 \times 10600 \]
Applications for $^6$He: Three-body resonances

- $R^{J\pi} \rightarrow U^{J\pi} \rightarrow (S^{-1}US = e^{2i\delta})$

- Information about three-body resonances is contained in the eigenphases $\delta$.

Fig. 2. Eigenphases for $^6$He for different $J$ values (From P. Descouvemont et al, Nucl. Phys. A 765 (2006) 370).
Applications for $^6$He: E1 strength distribution

$$\frac{dB_{E1}}{dE}(E) \propto \left| \langle \Psi_{k_x k_y}(E) | M^{E1} | \psi_{J_0 M_0 \pi_0} \rangle \right|^2$$

1$^-$ 3B cont. R-matrix  0$^+$ 3B bound state

Fig. 3. Electric dipole distribution for different $K_{\text{max}}$ values. From D. Baye et al, Phys. Rev. C 79, 024607 (2009).
Applications in $^{11}\text{Li}$: Conditions of the calculations

To calculate bound and scattering states of $^9\text{Li}+n+n$

$^9\text{Li}+n$ interaction

- Non-existent elastic scattering experimental data.
- Fitted to reproduce a presumed $p_{1/2}$ resonance at 540 keV and a s virtual state.
- $^9\text{Li}-n$ interaction multiplied by 1.0056 to reproduce G.S. energy of $^{11}\text{Li} = -0.378$ MeV.

$n+n$ potential

- Minnesota interaction

We those potentials we well reproduce r.m.s. radius of $^{11}\text{Li}$: 3.1 fm (exp. r.m.s of 3.16 ±0.11 fm).
Like-resonant behavior for $1^-$ and $2^+$ continuum

Rise of the $0^+$ phase shift with energy: “Like a superposition of resonances”
Convoluted E1 strength distribution of $^{11}$Li with the detector response

$R$-matrix (Red curve)

$$\frac{dE_1}{dE}(E) \propto \left| \frac{\psi_{k_x k_y}(E)}{M^{E_1}} \right|^2$$

$$\sigma = 0.17\sqrt{E}$$

Fig 5. The $\sigma$ value is in MeV. Experimental Data from T. Nakamura et. al, Phys. Rev. Lett. 252502 (2006).

- We overestimate the E1 distribution in the peak region.
Some applications by D. Baye, P. Capel, P. Descouvemont and Y. Suzuki, *Phys. Rev. C* 71, 024607 (2009). They described the elastic breakup cross section of $^{6}$He on $^{208}$Pb @ 70 A MeV.

**Qualities of the model:**

- Contributions different from the dipole.
- It does not require $^{6}$He-$^{208}$Pb potential: $\alpha$-$^{208}$Pb potential and n-$^{208}$Pb potential are well known.
- It takes nuclear and Coulomb effects and their interference on the same footing.
- There is not a adjustable parameter.
Conditions of the calculations for $^{11}\text{Li}$ on $^{208}\text{Pb}$

To calculate the breakup cross sections of $^{11}\text{Li}$ on $^{208}\text{Pb}$ @ 70 A MeV:

- $^9\text{Li}^\rightarrow^\text{208}\text{Pb}$ potential (lack of the potential):
  Renormalized $(9^{1/3}+208^{1/3})$ $\alpha^{\text{208Pb}}$ interaction @ 70 A MeV of B. Bonin et. al. (Following the same idea of P. Capel et. al, Phys. Rev. 68, 014612 (2003) for $^{10}\text{Be}$ on $^{208}\text{Pb}$).

- Variation of the $^9\text{Li}^\rightarrow^\text{208}\text{Pb}$ potential was checked but it did not provide a significant change to the breakup and angular distributions.

- $\text{n}^\rightarrow^\text{208Pb}$ potential:
Fig. 6. Partial and total eikonal breakup cross sections.

- Small correction of the $0^+$ and $2^+$ partial waves to the total cross section.
Convoluted breakup eikonal cross section with the detector response

$^{11}\text{Li on } ^{208}\text{Pb } @ 70 \text{ A MeV}$

Theoretical data convoluted with a Gaussian of $\sigma = 0.17\sqrt{E}$ MeV

Fig. 7. Exp. Data from T. Nakamura et. al, phys. Rev. Lett. 252502 (2006).
Fig. 8. Partial, total and convoluted total angular distributions. Experimental Data from T. Nakamura et. al, Phys. Rev. Lett. 252502 (2006).

- Very good agreement of the total convoluted curve for almost all angles.
- Appreciable $0^+$ and $2^+$ contributions after $\theta \gtrapprox 1$ deg.
Convoluted E1 strength distribution of $^{11}$Li with the detector response

$\sigma = 0.17\sqrt{E}$

**Why we overestimate the E1 distribution?**

Fig. 9. The $\sigma$ value is in MeV. Experimental Data from *T. Nakamura et. al, Phys. Rev. Lett.* 252502 (2006).
In the breakup reactions of $^{11}\text{Li}+^{208}\text{Pb} @ 70 \text{ A MeV}$

- $\frac{d\sigma}{dE}$ is measured directly $\rightarrow$ We fit the data
- $\frac{d\sigma}{d\theta}$ is measured directly $\rightarrow$ We fit the data
- $\frac{dB(E1)}{dE}$ is measured indirectly
  (It depends on model assumptions) $\rightarrow$ We do not fit the data
How is determined experimentally $dB(E1)/dE$?

It is extracted from the equivalent photon method as

$$
\frac{d\sigma^{\text{Exp}}}{dE} = \frac{16\pi^3}{9\hbar c} \frac{dB^{\text{Exp}}(E1)}{dE} \int_{b_{\text{min}}}^{\infty} 2\pi db b N_{E1}(b, E)
$$

- From $b_{\text{min}}$ to exclude nuclear excitation.
- $N_{E1}(b, E) \rightarrow$ Number of virtual photons incident on $^{11}\text{Li}$ by unit area.
- It is assumed to be one step and dominated by a single E1 multipolar transition.
- It comes from semi-classical perturbation theory.

$^{11}\text{Li}$ is excited by absorption of a virtual photon from the Coulomb field of the target.
Estimation of the $\theta_c$ dependence in the dipole distribution of $^{11}\text{Li}$

In non-relativistic regime

$$
\frac{dB^{\text{Exp}}(E1)}{dE} = \frac{9}{32\pi} \left( \frac{\hbar v}{Z_T e} \right)^2 \frac{1}{\xi_{\text{min}} K_0(\xi_{\text{min}}) K_1(\xi_{\text{min}})} \frac{d\sigma^{\text{Exp}}}{d\Omega}
$$

$\nu \rightarrow$ Projectile-target relative velocity,

$$
\xi_{\text{min}} = \frac{E - E_0}{\hbar v} b_{\text{min}},
$$

$E \rightarrow$ Excitation energy of $^{11}\text{Li}$, $E_0 \rightarrow$ G. S. energy of $^{11}\text{Li}$

$$
b_{\text{min}} = \frac{Z_P Z_T e^2}{2 \tan \left( \frac{\theta_c}{2} \right)} \rightarrow \text{Min. Impact parameter for the semi-classical Coulomb trajectory}
$$

$\theta_c \rightarrow$ maximum scattering angle (beyond $\theta_c$ nuclear interaction is important.)
Estimation of the $\theta_c$ dependence in the dipole distribution of $^{11}\text{Li}$

Fig. 10. The $\theta_c$ values of 0.9, 1.46 and 2 deg correspond to $b_{\text{min}}$ of 31, 19 and 14 fm respectively.

- Small $\theta_c$ provides a larger dipole distribution at low excitation energies.
Elastic scattering of $^{11}$Li on $^{208}$Pb @ 70 A MeV in the Eikonal method

- Reduction in the $^{11}$Li+$^{208}$Pb elastic scattering due to flux going to breakup.
- $0 \leq \theta \leq 1 \rightarrow$ Rutherford scattering.
Conclusions

- We have predicted a $1^-$ resonant eigenphase for $^{11}\text{Li}$.

- The maximal contribution for the total breakup cross section is coming from the $1^-$ partial wave.

- The breakup cross sections and angular distributions of $^{11}\text{Li}$ on $^{208}\text{Pb}$ are in good agreement with the experimental data.

- To test our model we suggest to experimentalists to measure elastic scattering of $^{11}\text{Li}$ at high-energies.

- We need to clarify why we overestimate the dipole strength distribution of $^{11}\text{Li}$ with the same $^{11}\text{Li}$ wave functions that we had successful results for the breakup and angular distributions. Ideas are welcome!

Thank you for your attention